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Numerical-analytical method for solving initial-boundary value problem for loaded parabolic equation

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An initial-boundary value problem for a loaded parabolic equation in a rectangular domain was considered. By discretization with respect to a spatial variable, the problem under study is reduced to the initial problem for a system of loaded ordinary differential equations. Based on the previously obtained results of Dzhumabaev and Assanova, an estimate for the solution of the original initial-boundary value problem for a loaded parabolic equation was established. An auxiliary initial problem for a system of loaded ordinary differential equations is solved by the Dzhumabaev parameterization method. Conditions of the unique solvability of the considering problem are obtained and algorithms for finding a solution are constructed. The results are illustrated with a numerical example.

Keywords: loaded parabolic equations, initial-boundary value problem, solvability conditions, parameterization method, polygonal method, numerical solution.

2020 Mathematics Subject Classification: 35K20, 34K10, 35R10, 65M22.

Introduction

The parabolic partial differential equations play a very important role in many branches of science and engineering. Applied problems involving boundary value problems for parabolic equations include: thermal analysis in engineering, groundwater flow, climate modeling, biological processes, chemical reactions, environmental engineering, material science and financial engineering [1–5].

Loaded parabolic equations are a type of parabolic partial differential equations that include additional terms or conditions representing external influences or interactions, which can vary over time and space. These "loads" can be functions or integrals that are added to the standard parabolic equation. Loaded parabolic equations often arise in various physical and engineering applications where external sources, sinks, or other dynamic interactions need to be modeled [6]. These equations are used to model more complex systems where simple diffusion or heat conduction is modified by additional processes such as external forces, reaction terms, or other dynamic effects. For information on various boundary value problems for loaded parabolic differential equations, refer to works [7–11].

A boundary value problem for a linear parabolic equation without loading was considered in works [12, 13]. Using the polygonal method, estimates of the solution and their derivatives were obtained in terms of the equation coefficients and boundary conditions [12]. Coefficient estimates of solutions and the first derivative with respect to x of a linear boundary value problem for a parabolic equation with one spatial variable were obtained [13].

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The numerical research of boundary value problems for a parabolic equations with and without loading are of great interest due to their broad range of applications. Several methods for solving these problems have been developed [14–18].

This work is aimed at development of methods for solving the initial-boundary value problem for parabolic equation, proposed in [12, 13] to the initial-boundary value problem for a loaded parabolic equation of the following form

$$\frac{\partial u}{\partial t} = p(t,x)\frac{\partial^2 u}{\partial x^2} + q(t,x)u(t,x) + \sum_{j=1}^{m+1} k_j(t,x)u(\xi_j,x) + f(t,x), \quad (t,x) \in \Omega = (0,T) \times (0,\omega), \tag{1}$$

$$u(0,x) = \varphi(x), \qquad x \in [0,\omega], \tag{2}$$

$$u(t,0) = \psi_0(t), \qquad u(t,\omega) = \psi_1(t), \qquad t \in [0,T], \tag{3}$$

where functions $p(t,x) > 0$, $q(t,x) \le 0$, $m \in \mathbb{N}$, $k_j(t,x)$, $j = \overline{1,m+1}$, $f(t,x)$ are continuous with respect to t and Holder continuous with respect to x; functions $\varphi(x)$, $\psi_0(t)$, $\psi_1(t)$ are sufficiently smooth and $\psi_0(0) = \varphi(0)$, $\psi_1(0) = \varphi(\omega)$ matching conditions are performed.

A solution to problem (1)–(3) is a function $u(t,x)$, continuous on $\bar{\Omega} = [0,T] \times [0,\omega]$ that has continuous first-order partial derivatives with respect to t and continuous second-order partial derivatives with respect to x. It satisfies the loaded differential equation (1) and boundary conditions (2), (3).

By discretizing with respect to the spatial variable x, problem (1)–(3) transforms to a problem for systems of loaded ordinary differential equations. An auxiliary problem for a system of loaded ordinary differential equations will be investigated. Based on the properties of solutions to the auxiliary problem, estimates for the solution of the original initial-boundary value problem for the loaded parabolic equation will be established. In this case, the approach proposed in works [12, 13] will be used. A numerical method for solving the initial problem for systems of loaded ordinary differential equations is also proposed. A numerical implementation of the algorithm for the initial-boundary value problem for a loaded parabolic equation is presented. The error between the exact solution of the problem under consideration and its numerical discrete solution has been established.

1 Problem formulation

We take $\forall\tau$ and produce a discretization by variable x: $x_i = i\tau$, $i = \overline{0,N}$, $N\tau = \omega$.

We present the following notations: $u_i(t) = u(t,i\tau)$, $p_i(t) = p(t,i\tau)$, $q_i(t) = q(t,i\tau)$, $k_j^i(t) = k_j(t,i\tau)$, $j = \overline{1,m}$, $f_i(t) = f(t,i\tau)$, $\varphi_i = \varphi(i\tau)$, $i = \overline{0,N}$.

Using these notations, the problem (1)–(3) is transformed into the following problem

$$\frac{du_i}{dt} = p_i(t)\frac{u_{i+1} - 2u_i + u_{i-1}}{\tau^2} + q_i(t)u_i + \sum_{j=1}^{m+1} k_j^i(t)u_i(\xi_j) + f_i(t), \quad i = \overline{1,N-1}, \tag{4}$$

$$u_i(0) = \varphi_i, \qquad i = \overline{1,N-1}, \tag{5}$$

$$u_0(0) = \varphi_0, \qquad u_N(0) = \varphi_N, \qquad u_0(t) = \psi_0(t), \qquad u_N(t) = \psi_1(t), \qquad t \in [0,T]. \tag{6}$$

Here from relation (6) it is clear that functions $u_0(t)$ and $u_N(t)$ are known.

Due to the linearity of the system for $\forall\tau > 0$ there is a unique solution to problem (4)–(6) defined on $[0,T]$: $\{u_1(t), u_2(t), \ldots, u_{N-1}(t)\}$. Relating the functions u_{i+1}, u_{i-1} to the right side of each i-th equation of system (4), we apply the estimate from work [19]

$$\|u_i\| = \max_{t\in[0,T]}\{|u_i(t)|\} \le \max\Bigg\{|\varphi_i|, \frac{1}{2}\left\|\frac{u_{i-1}(t)}{1+\frac{|q_i(t)|\tau^2}{2p_i(t)}}\right\| + \frac{1}{2}\left\|\frac{u_{i+1}(t)}{1+\frac{|q_i(t)|\tau^2}{2p_i(t)}}\right\|$$

$$+\frac{1}{2}\left\|\frac{f_i(t)}{p_i(t)\left[1+\frac{|q_i(t)|\tau^2}{2p_i(t)}\right]}\right\|\tau^2 + \frac{1}{2}\left\|\frac{\sum\limits_{j=1}^{m+1}k_j^i(t)u_i(\xi_j)}{p_i(t)\left[1+\frac{|q_i(t)|\tau^2}{2p_i(t)}\right]}\right\|\tau^2\Bigg\}$$

$$\le \max\left\{|\varphi_i|, \frac{1}{2}\|u_{i-1}\| + \frac{1}{2}\|u_{i+1}\| + \frac{1}{2}\max_{t\in[0,T]}\left\|\frac{f_i(t)}{p_i(t)}\right\|\tau^2 + \frac{1}{2}\max_{t\in[0,T]}\left\|\frac{\sum\limits_{j=1}^{m+1}k_j^i(t)u_i(\xi_j)}{p_i(t)}\right\|\tau^2\right\}.$$

From here it is easy to obtain the following inequality

$$\eta_i \le \frac{1}{2}\eta_{i-1} + \frac{1}{2}\eta_{i+1} + \frac{1}{2}\max_{t\in[0,T]}\left\|\frac{f_i(t)}{p_i(t)}\right\|\tau^2 + \frac{1}{2}\max_{t\in[0,T]}\sum_{j=1}^{m+1}\left\|\frac{k_j^i(t)}{p_i(t)}\right\|\cdot\|u_i\|\,\tau^2, \quad i=\overline{1,N-1}, \tag{7}$$

where $\eta_i = \max\{\hat{\varphi}, \|u_i\|\}$, $\hat{\varphi} = \max\limits_{i=\overline{1,N}}\{|\varphi_i|\}$.

Similarly to [12, 13], using the sweep down and up from (7), we get

$$\eta_i \le \frac{N-i}{N}\eta_0 + \frac{i}{N}\eta_N + \frac{N-i}{N}\sum_{l=1}^{i}\max_{t\in[0,T]}\left\|l\frac{f_l(t)}{p_l(t)}\right\|\tau^2 + \frac{i}{N}\sum_{l=i+1}^{N-1}\max_{t\in[0,T]}\left\|(N-l)\frac{f_l(t)}{p_l(t)}\right\|\tau^2$$

$$+\frac{N-i}{N}\sum_{l=1}^{i}\sum_{j=1}^{m+1}\max_{t\in[0,T]}\left\|l\frac{k_j^l(t)}{p_l(t)}\right\|\tau^2\cdot\eta_i + \frac{i}{N}\sum_{l=i+1}^{N-1}\sum_{j=1}^{m+1}\max_{t\in[0,T]}\left\|(N-l)\frac{k_j^l(t)}{p_l(t)}\right\|\tau^2\cdot\eta_i.$$

Considering that $\|u_i\| \le \eta_i$, we set

$$\|u_i\| \le \frac{N-i}{N}\max\{\hat{\varphi}, \|\psi_0\|\} + \frac{i}{N}\max\{\hat{\varphi}, \|\psi_1\|\} + \frac{N-i}{N}\sum_{l=1}^{i}\max_{t\in[0,T]}\left\|l\frac{f_l(t)}{p_l(t)}\right\|\tau^2$$

$$+\frac{i}{N}\sum_{l=i+1}^{N-1}\max_{t\in[0,T]}\left\|(N-l)\frac{f_l(t)}{p_l(t)}\right\|\tau^2 + \Theta_i\|u_i\|, \tag{8}$$

where $\Theta_i = \frac{N-i}{N}\sum\limits_{l=1}^{i}\sum\limits_{j=1}^{m+1}\max\limits_{t\in[0,T]}\left\|l\frac{k_j^l(t)}{p_l(t)}\right\|\tau^2 + \frac{i}{N}\sum\limits_{l=i+1}^{N-1}\sum\limits_{j=1}^{m+1}\max\limits_{t\in[0,T]}\left\|(N-l)\frac{k_j^l(t)}{p_l(t)}\right\|\tau^2 < 1$.

Then

$$\|u_i\| \le \frac{1}{1-\Theta_i}\Bigg\{\frac{N-i}{N}\max\{\varphi, \|\psi_0\|\} + \frac{i}{N}\max\{\varphi, \|\psi_1\|\}$$

$$+\frac{N-i}{N}\sum_{l=1}^{i}\max_{t\in[0,T]}\left\|l\frac{f_l(t)}{p_l(t)}\right\|\tau^2 + \frac{i}{N}\sum_{l=i+1}^{N-1}\max_{t\in[0,T]}\left\|(N-l)\frac{f_l(t)}{p_l(t)}\right\|\tau^2\Bigg\}.$$

From inequality (8), reasoning in the same way as in [13] and [19], we obtain the validity of the following statement:

Theorem 1. Let

a) the assumptions with respect to the data of problem (1)–(3) be fulfilled;

b) the inequality $\Theta(x) = \frac{\omega-x}{\omega}\int\limits_0^x z\sum\limits_{j=1}^{m+1}\max\limits_{t\in[0,T]}\left\|\frac{k_j(t,z)}{p(t,z)}\right\|dz + \frac{x}{\omega}\int\limits_x^\omega(\omega-z)\sum\limits_{j=1}^{m+1}\max\limits_{t\in[0,T]}\left\|\frac{k_j(t,z)}{p(t,z)}\right\|dz < 1$ be valid for all $x\in[0,\omega]$.

Then the problem (1)–(3) has a unique classical solution $u^*(t,x)$, and for it the following inequality holds:

$$\max_{t\in[0,T]}|u^*(t,x)| \le \frac{\omega-x}{\omega[1-\Theta(x)]}\max\Big\{\max_{x\in[0,\omega]}|\varphi(x)|, \max_{t\in[0,T]}|\psi_0(t)|\Big\}$$
$$+\frac{x}{\omega[1-\Theta(x)]}\max\Big\{\max_{x\in[0,\omega]}|\varphi(x)|, \max_{t\in[0,T]}|\psi_1(t)|\Big\}$$
$$+\frac{\omega-x}{\omega[1-\Theta(x)]}\int\limits_0^x z\max_{t\in[0,T]}\left\|\frac{f(t,z)}{p(t,z)}\right\|dz + \frac{x}{\omega[1-\Theta(x)]}\int\limits_x^\omega(\omega-z)\max_{t\in[0,T]}\left\|\frac{f(t,z)}{p(t,z)}\right\|dz.$$

The proof of Theorem 1, with minor modifications, follows the same principles as the proof of Theorem in [13].

Thus, we have established an estimate for the solution of the original initial-boundary value problem of the loaded parabolic equation (1)–(3).

Substituting $u_0(t)$ and $u_N(t)$ into the system of loaded equations (4), the discretized problem (4)–(6) can be written in the following matrix-vector form

$$\frac{d\tilde u}{dt} = \mathcal{A}(t)\tilde u + \sum_{j=1}^{m+1}\mathcal{K}_j(t)\tilde u(\xi_j) + \mathcal{F}(t), \quad \tilde u\in\mathbb{R}^{N-1}, \quad t\in[0,T), \tag{9}$$

$$\tilde u(0) = \Phi, \qquad \Phi\in\mathbb{R}^{N-1}, \tag{10}$$

where $\tilde u(t) = \big(u_1(t), u_2(t), \ldots, u_{N-1}(t)\big)$ is unknown function, the $(N-1)\times(N-1)$ matrices $\mathcal{A}(t)$, $\mathcal{K}_j(t)$, $j=\overline{1,m+1}$, and $(N-1)$ vector-function $\mathcal{F}(t)$ are continuous on $[0,T]$, $0=\xi_0<\xi_1<\ldots<\xi_m<\xi_{m+1}=T$. Here

$$\mathcal{A}(t) = \begin{bmatrix} -\frac{2p_1(t)}{\tau^2}+q_1(t) & \frac{p_1(t)}{\tau^2} & 0 & \ldots & 0\\ \frac{p_2(t)}{\tau^2} & -\frac{2p_2(t)}{\tau^2}+q_2(t) & \frac{p_2(t)}{\tau^2} & \ldots & 0\\ 0 & \frac{p_3(t)}{\tau^2} & -\frac{2p_3(t)}{\tau^2}+q_3(t) & \ldots & 0\\ \ldots & \ldots & \ldots & \ddots & \ldots\\ 0 & 0 & 0 & \ldots & -\frac{2p_{N-1}(t)}{\tau^2}+q_{N-1}(t)\end{bmatrix},$$

$$\mathcal{K}_j(t) = \begin{bmatrix} k_j^1(t) & 0 & 0 & \ldots & 0\\ 0 & k_j^2(t) & 0 & \ldots & 0\\ 0 & 0 & k_j^3(t) & \ldots & 0\\ \ldots & \ldots & \ldots & \ddots & \ldots\\ 0 & 0 & 0 & \ldots & k_j^{N-1}(t)\end{bmatrix}, \quad j=\overline{1,m+1},$$

$$\mathcal{F}(t) = \begin{bmatrix} \frac{p_1(t)}{\tau^2}\psi_0(t)+f_1(t)\\ f_2(t)\\ f_3(t)\\ \vdots\\ \frac{p_{N-1}(t)}{\tau^2}\psi_1(t)+f_{N-1}(t)\end{bmatrix}, \qquad \Phi = \begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \vdots\\ \varphi_{N-1}\end{bmatrix}.$$

A solution to problem (9), (10) is a vector function $\tilde{u}(t)$, which is continuous on $[0,T]$ and continuously differentiable on $(0,T)$. This function satisfies the loaded differential equation (9) and the condition (10).

2 *Solving problem* (9), (10) *by using the parameterization method*

We will use the approach proposed in [20–24] to solve the initial value problem for loaded differential equations (9), (10). This approach relies on the algorithms of the Dzhumabaev parametrization method [19,25] and numerical methods. The implementation and efficiency of this method for finding analytical and numerical solutions to boundary value problems for various differential equations are shown in [26–32].

The interval $[0,T]$ is partitioned into subintervals by loading points: $[0,T)=\bigcup\limits_{s=1}^{m+1}[\xi_{s-1},\xi_s)$.

Define the space $C([0,T],\xi_s,\mathbb{R}^{(N-1)(m+1)})$ consisting of system functions $\tilde{u}[t]=(\tilde{u}_1(t),\tilde{u}_2(t),\ldots,\tilde{u}_{m+1}(t))$, where each $\tilde{u}_s:[\xi_{s-1},\xi_s)\to\mathbb{R}^{N-1}$ is continuous on $[\xi_{s-1},\xi_s)$ and has finite left-sided limits $\lim\limits_{t\to\xi_s-0}\tilde{u}_s(t)$ for all $s=\overline{1,m+1}$. The norm on this space is defined as $||\tilde{u}[\cdot]||_2=\max\limits_{s=\overline{1,m+1}}\sup\limits_{t\in[\xi_{s-1},\xi_s)}||\tilde{u}_s(t)||$.

The restriction of the function $\tilde{u}(t)$ to the interval $[\xi_{s-1},\xi_s)$ is denoted by $\tilde{u}_s(t)$, meaning $\tilde{u}_s(t)=\tilde{u}(t)$ for $t\in[\xi_{s-1},\xi_s)$, $s=\overline{1,m+1}$. We introduce additional parameters $\mu_s=\tilde{u}_{s+1}(\xi_s)$, $s=\overline{1,m}$, $\mu_{m+1}=\tilde{u}(\xi_{m+1})$. By making the substitution $\tilde{u}_1(t)=v_1(t)+\Phi$ on $[\xi_0,\xi_1)$ and $\tilde{u}_s(t)=v_s(t)+\mu_{s-1}$ on each interval $[\xi_{s-1},\xi_s)$, $s=\overline{2,m+1}$, we obtain multi-point initial value problem with parameters

$$\frac{dv_1}{dt}=\mathcal{A}(t)(v_1+\Phi)+\sum_{j=1}^{m+1}\mathcal{K}_j(t)\mu_j+\mathcal{F}(t),\quad t\in[\xi_0,\xi_1),\tag{11}$$

$$\frac{dv_s}{dt}=\mathcal{A}(t)(v_s+\mu_{s-1})+\sum_{j=1}^{m+1}\mathcal{K}_j(t)\mu_j+\mathcal{F}(t),\quad t\in[\xi_{s-1},\xi_s),\quad s=\overline{2,m+1},\tag{12}$$

$$v_s(\xi_{s-1})=0,\qquad s=\overline{1,m+1},\tag{13}$$

$$\Phi+\lim_{t\to\xi_1-0}v_1(t)=\mu_1,\tag{14}$$

$$\mu_{s-1}+\lim_{t\to\xi_s-0}v_s(t)=\mu_s,\qquad s=\overline{2,m+1}.\tag{15}$$

A pair $(v^*[t],\mu^*)$, where the elements are $v^*[t]=\big(v_1^*(t),v_2^*(t),\ldots,v_{m+1}^*(t)\big)\in C([0,T],\xi_s,R^{(N-1)(m+1)})$, $\mu^*=(\mu_1^*,\mu_2^*,\ldots,\mu_{m+1}^*)\in R^{(N-1)(m+1)}$, is said to be a solution to problem (11)–(15) if the functions $v_s^*(t)$, $s=\overline{1,m+1}$, are continuously differentiable on $[\xi_{s-1},\xi_s)$ and satisfy (11), (12) and additional conditions (14), (15) with $\mu_j=\mu_j^*$, $j=\overline{1,m+1}$, and initial conditions (13).

Problems (9), (10) and (11)–(15) are equivalent. If the $\tilde{u}^*(t)$ is a solution of problem (9), (10), then the pair $(v^*[t],\mu^*)$, where $v^*[t]=\Big(\tilde{u}^*(t)-\Phi,\tilde{u}^*(t)-\tilde{u}^*(\xi_1),\ldots,\tilde{u}^*(t)-\tilde{u}^*(\xi_m)\Big)$, and $\mu^*=\Big(\tilde{u}^*(\xi_1),\tilde{u}^*(\xi_2),\ldots,\tilde{u}^*(\xi_m),\tilde{u}^*(\xi_{m+1})\Big)$, is a solution to problem (11)–(15). Conversely, if the pair $(\widetilde{v}[t],\widetilde{\mu})$ with elements $\widetilde{v}[t]=\big(\widetilde{v}_1(t),\widetilde{v}_2(t),\ldots,\widetilde{v}_{m+1}(t)\big)\in C([0,T],\xi_s,R^{(N-1)(m+1)})$, $\widetilde{\mu}=(\widetilde{\mu}_1,\widetilde{\mu}_2,\ldots,\widetilde{\mu}_{m+1})\in R^{(N-1)(m+1)}$ is a solution to problem (11)–(15), then the function $\widetilde{\tilde{u}}(t)$ defined by the equalities $\widetilde{\tilde{u}}(t)=\widetilde{v}_1(t)+\Phi$, $t\in[\xi_0,\xi_1)$, $\widetilde{\tilde{u}}(t)=\widetilde{v}_s(t)+\widetilde{\mu}_{s-1}$, $t\in[\xi_{s-1},\xi_s)$, $s=\overline{2,m+1}$, and $\widetilde{\tilde{u}}(T)=\widetilde{\mu}_{m+1}$, will be the solution of the original problem (9), (10).

By employing the fundamental matrix $\mathcal{X}_s(t)$ of differential equation $\frac{d\tilde{u}}{dt} = \mathcal{A}(t)\tilde{u}$ on $[\xi_{s-1}, \xi_s]$, $s = \overline{1, m+1}$, we transform the solution of an initial value problem for a differential equations with parameters (11)–(13) into an equivalent system of integral equations:

$$v_1(t) = \mathcal{X}_1(t) \int\limits_{\xi_0}^{t} \mathcal{X}_1^{-1}(\eta)\mathcal{A}(\eta)d\eta \cdot \Phi + \mathcal{X}_1(t) \int\limits_{\xi_0}^{t} \mathcal{X}_1^{-1}(\eta) \sum_{j=1}^{m+1} \mathcal{K}_j(\eta)d\eta\mu_j$$
$$+ \mathcal{X}_1(t) \int\limits_{\xi_0}^{t} \mathcal{X}_1^{-1}(\eta)\mathcal{F}(\eta)d\eta, \quad t \in [\xi_0, \xi_1), \tag{16}$$

$$v_s(t) = \mathcal{X}_s(t) \int\limits_{\xi_{s-1}}^{t} \mathcal{X}_s^{-1}(\eta)\mathcal{A}(\eta)d\eta \cdot \mu_{s-1} + \mathcal{X}_s(t) \int\limits_{\xi_{s-1}}^{t} \mathcal{X}_s^{-1}(\eta) \sum_{j=1}^{m+1} \mathcal{K}_j(\eta)d\eta\mu_j$$
$$+ \mathcal{X}_s(t) \int\limits_{\xi_{s-1}}^{t} \mathcal{X}_s^{-1}(\eta)\mathcal{F}(\eta)d\eta, \quad t \in [\xi_{s-1}, \xi_s), \quad s = \overline{2, m+1}. \tag{17}$$

By substituting the respective expressions from (16), (17) into the conditions (14) and (15), we get a system of linear algebraic equations with respect to the parameters μ_s, $s = \overline{1, m+1}$:

$$\mathcal{X}_1(\xi_1) \int\limits_{\xi_0}^{\xi_1} \mathcal{X}_1^{-1}(\eta) \sum_{j=1}^{m+1} \mathcal{K}_j(\eta)d\eta\mu_j - \mu_1$$
$$= -\Phi - \mathcal{X}_1(\xi_1) \int\limits_{\xi_0}^{\xi_1} \mathcal{X}_1^{-1}(\eta)\mathcal{A}(\eta)d\eta \cdot \Phi - \mathcal{X}_1(\xi_1) \int\limits_{\xi_0}^{\xi_1} \mathcal{X}_1^{-1}(\eta)\mathcal{F}(\eta)d\eta, \tag{18}$$

$$\mu_{s-1} + \mathcal{X}_s(\xi_s) \int\limits_{\xi_{s-1}}^{\xi_s} \mathcal{X}_s^{-1}(\eta)\mathcal{A}(\eta)d\eta \cdot \mu_{s-1} - \mu_s + \mathcal{X}_s(\xi_s) \int\limits_{\xi_{s-1}}^{\xi_s} \mathcal{X}_s^{-1}(\eta) \sum_{j=1}^{m+1} \mathcal{K}_j(\eta)d\eta\mu_j$$
$$= -\mathcal{X}_s(\xi_s) \int\limits_{\xi_{s-1}}^{\xi_s} \mathcal{X}_s^{-1}(\eta)\mathcal{F}(\eta)d\eta, \quad t \in [\xi_{s-1}, \xi_s), \quad s = \overline{2, m+1}. \tag{19}$$

Unknown parameters μ_s, $s = \overline{1, m+1}$ can be found using the system (18), (19). Using $O \in \mathbb{R}^{N-1, N-1}$ zero matrix, $I \in \mathbb{R}^{N-1, N-1}$ identity matrix and

$$y_s(B, t) = \mathcal{X}_s(t) \int\limits_{\xi_{s-1}}^{t} \mathcal{X}_s^{-1}(\eta)B(\eta)d\eta, \quad s = \overline{1, m}$$

notations, we write system (18), (19) in the following form

$$Q_*(\xi)\mu = F_*(\xi), \quad \mu \in \mathbb{R}^{(N-1)(m+1)}, \tag{20}$$

where

$$Q_*(\xi) = \begin{bmatrix} y_1(\mathcal{K}_1, \xi_1) - I & y_1(\mathcal{K}_2, \xi_1) & y_1(\mathcal{K}_3, \xi_1) & \dots & y_1(\mathcal{K}_{m+1}, \xi_1) \\ I + y_2(\mathcal{A}, \xi_2) + y_2(\mathcal{K}_1, \xi_2) & y_2(\mathcal{K}_2, \xi_2) - I & y_2(\mathcal{K}_3, \xi_2) & \dots & y_2(\mathcal{K}_{m+1}, \xi_2) \\ y_3(\mathcal{K}_1, \xi_3) & I + y_3(\mathcal{A}, \xi_3) + y_3(\mathcal{K}_2, \xi_3) & y_3(\mathcal{K}_3, \xi_3) - I & \dots & y_3(\mathcal{K}_{m+1}, \xi_3) \\ \dots & \dots & \dots & \ddots & \dots \\ y_{m+1}(\mathcal{K}_1, \xi_{m+1}) & y_{m+1}(\mathcal{K}_2, \xi_{m+1}) & y_{m+1}(\mathcal{K}_3, \xi_{m+1}) & \dots & y_{m+1}(\mathcal{K}_{m+1}, \xi_{m+1}) - I \end{bmatrix},$$

$$F_*(\xi) = \Big(-\Phi - y_1(\mathcal{A},\xi_1)\Phi - y_1(\mathcal{F},\xi_1), -y_2(\mathcal{F},\xi_2),\dots,-y_m(\mathcal{F},\xi_m), -y_{m+1}(\mathcal{F},\xi_{m+1})\Big)'.$$

It can be readily shown that solving the boundary value problem (9), (10) is equivalent to solving the system (20).

Theorem 2. Let the matrix $Q_*(\xi): \mathbb{R}^{(N-1)(m+1)} \to \mathbb{R}^{(N-1)(m+1)}$ be invertible. Then, for any $\mathcal{F}(t)$ and $\Phi \in \mathbb{R}^{(N-1)}$, the problem (9), (10) has a unique solution $\tilde{u}^\star(t)$ and this solution satisfies the estimate

$$\|\tilde{u}^\star\|_1 \leq M \max(\|\Phi\|, \|\mathcal{F}\|_1),$$

$$M = e^{\alpha\bar{\xi}}\bar{\xi}\Big\{\alpha \max\Big(1, \gamma(\xi)\Big[1+e^{\alpha\bar{\xi}}\bar{\xi}\alpha + e^{\alpha\bar{\xi}}\bar{\xi}\Big]\Big) + \Big(\sum_{j=1}^{m+1}\beta_j\gamma(\xi)+1\Big)\Big[1+e^{\alpha\bar{\xi}}\bar{\xi}\alpha+e^{\alpha\bar{\xi}}\bar{\xi}\Big]\Big\}$$
$$+\gamma(\xi)\Big[1+e^{\alpha\bar{\xi}}\bar{\xi}\alpha+e^{\alpha\bar{\xi}}\bar{\xi}\Big],$$

where $\gamma(\xi) = \|[Q_*(\xi)]^{-1}\|$, $\alpha = \max\limits_{t\in[0,T]}\|\mathcal{A}(t)\|$, $\beta_j = \max\limits_{t\in[0,T]}\|\mathcal{K}_j(t)\|$, $j=\overline{1,m+1}$, $\bar{\xi} = \max\limits_{s=\overline{1,m+1}}(\xi_s - \xi_{s-1})$, $\|\tilde{u}^\star\|_1 = \max\limits_{t\in[0,T]}\|\tilde{u}^\star(t)\|$.

The proof of Theorem 2, with minor modifications, follows the same principles as the proof of Theorem 1.1. in [32].

3 *Algorithm for numerical solving of problem* (9), (10) *and* (1)–(3)

The proposed numerical algorithm is based on the construction and solving of system (20). The coefficients and the right-hand side of this system (20) are found as solutions to Cauchy problems.

Algorithm for numerical solving of problem (9), (10):

1. Assume we have a partition: $0 = \xi_0 < \xi_1 < ... < \xi_m < \xi_{m+1} = T$. Divide every interval $[\xi_{s-1}, \xi_s]$, $s=\overline{1,m+1}$, with step $h_s = (\xi_s - \xi_{s-1})/l$, $l \in \mathbb{N}$, $s=\overline{1,m+1}$.

2. To determine the values of matrix $Q_*(\xi)$ and the vector $F_*(\xi)$ in system (20) we compute the values $y_s(\mathcal{A}, h_s)$, $y_s(\mathcal{K}_j, h_s)$, $j=\overline{1,m+1}$, $y_s(\mathcal{F}, h_s)$, $s=\overline{1,m+1}$, using the Runge Kutta RK4 Method with step size h_s in each subinterval.

3. Solve the system of linear algebraic equations

$$Q_*^{\tilde{h}}(\xi)\mu = F_*^{\tilde{h}}(\xi), \quad \mu \in R^{(N-1)(m+1)}, \tag{21}$$

here $\tilde{u}^{\tilde{h}_r}(\xi_s) = \mu_s^{\tilde{h}}$, $s=\overline{1,m+1}$.

4. To define the values of approximate solution at the remaining points, we solve the Cauchy problems

$$\frac{d\tilde{u}}{dt} = \mathcal{A}(t)\tilde{u} + \sum_{j=1}^{m+1}\mathcal{K}_j(t)\mu_j^{\tilde{h}} + \mathcal{F}(t), \quad \tilde{u}(0) = \Phi, \quad t\in[\xi_0,\xi_1], \tag{22}$$

$$\frac{d\tilde{u}}{dt} = \mathcal{A}(t)\tilde{u} + \sum_{j=1}^{m+1}\mathcal{K}_j(t)\mu_j^{\tilde{h}} + \mathcal{F}(t), \quad \tilde{u}(\xi_{s-1}) = \mu_{s-1}^{\tilde{h}}, \quad t\in[\xi_{s-1},\xi_s), \quad s=\overline{2,m+1}. \tag{23}$$

Solving Cauchy problems (22), (23) also using the Runge Kutta RK4 Method, we obtain a numerical solution to linear initial-boundary value problem for loaded differential equations (9), (10).

If $\tilde{u}^\star(\xi_j^{\tilde{h}}) = \big(u_1^\star(\xi_j^{\tilde{h}}), u_2^\star(\xi_j^{\tilde{h}}),\dots,u_{N-1}^\star(\xi_j^{\tilde{h}})\big)'$, $j=\overline{0,(m+1)l}$ is a numerical solution to linear initial value problem for loaded differential equations (9), (10), then $u^\star(\xi_j^{\tilde{h}}, s\tau) = u_s^\star(\xi_j^{\tilde{h}})$, $s=\overline{1,N-1}$ will be a numerical solution to linear initial value problem for loaded differential equation of parabolic type (1)–(3).

4 Example

To provide a clear overview of our investigation, we selected a specific test problem. Thus, the numerical method discussed in earlier sections were applied to the following initial-boundary value problem for a loaded parabolic equation

$$\begin{aligned}\frac{\partial u}{\partial t} &= xt\frac{\partial^2 u}{\partial x^2} - (x+2t)u(t,x)\\ &+ xtu(0.02,x) + 2x^2tu(0.04,x) + (x+t)u(0.06,x) + 4xtu(0.08,x) + 5xt^2u(0.1,x)\\ &+ (4x^3+8tx^2-8tx)cos25\pi t - 100\pi x^2 sin25\pi t + \Big(2+\frac{7}{2}x - x^2\Big)t^2 + \frac{97}{50}x - \frac{3}{25}x^2\\ &+ \Big(8x^4 - \frac{404}{25}x^3 + \frac{31}{25}x^2 + \frac{27}{50}x - \frac{3}{50}\Big)t + 1, \quad (t,x)\in(0,0.1)\times(0,0.5),\end{aligned} \tag{24}$$

$$u(0,x) = 4x^2, \qquad x\in[0,0.5], \tag{25}$$

$$u(t,0) = t, \qquad u(t,\omega) = 2t + cos25\pi t, \qquad t\in[0,0.1]. \tag{26}$$

The analytical solution of the given problem (24)–(26) is $\hat{u}(t,x) = 2xt + 4x^2cos25\pi t + t$.

We take $\tau = 0.1$ and produce a discretization by x: $x_i = i\tau$, $i=\overline{0,5}$. Using this, the problem (24)–(26) is replaced by the following boundary value problem for loaded differential equation

$$\frac{d\tilde{u}}{dt} = \mathcal{A}(t)\tilde{u} + \sum_{j=1}^{5}\mathcal{K}_j(t)\tilde{u}(\xi_j) + \mathcal{F}(t), \quad t\in(0,0.1), \tag{27}$$

$$\tilde{u}(0) = \Phi, \quad \tilde{u}\in\mathbb{R}^4. \tag{28}$$

Here $\xi_1 = 0.02$, $\xi_2 = 0.04$, $\xi_3 = 0.06$, $\xi_4 = 0.08$, $\xi_5 = 0.1$,

$$\tilde{u} = \begin{pmatrix}u_1(t)\\u_2(t)\\u_3(t)\\u_4(t)\end{pmatrix}, \mathcal{A}(t) = \begin{pmatrix}-22t-0.1 & 10t & 0 & 0\\ 20t & -42t-0.2 & 20t & 0\\ 0 & 30t & -62t-0.3 & 30t\\ 0 & 0 & 40t & -82t-0.4\end{pmatrix}, \Phi = \begin{pmatrix}0.04\\0.16\\0.36\\0.64\end{pmatrix},$$

$$\mathcal{K}_j(t) = \begin{pmatrix}j\cdot 0.1^j\cdot t & 0 & 0 & 0\\ 0 & j\cdot 0.2^j\cdot t & 0 & 0\\ 0 & 0 & j\cdot 0.3^j\cdot t & 0\\ 0 & 0 & 0 & j\cdot 0.4^j\cdot t\end{pmatrix}, \quad j=\overline{1,2},$$

$$\mathcal{K}_3(t) = \begin{pmatrix}0.1+t & 0 & 0 & 0\\ 0 & 0.2+t & 0 & 0\\ 0 & 0 & 0.3+t & 0\\ 0 & 0 & 0 & 0.4+t\end{pmatrix},$$

$$\mathcal{K}_i(t) = \begin{pmatrix}i\cdot 0.1\cdot t^{i-3} & 0 & 0 & 0\\ 0 & i\cdot 0.2\cdot t^{i-3} & 0 & 0\\ 0 & 0 & i\cdot 0.3\cdot t^{i-3} & 0\\ 0 & 0 & 0 & i\cdot 0.4\cdot t^{i-3}\end{pmatrix}, \quad i=\overline{4,5},$$

$$\mathcal{F}(t) = \begin{pmatrix}\Big(\frac{1}{250}-\frac{18t}{25}\Big)cos25\pi t - \pi sin25\pi t + \frac{617}{50}t^2 - \frac{28t}{3125} + \frac{1491}{1250}\\ \Big(\frac{4}{125}-\frac{32t}{25}\Big)cos25\pi t - 4\pi sin25\pi t + \frac{133}{50}t^2 - \frac{59t}{3125} + \frac{1729}{1250}\\ \Big(\frac{27}{250}-\frac{42t}{25}\Big)cos25\pi t - 9\pi sin25\pi t + \frac{74}{25}t^2 - \frac{987t}{6250} + \frac{982}{625}\\ \Big(\frac{32}{125}+\frac{952t}{25}\Big)cos25\pi t - 16\pi sin25\pi t + \frac{2081}{25}t^2 - \frac{2969t}{6250} + \frac{1098}{625}\end{pmatrix}.$$

To solve linear initial value problem for loaded differential equations (27), (28) we will use the algorithm of Dzhumabaev parameterization method. According to the scheme of this method, the interval $[0, 0.1)$ is partitioned into subintervals by loading points: $[0, 0.1) = \bigcup_{s=1}^{5} [\xi_{s-1}, \xi_s)$. Using the above proposed algorithm, we compose system (21) and by solving this system (21) we find the numerical values of the unknown parameters μ:

$$\mu_1 = \begin{pmatrix} 0.02399999 \\ 0.02799997 \\ 0.03199992 \\ 0.03599988 \end{pmatrix}, \ \mu_2 = \begin{pmatrix} 0.00799998 \\ -0.10400007 \\ -0.29600015 \\ -0.56800025 \end{pmatrix}, \ \mu_3 = \begin{pmatrix} 0.07199999 \\ 0.08399997 \\ 0.09599992 \\ 0.10799986 \end{pmatrix},$$

$$\mu_4 = \begin{pmatrix} 0.136 \\ 0.272 \\ 0.488 \\ 0.78400001 \end{pmatrix}, \ \mu_5 = \begin{pmatrix} 0.11999999 \\ 0.13999996 \\ 0.15999993 \\ 0.17999994 \end{pmatrix}.$$

We find the values of approximate solution at the remaining points of the subintervals of problem (27), (28) by solving Cauchy problems (22), (23). Then $u^\star(\xi_j^{\widetilde{h}}, s\tau) = u_s^\star(\xi_j^{\widetilde{h}})$, $s = \overline{1, 4}$, $j = \overline{0, 50}$, $\tau = 0.1$, will be a numerical solution to linear boundary value problem for loaded differential equation of parabolic type (24)–(26).

The graph of the exact solution $\hat{u}(t, x)$ and the found numerical solutions $u^\star(t, x)$ of the boundary value problem for a parabolic equation (24)–(26) are shown in Figure.

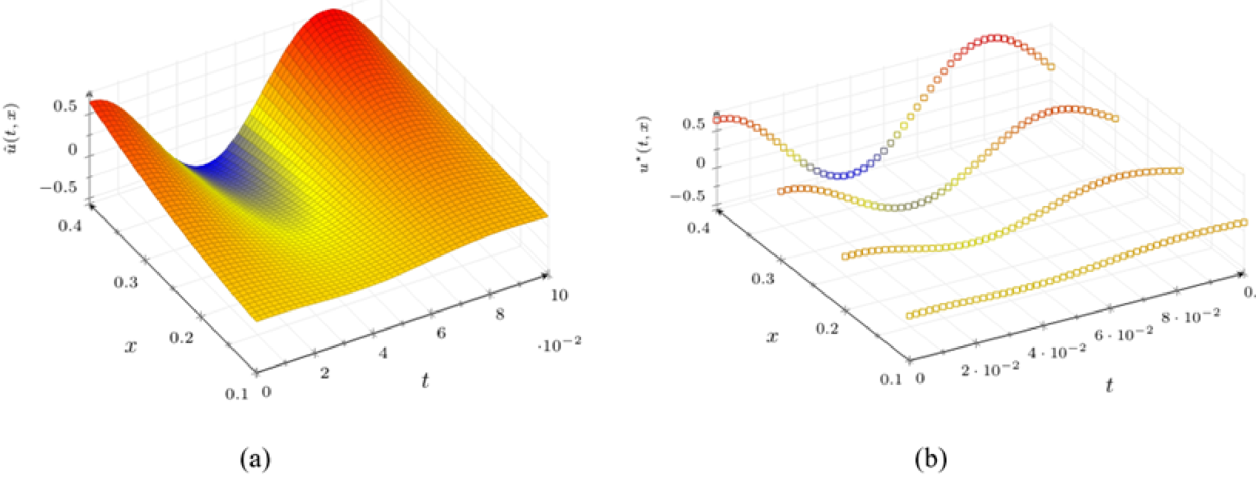

Figure. Exact (a) and numerical (b) solutions for example

For the difference of the corresponding values of the exact $\hat{u}(t, x)$ and constructed solutions $u^\star(t, x)$ of the boundary value problem for a parabolic equation (24)–(26) the following estimate is true:

$$\max_{j=\overline{0,50}, i=\overline{1,4}} \|\hat{u}(\xi_j^{\widetilde{h}}, x_i) - u^\star(\xi_j^{\widetilde{h}}, x_i)\| < 0.0000003.$$

Conclusion

The initial-boundary value problem for a loaded parabolic equation in a rectangular domain is investigated. Using discretization with respect to the variable x, the problem under consideration is reduced to the initial problem for a system of loaded ordinary differential equations. Using the results of works [12, 13], an estimate for the solution of the initial-boundary value problem for a loaded parabolic equation is established. The parameterization method is used to solve the initial problem for a system of loaded ordinary differential equations. Algorithms for finding a solution to the problem under study are constructed and their convergence is shown. Conditions for the unique solvability of the initial problem for a system of loaded ordinary differential equations are established. Further, the proposed approach will be applied to solving boundary value problems for a loaded parabolic equation.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

This work does not have any conflicts of interest.

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