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On the solvability of one inverse problem for a fourth-order equation

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In this paper, for a fourth-order equation in a rectangular domain, an inverse problem of finding the unknown right-hand side, which depends on one variable, is considered. Criteria for the uniqueness and existence of a solution to the inverse problem under consideration for a fourth-order equation are established. The solution to the problem is constructed as the sum of a series in eigenfunctions of the corresponding spectral problem. The uniqueness of the solution to the inverse problem follows from the completeness of the system of eigenfunctions. Sufficient conditions are established for the boundary functions that guarantee theorems of existence and stability of the solution to the problem. In a closed domain, absolute and uniform convergence of the found solution to the inverse problem in the form of a series in the class of regular solutions is shown, as well as series obtained by term-by-term differentiation with respect to t and x three and four times, respectively. The stability of the solution of the inverse problem in the norms of the space of square-summable functions and in the space of continuous functions with respect to changes in the input data has also been proven.

Keywords: fourth-order equation, inverse problem, classical solution, method of separation of variables, uniform convergence of the solution, uniqueness, existence, stability of the solution.

2020 Mathematics Subject Classification: 35R30.

Introduction

Boundary value and inverse problems for fourth-order differential equations are widely used in modeling processes in various fields of science and technology: in studying the dynamics of compressible stratified fluid, wave propagation in dispersive media, ship vibrations, oscillations of rods, beams and plates. Such problems are often reduced to studying fourth-order equations with various types of conditions. Numerous studies have been devoted to boundary value problems for fourth-order equations.

In the monograph by Smirnov [1], problems for a model equation of mixed type of the fourth order in various geometric domains are considered. In the work by Amirov and Khojanov [2], the global solvability of initial-boundary value problems for nonlinear analogues of the Boussinesq equation is proved, which expands the range of studied problems of mathematical physics.

The inverse problem for a parabolic equation of the fourth order with a complex-valued coefficient is considered in [3], where a theorem on the existence and uniqueness of a solution is proved. The articles [4–6] consider boundary value problems with local conditions for fourth-order equations in rectangular domains. Thus, in [4] the problems with the third derivative with respect to time are analyzed, and in [5] and [6] – problems with the lowest term and mixed type of equation, respectively.

In the works [7, 8] the boundary value problems with nonlocal conditions are studied. The authors prove that the eigenfunctions and associated functions of the corresponding spectral problem form a Riesz basis, and the solution to the problem is expressed as a biorthogonal series. This is important for constructing analytical solutions in complex domains.

In the works [9, 10] the boundary value problems for fourth-order mixed-type equations are studied.

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Initial-boundary value problems for beam and plate vibrations are studied in the works of Sabitov and co-authors [11–13]. The use of the method of separation of variables allows us to establish solvability conditions and construct explicit representations of solutions for equations taking into account rotational motion and various types of fixings.

Some boundary-value problems for nonhomogeneous biharmonic equation is presented in [14], where the conditions for periodic boundary are studied. In the work of Urinov and Azizov [15], an inverse problem for a fourth-order equation with an unknown right-hand side is considered, a uniqueness theorem is proved, and a constructive solution method is given.

A classification of fourth-order equations with two independent variables is given in the monograph by Dzhuraev and Sopuev [16], where an extensive bibliography on this topic is also presented and various types of boundary value problems are considered.

Inverse problems, as shown in [17–19], have numerous applications in seismology, geophysics, biomedicine, and computed tomography. Here, both problems of restoring the right-hand side and coefficient inverse problems are considered. In particular, Sabitov and Martemyanova [17] investigated a nonlocal inverse problem for a mixed-type equation, and Khojanov [18, 19] proposed methods for restoring special types of right-hand sides in parabolic equations.

General approaches to solving inverse problems and theoretical foundations are presented in classical monographs [20–22], which present regularization methods, a functional-analytical apparatus, and examples of formulations in mathematical physics.

Thus, the present study continues the development of the theory of fourth-order equations, relying on the indicated scientific achievements, and is aimed at formulating and solving new classes of boundary and inverse problems with practical significance.

1 *Formulation of the problem*

In the domain $\Omega = \{(x,t) : 0 < x < p,\ 0 < t < \beta\}$, we consider the equation

$$Lu \equiv u_{ttt} - u_{xxxx} - b^2 u = f(x), \tag{1}$$

where $b = const$.

Problem 1. Find functions $u(x,t)$ and $f(x)$ in the domain Ω that satisfy the conditions

$$u(x,t) \in C^{3,1}_{x,t}\left(\bar{\Omega}\right) \cap C^{4,3}_{x,t}(\Omega), \quad f(x) \in C(0,p) \cap L_2(0,p), \tag{2}$$

$$Lu(x,t) = f(x), \tag{3}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad u(x,\beta) = \xi(x), \quad u_t(x,\beta) = \mu(x), \quad 0 \le x \le p, \tag{4}$$

$$u(0,t) = u(p,t) = 0, \quad u_{xx}(0,t) = u_{xx}(p,t) = 0, \quad 0 \le t \le \beta, \tag{5}$$

where $\varphi(x), \xi(x), \psi(x), \mu(x)$ are the given functions, and $\varphi^{(i)}(0) = \varphi^{(i)}(p) = 0$, $\xi^{(i)}(0) = \xi^{(i)}(p) = 0$, $i = 0, 2$, $\psi(0) = \psi(p) = 0$, $\mu(0) = \mu(p) = 0$.

By the classical solution of the inverse boundary value problem (2)–(5) we mean a pair $\{u(x,t), f(x)\}$ of functions $u(x,t) \in C^{4,3}_{x,t}(\Omega)$ and $f(x) \in C(0,\beta)$, satisfying conditions (2)–(5) in the usual sense.

2 *Uniqueness and existence of a solution to the inverse problem*

We solve problems (2)–(5) at $f(x) \equiv 0$ using the method of separation of variables $u(x,t) = X(x)T(t)$. Then we have the following spectral problem for the function $X(x)$:

$$\begin{array}{l} X^{IV}(x) - \eta X(x) = 0, \ \ 0 < x < p, \\ X(0) = X(p) = X''(0) = X''(p) = 0, \end{array} \tag{6}$$

where η is the separation constant. Problem (6) has a solution

$$X_k(x) = \sqrt{\frac{2}{p}} \sin \lambda_k x, \quad \lambda_k = \sqrt[4]{\eta_k} = \frac{k\pi}{p}, \quad k = 1, 2, \dots . \tag{7}$$

We look for a solution to problem (2)–(5) in the form

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t) X_k(x), \tag{8}$$

$$f(x) = \sum_{k=1}^{\infty} f_k X_k(x), \tag{9}$$

where

$$T_k(t) = \int_0^p u(x,t) X_k(x)\, dx, \tag{10}$$

$$f_k = \int_0^p f(x) X_k(x)\, dx. \tag{11}$$

Based on (10), we introduce the functions

$$T_{k,\varepsilon}(t) = \int_{\varepsilon}^{p-\varepsilon} u(x,t) X_k(x)\, dx, \tag{12}$$

where ε is a fairly small number. We differentiate equalities (12) three times and take into account (1), we have

$$T''_{k,\varepsilon}(t) = \int_{\varepsilon}^{p-\varepsilon} \left[f(x) + u_{xxxx}(x,t) + b^2 u(x,t) \right] X_k(x)\, dx. \tag{13}$$

In integral (13), integrating four times by parts and passing to the limit at $\varepsilon \to 0$ taking into account boundary conditions (5), we have the differential equations:

$$T'''_k(t) - v_k^3 T_k(t) = f_k, \tag{14}$$

where $v_k^3 = \lambda_k^4 + b^2$.

General solutions of equation (14) take the form

$$T_k(t) = a_k e^{v_k t} + e^{-\frac{1}{2} v_k t} \left(b_k \cos \frac{\sqrt{3}}{2} v_k t + c_k \sin \frac{\sqrt{3}}{2} v_k t \right) - v_k^{-3} f_k, \tag{15}$$

where $a_k,\ b_k,\ c_k$ are arbitrary constants.

To determine the coefficients $a_k,\ b_k,\ c_k$ and f_k we use conditions (4), which go over

$$T_k(0) = \varphi_k, \quad T_k^{'}(0) = \psi_k, \quad T_k(\beta) = \xi_k, \quad T_k^{'}(\beta) = \mu_k, \tag{16}$$

where

$$\begin{aligned} \varphi_k &= \textstyle\int_0^p \varphi(x) X_k(x)\, dx, \quad \psi_k = \int_0^p \psi(x) X_k(x)\, dx, \\ \xi_k &= \textstyle\int_0^p \xi(x) X_k(x)\, dx, \quad \mu_k = \int_0^p \mu(x) X_k(x)\, dx. \end{aligned} \tag{17}$$

Substituting solutions (15) into (16), we obtain a system of equations for determining a_k, b_k, c_k and f_k:

$$\begin{cases} a_k + b_k - v_k^{-3} f_k = \varphi_k, \\ 2a_k - b_k + \sqrt{3}c_k = 2v_k^{-1}\psi_k, \\ a_k e^{\frac{3}{2}v_k\beta} + b_k \cos\frac{\sqrt{3}}{2}v_k\beta + c_k \sin\frac{\sqrt{3}}{2}v_k\beta - v_k^{-3}e^{\frac{1}{2}v_k\beta} f_k = \xi_k e^{\frac{1}{2}v_k\beta}, \\ a_k e^{\frac{3}{2}v_k\beta} - b_k \cos\left(\frac{\sqrt{3}}{2}v_k\beta - \frac{\pi}{3}\right) - c_k \sin\left(\frac{\sqrt{3}}{2}v_k\beta - \frac{\pi}{3}\right) = v_k^{-1}\mu_k e^{\frac{1}{2}v_k\beta}. \end{cases} \tag{18}$$

The determinant of system (18) takes the form:

$$\Delta_k(\beta) = 2e^{v_k\beta}\left(chv_k\beta - \cos\frac{\sqrt{3}}{2}v_k\beta \cdot ch\frac{1}{2}v_k\beta - \sqrt{3}\sin\frac{\sqrt{3}}{2}v_k\beta \cdot sh\frac{1}{2}v_k\beta\right). \tag{19}$$

Now we represent (19) in the form

$$\Delta_k(\beta) = 2e^{v_k\beta}chv_k\beta \cdot \left[1 - A_k \sin\left(\frac{\sqrt{3}}{2}v_k\beta + \gamma_k\right)\right], \tag{20}$$

where $\gamma_k = \arcsin\frac{ch\frac{1}{2}v_k\beta}{\sqrt{2chv_k\beta - 1}}$, $A_k = \frac{\sqrt{2chv_k\beta - 1}}{chv_k\beta}$.

Lemma 1. For any $\beta > 0$ the following estimate is valid

$$|\Delta_k(\beta)| \geq C_0 e^{2v_k\beta}, \tag{21}$$

where C_0 is a positive constant.

Proof. Taking into account $A_k < 1$, from (20), we have

$$|\Delta_k(\beta)| \geq e^{v_k\beta} \cdot \left(e^{v_k\beta} + e^{-v_k\beta}\right) \cdot \left[1 - A_k\left|\sin\left(\frac{\sqrt{3}}{2}v_k\beta + \gamma_k\right)\right|\right] \geq e^{2v_k\beta} \cdot [1 - A_k] \geq C_0 e^{2v_k\beta}.$$

The lemma is proved.

Then system (18) has a unique solution

$$\begin{aligned} a_k = \frac{1}{\sqrt{3}v_k\Delta_k(\beta)} \Big[& 2v_k\varphi_k e^{\frac{1}{2}v_k\beta} \sin\frac{\sqrt{3}}{2}v_k\beta + \sqrt{3}\psi_k + 2\psi_k e^{\frac{1}{2}v_k\beta}\sin\left(\frac{\sqrt{3}}{2}v_k\beta - \frac{\pi}{3}\right) - \\ & -2v_k\xi_k e^{\frac{1}{2}v_k\beta}\sin\frac{\sqrt{3}}{2}v_k\beta + \sqrt{3}\mu_k e^{v_k\beta} - 2\mu_k e^{\frac{1}{2}v_k\beta}\sin\left(\frac{\sqrt{3}}{2}v_k\beta + \frac{\pi}{3}\right)\Big], \end{aligned} \tag{22}$$

$$\begin{aligned} b_k = \frac{e^{\frac{1}{2}v_k\beta}}{\sqrt{3}v_k\Delta_k(\beta)} \Big[& \sqrt{3}v_k\varphi_k e^{\frac{3}{2}v_k\beta} + 2v_k\varphi_k \sin\left(\frac{\sqrt{3}}{2}v_k\beta - \frac{\pi}{3}\right) - \\ & -2\psi_k \sin\left(\frac{\sqrt{3}}{2}v_k\beta - \frac{\pi}{3}\right) - 2\sqrt{3}\psi_k e^{v_k\beta}\cos\left(\frac{\sqrt{3}}{2}v_k\beta + \frac{\pi}{3}\right) - \sqrt{3}v_k\xi_k e^{\frac{3}{2}v_k\beta} - \\ & -2v_k\xi_k \sin\left(\frac{\sqrt{3}}{2}v_k\beta - \frac{\pi}{3}\right) - 2\mu_k \sin\frac{\sqrt{3}}{2}v_k\beta + 2\sqrt{3}\mu_k e^{v_k\beta} sh\frac{1}{2}v_k\beta\Big], \end{aligned} \tag{23}$$

$$c_k = \frac{e^{\frac{1}{2}v_k\beta}}{\sqrt{3}v_k\Delta_k(\beta)}\left[v_k\varphi_k e^{\frac{3}{2}v_k\beta} - 2v_k\varphi_k\cos\left(\frac{\sqrt{3}}{2}v_k\beta - \frac{\pi}{3}\right) + 2\psi_k e^{\frac{3}{2}v_k\beta} + \right.$$

$$+2\psi_k\cos\left(\frac{\sqrt{3}}{2}v_k\beta - \frac{\pi}{3}\right) - 2\sqrt{3}\psi_k e^{v_k\beta}\sin\left(\frac{\sqrt{3}}{2}v_k\beta + \frac{\pi}{3}\right) - v_k\xi_k e^{\frac{3}{2}v_k\beta} +$$

$$\left. +2v_k\xi_k\cos\left(\frac{\sqrt{3}}{2}v_k\beta - \frac{\pi}{3}\right) + \mu_k e^{\frac{3}{2}v_k\beta} + 2\mu_k\cos\frac{\sqrt{3}}{2}v_k\beta - 3\mu_k e^{\frac{1}{2}v_k\beta}\right], \tag{24}$$

$$f_k = \frac{v_k^2}{\Delta_k(\beta)}\left\{v_k\varphi_k\left[2e^{\frac{3}{2}v_k\beta}\cos\left(\frac{\sqrt{3}}{2}v_k\beta - \frac{\pi}{3}\right) - 1\right] + \right.$$

$$+\psi_k\left[-2e^{\frac{3}{2}v_k\beta}\cos\left(\frac{\sqrt{3}}{2}v_k\beta + \frac{\pi}{3}\right) + 1\right] + v_k\xi_k\left[2e^{\frac{1}{2}v_k\beta}\cos\left(\frac{\sqrt{3}}{2}v_k\beta + \frac{\pi}{3}\right) - e^{2v_k\beta}\right] + \tag{25}$$

$$\left. +\mu_k\left[-2e^{\frac{1}{2}v_k\beta}\cos\left(\frac{\sqrt{3}}{2}v_k\beta - \frac{\pi}{3}\right) + e^{2v_k\beta}\right]\right\}.$$

So we obtained a solution of problem (2)–(5) in the form (8)–(9), where $X_k(x)$, $T_k(t)$ and f_k are determined from (7), (15) and (25), respectively, and the coefficients a_k, b_k, c_k are determined from (22)–(24).

Now we will prove the uniqueness of the solution of problem (2)–(5). Let $\xi(x) \equiv 0$, $\mu(x) \equiv 0$, $\varphi(x) \equiv 0$, $\psi(x) \equiv 0$ on $[0, p]$. Then $\varphi_k \equiv 0$, $\psi_k \equiv 0$, $\xi_k \equiv 0$, $\mu_k \equiv 0$ from (15) and (25) it follows that $T_k(t) \equiv 0$ on $[0, \beta]$ and $f_k \equiv 0$ for all $k \in N$. Then from equalities (10)–(11) we have that for all $t \in [0, \beta]$

$$\int_0^p u(x,t)\,X_k(x)\,dx \equiv 0, \quad \int_0^p f(x)\,X_k(x)\,dx \equiv 0, \ k \in N.$$

From here, due to the completeness of (7) in space $L_2[0, p]$ and the continuity of the function $u(x,t)$ and $f(x)$ respectively, on the domain $\bar{\Omega}$ and $(0, p)$, it follows that $u(x,t) \equiv 0$ in $\bar{\Omega}$ and $f(x) \equiv 0$ on $(0, p)$. So it's proven.

Theorem 1. If there is a solution of the problem (2)–(5), then it is unique.

Lemma 2. For large natural k, the following estimates are valid:

$$\begin{gathered}
|T_k(t)| \le C_1\left[|\varphi_k| + k^{-1\frac{1}{3}}|\psi_k| + |\xi_k| + k^{-1\frac{1}{3}}|\mu_k|\right], \\
\left|T_k{}'(t)\right| \le C_2\left[k^{1\frac{1}{3}}|\varphi_k| + |\psi_k| + k^{1\frac{1}{3}}|\xi_k| + |\mu_k|\right], \\
\left|T_k{}''(t)\right| \le C_3\left[k^{2\frac{2}{3}}|\varphi_k| + k^{1\frac{1}{3}}|\psi_k| + k^{2\frac{2}{3}}|\xi_k| + k^{1\frac{1}{3}}|\mu_k|\right], \\
\left|T_k{}'''(t)\right| \le C_4\left(k^4|\varphi_k| + k^{2\frac{2}{3}}|\psi_k| + k^4|\xi_k| + k^{2\frac{2}{3}}|\mu_k|\right).
\end{gathered} \tag{26}$$

Here and below C_i are positive constants.

Proof. From (22)–(25), taking Lemma 1 into account, we obtain the following estimates:

$$\begin{aligned}
&|a_k| \le C_5 e^{-2k\beta}\left(|\varphi_k| + k^{-1\frac{1}{3}}|\psi_k| + |\xi_k| + k^{-1\frac{1}{3}}|\mu_k|\,e^{\frac{2}{3}k\beta}\right), \\
&|b_k| \le C_6\left(|\varphi_k| + k^{-1\frac{1}{3}}|\psi_k| + |\xi_k| + k^{-1\frac{1}{3}}|\mu_k|\right), \\
&|c_k| \le C_7\left(|\varphi_k| + k^{-1\frac{1}{3}}|\psi_k| + |\xi_k| + k^{-1\frac{1}{3}}|\mu_k|\right), \\
&|f_k| \le C_8\left(k^4|\varphi_k| + k^{2\frac{2}{3}}|\psi_k| + k^4|\xi_k| + k^{2\frac{2}{3}}|\mu_k|\right).
\end{aligned} \tag{27}$$

From (15) we have

$$|T_k(t)| \le |a_k| e^{v_k t} + |b_k| e^{-\frac{1}{2}v_k t} + |c_k| e^{-\frac{1}{2}v_k t} + v_k^{-3} |f_k| . \quad (28)$$

We substitute (27) into (28). This estimate implies the validity of the first estimate required in the lemma. The proof of the validity of the remaining estimates is shown similarly. The lemma is proved.

Lemma 3. Let $\varphi(x), \xi(x) \in C^5[0,p]$, $\varphi^{(2i)}(0) = \varphi^{(2i)}(p) = 0$, $\xi^{(2i)}(0) = \xi^{(2i)}(p) = 0$, $i = 0,1,2$; $\psi(x), \mu(x) \in C^4[0,p]$, $\psi^{(2i)}(0) = \psi^{(2i)}(p) = 0$, $\mu^{(2i)}(0) = \mu^{(2i)}(p) = 0$, $i = 0,1$. Then the representations are valid

$$\varphi_k = \frac{1}{\lambda_k^5}\bar{\varphi}_k^{(5)}, \quad \psi_k = \frac{1}{\lambda_k^4}\bar{\psi}_k^{(4)}, \quad \xi_k = \frac{1}{\lambda_k^5}\bar{\xi}_k^{(5)}, \quad \xi_k = \frac{1}{\lambda_k^4}\bar{\xi}_k^{(4)}, \quad (29)$$

where

$$\bar{\varphi}_k^{(5)} = \sqrt{\tfrac{2}{p}} \int_0^p \varphi^{(5)}(x)\cos\lambda_k x dx, \quad \bar{\psi}_k^{(4)} = \sqrt{\tfrac{2}{p}} \int_0^p \psi^{(4)}(x)\sin\lambda_k x dx,$$

$$\bar{\xi}_k^{(5)} = \sqrt{\tfrac{2}{p}} \int_0^p \xi^{(5)}(x)\cos\lambda_k x dx, \quad \bar{\mu}_k^{(4)} = \sqrt{\tfrac{2}{p}} \int_0^p \mu^{(4)}(x)\sin\lambda_k x dx.$$

Integrating the first and third integrals in (17) by parts five times, and the second and fourth integrals by parts four times, taking into account the conditions of the lemma, we obtain representations (29).

Theorem 2. Let the functions $\varphi(x)$, $\psi(x)$, $\xi(x)$ and $\mu(x)$ satisfy the conditions of Lemma 3. Then there is a unique solution of problem (2)–(5), which is determined by the series (8)–(9).

Proof. We formally differentiate series (8) term by t three times and by x four times and have

$$u_{ttt}(x,t) = \sum_{k=1}^{\infty} T_k'''(t) X_k(x), \quad (30)$$

$$u_{xxxx}(x,t) = \sum_{k=1}^{\infty} \lambda_k^4 T_k(t) X_k(x). \quad (31)$$

From (26) we have

$$C_4\sqrt{\frac{2}{p}}\sum_{k=1}^{\infty}\left(k^4|\varphi_k| + k^{2\frac{2}{3}}|\psi_k| + k^4|\xi_k| + k^{2\frac{2}{3}}|\mu_k|\right). \quad (32)$$

Based on (29), the convergence of series (32) is proved, i.e., the following series

$$\bar{C}_4\sum_{k=1}^{\infty}\left(\frac{1}{k}\left|\bar{\varphi}_k^{(5)}\right| + \frac{1}{k^{1\frac{1}{3}}}\left|\bar{\psi}_k^{(4)}\right| + \frac{1}{k}\left|\bar{\xi}_k^{(5)}\right| + \frac{1}{k^{1\frac{1}{3}}}\left|\bar{\mu}_k^{(4)}\right|\right)$$

converges. From convergence (32), due to the Weierstrass criterion, series (8), (30), (31) uniformly converge in the domain $\bar{\Omega}$ and series (9) on [0, p]. The theorem is proved.

3 *Stability of the solution*

Let us introduce the following norms:

$$\|u(x,t)\|_{L_2[0,p]} = \left(\int_0^p |u(x,t)|^2 dx\right)^{\frac{1}{2}}, \quad \|u(x,t)\|_{C(\bar{\Omega})} = \max_{\bar{\Omega}}|u(x,t)|,$$

$$\|f(x)\|_{W_2^n[0,p]} = \left(\int_0^p \left(\sum_{k=0}^{n}\left|f^{(k)}(x)\right|^2 dx\right)\right)^{\frac{1}{2}}, \quad n \in N.$$

Theorem 3. Let the conditions of Theorem 2 be satisfied, then for solution (8), (9) of Problem 1 the following estimates are valid:

$$\|u(x,t)\|_{L_2[0,p]} \le C_9\left[\|\varphi\|_{L_2} + \|\psi\|_{L_2} + \|\xi\|_{L_2} + \|\mu\|_{L_2}\right], \tag{33}$$

$$\|f(x)\|_{L_2[0,p]} \le C_{10}\left[\|\varphi\|_{W_2^4} + \|\psi\|_{W_2^3} + \|\xi\|_{W_2^4} + \|\mu\|_{W_2^3}\right], \tag{34}$$

$$\|u(x,t)\|_{C(\bar{\Omega})} \le C_{11}\left[\|\varphi\|_{W_2^1} + \|\psi\|_{W_2^0} + \|\xi\|_{W_2^1} + \|\mu\|_{W_2^0}\right], \tag{35}$$

$$\|f(x)\|_{C[0,p]} \le C_{12}\left[\|\varphi\|_{W_2^5} + \|\psi\|_{W_2^4} + \|\xi\|_{W_2^5} + \|\mu\|_{W_2^4}\right]. \tag{36}$$

Proof. From (8), (21) and the first inequality of Lemma 2 we have

$$\begin{aligned}\|u(x,t)\|_{L_2}^2 = \sum_{k=1}^{\infty} T_k^2(t) \le C_1^2 \sum_{k=1}^{\infty}\left[|\varphi_k| + k^{-1\frac{1}{3}}|\psi_k| + |\xi_k| + k^{-1\frac{1}{3}}|\mu_k|\right]^2 \le \\ \le 4C_1^2\sum_{k=1}^{\infty}\left[|\varphi_k|^2 + k^{-2\frac{2}{3}}|\psi_k|^2 + |\xi_k|^2 + k^{-2\frac{2}{3}}|\mu_k|^2\right] \le \\ \le C_9^2\left[\|\varphi\|_{L_2}^2 + \|\psi\|_{L_2}^2 + \|\xi\|_{L_2}^2 + \|\mu\|_{L_2}^2\right].\end{aligned} \tag{37}$$

From inequality (37) estimate (33) follows:

$$\|f(x)\|_{L_2[0,p]}^2 = \sum_{k=1}^{\infty}|f_k|^2 \le C_8^2\sum_{k=1}^{\infty}\left(k^4|\varphi_k| + k^{2\frac{2}{3}}|\psi_k| + k^4|\xi_k| + k^{2\frac{2}{3}}|\mu_k|\right)^2 \le$$

$$\le 4C_8^2\sum_{k=1}^{\infty}\left[\left(k^4|\varphi_k|\right)^2 + \left(k^{2\frac{2}{3}}|\psi_k|\right)^2 + \left(k^4|\xi_k|\right)^2 + \left(k^{2\frac{2}{3}}|\mu_k|\right)^2\right].$$

The coefficients φ_k, ψ_k, ξ_k and μ_k can be represented in the form

$$\varphi_k = \frac{1}{\lambda_k^4}\bar{\varphi}_k^{(4)}, \quad \psi_3 = \frac{1}{\lambda_k^3}\bar{\psi}_k^{(3)}, \quad \xi_k = \frac{1}{\lambda_k^4}\bar{\xi}_k^{(4)}, \quad \mu_3 = \frac{1}{\lambda_k^3}\bar{\mu}_k^{(3)},$$

where

$$\bar{\varphi}_k^{(4)} = \sqrt{\tfrac{2}{p}}\int_0^p \varphi^{(4)}(x)\sin\lambda_k x dx, \quad \bar{\psi}_k^{(3)} = -\sqrt{\tfrac{2}{p}}\int_0^p \psi^{(3)}(x)\cos\lambda_k x dx,$$

$$\bar{\xi}_k^{(4)} = \sqrt{\tfrac{2}{p}}\int_0^p \xi^{(4)}(x)\sin\lambda_k x dx, \quad \bar{\mu}_k^{(3)} = -\sqrt{\tfrac{2}{p}}\int_0^p \mu^{(3)}(x)\cos\lambda_k x dx.$$

Then

$$\begin{aligned}\|f(x)\|_{L_2[0,p]}^2 \le 4\bar{C}_8^2\sum_{k=1}^{\infty}\left[\left|\bar{\varphi}_k^{(4)}\right|^2 + \left|\bar{\psi}_k^{(3)}\right|^2 + \left|\bar{\varphi}_k^{(4)}\right|^2 + \left|\bar{\mu}_k^{(3)}\right|^2\right] \le \\ \le 4\bar{C}_8^2\left[\left\|\varphi^{(4)}\right\|_{L_2}^2 + \left\|\psi^{(3)}\right\|_{L_2}^2 + \left\|\xi^{(4)}\right\|_{L_2}^2 + \left\|\mu^{(3)}\right\|_{L_2}^2\right] \le \\ \le C_{10}^2\left[\|\varphi\|_{W_2^4}^2 + \|\psi\|_{W_2^3}^2 + \|\xi\|_{W_2^4}^2 + \|\mu\|_{W_2^3}^2\right].\end{aligned} \tag{38}$$

The validity of estimate (34) follows from (38).

Let $(x,\, t)$ be an arbitrary point from the domain $\bar{\Omega}$. From the first estimate (26) we have

$$|u(x,t)| \le C_1\sum_{k=1}^{\infty}\left(|\varphi_k| + k^{-1\frac{1}{3}}|\psi_k| + |\xi_k| + k^{-1\frac{1}{3}}|\mu_k|\right). \tag{39}$$

The coefficients φ_k, ξ_k are presented in the form

$$\varphi_k = \frac{1}{\lambda_k}\bar{\varphi}_k^{(1)}, \quad \xi_k = \frac{1}{\lambda_k}\bar{\xi}_k^{(1)},$$

where

$$\bar{\varphi}_k^{(1)} = \sqrt{\frac{2}{p}}\int_0^p \varphi'(x)\cos\lambda_k x dx, \quad \bar{\xi}_k^{(1)} = \sqrt{\frac{2}{p}}\int_0^p \xi'(x)\cos\lambda_k x dx.$$

From (39) we have

$$|u(x,t)| \le C_1 \sum_{k=1}^{\infty}\left(k^{-1}\left|\bar{\varphi}_k^{(1)}\right| + k^{-1\frac{1}{3}}|\psi_k| + k^{-1}\left|\bar{\xi}_k^{(1)}\right| + k^{-1\frac{1}{3}}|\mu_k|\right) \le$$

$$\le C_{12}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left[\left(\sum_{k=1}^{\infty}\left|\bar{\varphi}_k^{(1)}\right|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}|\psi_k|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}\left|\bar{\xi}_k^{(1)}\right|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}|\mu_k|^2\right)^{\frac{1}{2}}\right] \le$$

$$\le C_{13}\left[\|\varphi'\|_{L_2} + \|\psi\|_{L_2} + \|\xi'\|_{L_2} + \|\mu\|_{L_2}\right] \le C_{11}\left[\|\varphi\|_{W_2^1} + \|\psi\|_{W_2^0} + \|\xi\|_{W_2^1} + \|\mu\|_{W_2^0}\right].$$

This implies estimate (35). Based on the last estimate (27), we have

$$|f(x)| \le C_8 \sum_{k=1}^{\infty}\left(k^4|\varphi_k| + k^{2\frac{2}{3}}|\psi_k| + k^4|\xi_k| + k^{2\frac{2}{3}}|\mu_k|\right) \le$$

$$\le C_{14}\sum_{k=1}^{\infty}\frac{1}{k}\left(\left|\bar{\varphi}_k^{(5)}\right| + \left|\bar{\psi}_k^{(4)}\right| + \left|\bar{\xi}_k^{(5)}\right| + \left|\bar{\mu}_k^{(4)}\right|\right) \le$$

$$\le C_{14}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left[\left(\sum_{k=1}^{\infty}\left|\bar{\varphi}_k^{(5)}\right|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}\left|\bar{\psi}_k^{(4)}\right|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}\left|\bar{\xi}_k^{(5)}\right|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}\left|\bar{\mu}_k^{(4)}\right|^2\right)^{\frac{1}{2}}\right] \le$$

$$\le C_{15}\left(\left\|\varphi^V\right\|_{L_2} + \left\|\psi^{IV}\right\|_{L_2} + \left\|\xi^V\right\|_{L_2} + \left\|\mu^{IV}\right\|_{L_2}\right) \le C_{12}\left(\|\varphi\|_{W_2^5} + \|\psi\|_{W_2^4} + \|\xi\|_{W_2^5} + \|\mu\|_{W_2^4}\right).$$

From this inequality follows (36). The theorem is proved.

Conclusion

In this paper, the inverse problem for a fourth-order equation is considered. The solution is constructed as a series. The uniqueness of the solution to the inverse problem follows from the completeness of the system of eigenfunctions. The stability of the solution to the inverse problem is proven. The results obtained can be used for further development of various direct and inverse problems for a fourth-order equation.

Conflict of Interest

The authors declare no conflict of interest.

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