Modeling of dynamics processes and dynamics control

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Equations and methods of classical mechanics are used to describe the dynamics of technical systems containing elements of various physical nature, planning and management tasks of production and economic objects. The direct use of known dynamics equations with indefinite multipliers leads to an increase in deviations from the constraint equations in the numerical solution. Common methods of constraint stabilization, known from publications, are not always effective. In the general formulation, the problem of constraint stabilization was considered as an inverse problem of dynamics and it requires the determination of Lagrange multipliers or control actions, in which holonomic and differential constraints are partial integrals of the equations of the dynamics of a closed system. The conditions of stability of the integral manifold determined by the constraint equations and stabilization of the constraint in the numerical solution of the dynamic equations were formulated.

Keywords: constraint stabilization, numerical methods, nonholonomic constraints, Helmholtz conditions.

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Introduction

The main task of modeling the dynamics is the construction of differential equations of a closed system, the solutions of which have the required properties. The kinematic properties of the motion of a mechanical system and the required properties of the state change of the controlled system are usually given by the constraint equations. The problem of determining the right-hand sides of the equations of dynamics of controlled systems due to the formation of control functions, in essence, refers to the inverse problems of dynamics [1–9]. Methods of classical mechanics are successfully applied to construct the equations of dynamics of a system consisting of elements of various physical nature [10]. The description of analytical dynamics and systems of differential-algebraic equations is proposed in [11]. The analogy between the dynamics of a point of variable mass and the process of change of the simplest economic object allows us to use the equations of classical mechanics to solve problems of control of economic objects and securities portfolios [12–14]. The works [15–17] are devoted to the study of direct and inverse problems of stochastic differential equations describing the dynamics of mechanical systems subject to random influences. In classical mechanics, contact constraints are used, meaning that the initial state and subsequent motion of the system correspond to the constraint equations [18]. In control systems, the equations of servoconstraints [19] are usually introduced, supported by additional control forces. Additional conditions imposed on the solutions of the dynamics equations corresponding to the motion of the image point along the manifold described by the constraint equations and in its vicinity lead to the need to introduce the concepts of program constraint and equations of perturbations of constraints in control systems [20]. The expressions of the controlling influences that ensure the fulfillment of the constraint equations are determined by the relations between the phase coordinates of the system.

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1   Problem Statement

The dynamics of a controlled system with mechanical constraints, the phase state of which is determined by the vectors $q = (q^1, ..., q^n)$, $v = (v^1, ..., v^n)$, is usually described by a system of differential equations

$$\frac{dq^i}{dt} = a^i\left(q, v, t\right), \quad \frac{dv^i}{dt} = b^i\left(q, v, t\right) + b^{i\kappa}\left(q, v, t\right) u_\kappa, \tag{1}$$

with initial conditions

$$q^i\left(t_0\right) = q_0^i, v^i\left(t_0\right) = v_0^i, \quad i = 1, \ldots, n, \quad \kappa = 1, \ldots, s. \tag{2}$$

In equations (and further) summation is assumed for the repeated indices. Control forces are chosen so they satisfy the consraint equations

$$f^\mu\left(q, t\right) = 0, \quad \varphi^\nu\left(q, v, t\right) = 0, \quad \mu = 1, \ldots, m, \quad \nu = m+1, \ldots, r, \quad r \le s, \tag{3}$$

along with a given accuracy in the numerical solution of a system of equations (1), (2).

In particular, the dynamics of the mechanical system on which the constraints are imposed is described by the equations

$$\frac{dq^i}{dt} = v^i, \quad \frac{d}{dt}\frac{\partial L}{\partial v^i} = \frac{\partial L}{\partial q^i} + Q_i\left(q, v, t\right) + \frac{\partial \varphi^\kappa}{\partial v^i}\lambda_\kappa, \quad \varphi^\mu = \frac{\partial f^\mu}{\partial q^i}v^i + \frac{\partial f^\mu}{\partial t}, \quad \kappa = 1, \ldots, r \le n. \tag{4}$$

Here $L = T - P(q)$ is the Lagrangian, the doubled kinetic energy $2T = m_{ij}(q)v^i v^j$, $i, j = 1, ..., n$, $P = P(q)$ is the potential energy, $Q_i = Q_i(q, v, t)$ are non-potential generalized forces. Lagrange multipliers λ_k are considered as control functions, which must be selected so that the coordinates q^i and the velocities v^i of the system satisfy the constraint equations (3). The system of equations (4) resolved with respect to derivatives is reduced to the form (1) with notation

$$b^l\left(q, v, t\right) = m^{lk}\left(Q_k\left(q, v, t\right) - \frac{1}{2}\left(\frac{\partial m_{ik}}{\partial q^j} + \frac{\partial m_{jk}}{\partial q^i} - \frac{\partial m_{ij}}{\partial q^k}\right)v^i v^j\right),$$

$$b^{l\kappa}\left(q, v, t\right) = m^{lk}\frac{\partial \varphi^\kappa}{\partial v^k}, \quad m^{lk}m_{kj} = \delta_j^l,$$

$$\delta_j^l = 1, \; l = j, \; \delta_j^l = 0, \; l \ne j, \; i, j, k, l = 1, \ldots, n.$$

2   Formulas and theorems

In the case of contact constraints, the initial conditions are

$$q^i\left(t_0\right) = q_0^i, \quad v^i\left(t_0\right) = v_0^i \tag{5}$$

satisfy the constraint equations: $f^\mu(q_0, t_0) = 0$, $\varphi^k(q_0, v_0, t_0) = 0$, and the Lagrange multipliers are determined from the conditions

$$\frac{d\varphi^\rho\left(q, v, t\right)}{dt} = 0, \quad \rho = 1, ..., r. \tag{6}$$

From the equalities (6), taking into account the equations (1), a system of linear algebraic equations follows to determine the expression:

$$\frac{\partial \varphi^\rho}{\partial v^i}\left(b^i\left(q, v, t\right) + b^{i\kappa}\left(q, v, t\right)u_\kappa\right) + \frac{\partial \varphi^\rho}{\partial q^i}a^i + \frac{\partial \varphi^\rho}{\partial t} = 0.$$

If the initial conditions (5) are not consistent with the coupling equations (3):

$$f^\mu(q_0, t_0) = f_0^\mu, \quad \varphi^\rho(q_0, v_0, t_0) = \varphi_0^\rho, \quad \rho = 1, \ldots, r, \tag{7}$$

it follows from the equalities (6), (7) that with the numerical solution of the system (1), deviations from the coupling equations increase over time:

$$f^\mu = f_0^\mu + \varphi_0^\mu t, \quad \varphi^\rho(q_0, v_0, t_0) = \varphi_0^\rho.$$

The problem of constraint stabilization arises, for the solution of which it was proposed [21] to use a linear combination of constraint equations with their derivatives:

$$\frac{d^2 f^\mu}{dt^2} + k_1 \frac{df^\mu}{dt} + k_0 f^\mu = 0, \quad \frac{d\varphi^\rho}{dt} = \gamma(q, v, t)\varphi^\rho. \tag{8}$$

In essence, equalities (8) are equations of perturbations of constraints. Obviously, when the constraints are satisfied $k_1 - const, k_0 - const, k_1 > 0, \ k_0 > 0, \ \gamma(q, v, t) > 0$ trivial solutions of $f^\mu = 0, \ \varphi^\rho = 0$ of equations (8) are asymptotically stable. So in the simplest case, the equations with respect to perturbations of holonomic constraints of equation (8) can be represented by a linear system with constant coefficients [22]

$$\frac{df^\mu}{dt} = \varphi^\mu, \quad \frac{d\varphi^\rho}{dt} = k_\mu^\rho f^\mu + k_\kappa^\rho \varphi^\kappa, \quad \mu = 1, \ldots, m, \quad \rho, \kappa = 1, \ldots, r.$$

To limit deviations from the coupling equations in the numerical solution of the dynamics equations, additional conditions should be imposed on the coefficients of the equations (8). Various modifications of the J. Baumgarte method were proposed, for example, [22, 23], which were reduced to the selection of numerical methods for solving dynamic equations and recommendations for the selection of coefficients of the equations of the system (8). In [22], a hybrid scheme of integration of a controlled system consisting of a non-rigid mechanical subsystem and a rigid controlled subsystem is described. J. Baumgarte is also used to stabilize constraints in higher-order control systems [23]. To determine the expression of the multiplier λ in the right side of the equation of the system

$$\frac{d^m q}{dt^m} = Q\left(q, \frac{dq}{dt}, \ldots, \frac{d^{m-1}q}{dt^{m-1}}, t\right) + B(q, t)\lambda, \quad f(q, t) = 0$$

a linear combination of the constraint equation with derivatives up to the order of $m \geq 2$ is used, which leads to a differential equation of the constraint perturbations:

$$\alpha_\mu y^{(\mu)} = 0, \quad y = f(x, t), \quad y^{(\mu)} = \frac{d^\mu y}{dt^\mu}, \quad \mu = 0, \ldots, m.$$

The coefficients α_μ of the differential equation should be chosen so that the roots of the characteristic equation $\alpha_\mu \kappa^\mu = 0$ have negative real parts, for which it is proposed to use a polynomial of the form $\alpha_\mu \kappa^\mu = (\kappa + k)^m, \ k - const$. In this case, the solution of the equation of constraints perturbations is represented by the expression

$$y = (A_\mu t^\mu) e^{-kt}, \quad \mu = 0, \ldots, m - 1.$$

The integration constants are determined by the choice of the initial conditions $y^{(\mu)}(t_0) = y_0^\mu$, $\mu = 0, \ldots, m - 1$, and for small values of t, the value of y may be significant. So, for $m = 2$ and initial conditions (5) corresponding to the equalities

$$f(q_0, t_0) = 0, \quad \left(\frac{\partial f}{\partial q^i}\right)_0 v_0^i + \left(\frac{\partial f}{\partial t}\right)_0 = v_0,$$

change deviation from the constraint equation $f(q,t) = 0$ defined by the expression $y = v_0 t e^{-kt}$ and it can reach a significant value when numerically solving the dynamical equations (Fig. 1).
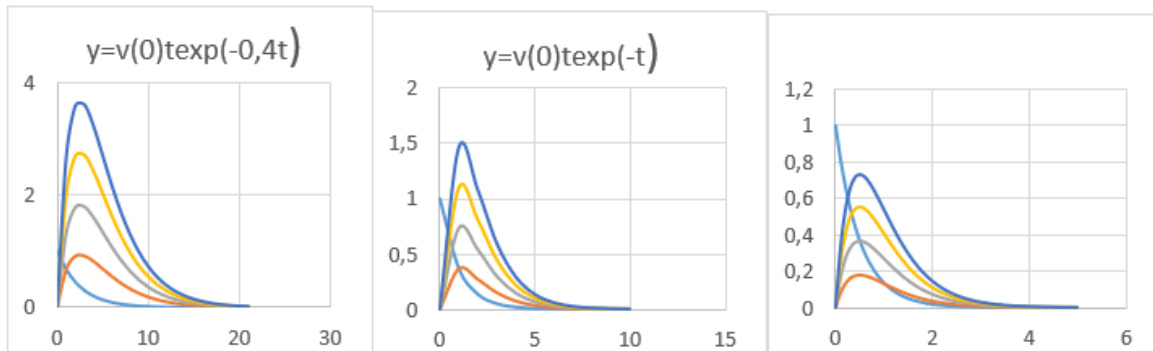

Figure 1. Graphs of the change in the value $y = f(q,t)$ at $v_0 = 1; 2; 3; 4$ and $k = 0, 4, k = 1, k = 2$ correspondingly

3 Construction of systems of differential equations

The concept of program constraints is associated with the construction of systems of differential equations with given partial integrals [24, 25] and to stabilize the constraints it is necessary that the constraints equations constitute partial integrals of the dynamics equations. The behavior of solutions in the vicinity of a set of points determined by the constraint equations must correspond to the operating conditions of a real system.

If the values of control actions u_k are defined as functions $u_k = u_k(q, v, t)$ of variables q, v, then by introducing the phase state vector $x = (q, v)$, the system of equations (1) and constraint equations (3) can be represented equalities

$$\frac{dx^i}{dt} = X^i(x, t), \tag{9}$$

$$\varphi^\rho(x, t) = 0, \ x = (x^1, \ldots, x^{2n}), \ \rho = 1, \ldots, r \le 2n, \ i = 1, \ldots, 2n. \tag{10}$$

Since functions (10) are partial integrals of the system of differential equations (9), the right-hand sides of X^i must satisfy the conditions

$$\frac{\partial \varphi^\rho}{\partial x^i} X^i + \frac{\partial \varphi^\rho}{\partial t} = F^\rho(f, \varphi, x, t), \quad \varphi = (\varphi^1, \ldots, \varphi^r), \tag{11}$$

where $F^\rho(f, \varphi, x, t)$ are arbitrary functions that satisfy the equalities $F^\rho(0, 0, x, t) = 0$. From equality (11) it follows that the right-hand sides of the equations of system (9) should have the following structure

$$X^i = \varphi^0 X^i_\tau + X^i_\nu,$$

where φ^0 is an arbitrary value, X^i_τ is the corresponding component of the vector product

$$X_\tau = \varphi^0 \left[\nabla \varphi^1 \ldots \nabla \varphi^r c^{r+1} \ldots c^{2n-1} \right], \quad \nabla \varphi^\rho = \left(\frac{\partial \varphi^\rho}{\partial x^1}, \ldots, \frac{\partial \varphi^\rho}{\partial x^{2n}} \right),$$

$$\nabla \varphi^\rho = (\varphi^\rho_1, \ldots, \varphi^\rho_{2n}), \quad \varphi^\rho_i = \frac{\partial \varphi^\rho}{\partial x^i},$$

$c^{r+1}, \ldots, c^{2n-1}$ are arbitrary vectors $c^\sigma = (c^\sigma_1, \ldots, c^\sigma_{2n})$, $\sigma = r + 1, \ldots, 2n - 1$,

$$X^i_\nu = \delta^{ij} \varphi^\alpha_j \omega_{\alpha\rho} F^\rho, \quad \delta^{ij} = 0, \quad i \ne j,$$

$$\delta^{ii} = 1, \quad (\omega_{\alpha\rho}) = (\omega^{\rho\gamma})^{-1}, \quad \omega^{\rho\gamma} = \varphi^\rho_i \delta^{ij} \varphi^\gamma_j, \quad i = 1, \ldots 2n, \quad \alpha, \gamma, \rho = 1, \ldots, r.$$

The system of equations represented by equalities (1), (3) constitutes a system of differential algebraic equations. The functions u_k in equations (1) are control actions that ensure the fulfillment of constraint equations (3). To stabilize the constraints (3), we determine possible deviations from the constraint equations (3) by the quantities

$$y^\mu = f^\mu(q, t), \quad z^\rho = \varphi^\rho(q, v, t), \quad \mu = 1, \ldots, m, \quad \rho = 1, \ldots, r, \quad r \leq s. \tag{12}$$

We define new variables y^μ, z^ρ as solutions to the system of constraint perturbation equations

$$\frac{dy^\mu}{dt} = z^\mu, \qquad \frac{dz^\rho}{dt} = Z^\rho(y, z, q, v, t), \tag{13}$$

satisfying the equalities $Z^\rho(0, 0, q, v, t) = 0$ and the initial conditions

$$y_0^\mu = f^\mu(q_0, t_0), \quad z_0^\rho = \varphi^\rho(q_0, v_0, t_0), \quad \mu = 1, \ldots, m, \quad \rho = 1, \ldots, r. \tag{14}$$

Equalities (13) define a system of equations for constraint perturbations, which, when

$$Z^\mu = -\omega^2 y^\mu - 2\alpha z^\mu, \quad Z^\nu = -\gamma(q, v, t) z^\nu, \quad \alpha, \omega - const,$$

corresponds to the method of J. Baumgarte [21]. Constraint equations (12), supplemented with conditions (13), (14), constitute the program coupling equations.

From equalities (1), (12), (13) follows a system of equations for determining the control actions u_k:

$$
\begin{aligned}
&p^{\rho\kappa} u_\kappa = h^\rho, \\
&p^{\rho\kappa} = \frac{\partial \varphi^\rho}{\partial v^i} b^{i\kappa}, \ h^\rho = Z^\rho(f, \varphi, q, v, t) - \frac{\partial \varphi^\rho}{\partial v^i} b^i(q, v, t) - \frac{\partial \varphi^\rho}{\partial q^i} a^i - \frac{\partial \varphi^\rho}{\partial t}, \\
&f = (f^1, \ldots, f^m), \ \varphi = (\varphi^1, \ldots, \varphi^\rho), \\
&\rho = 1, \ldots, r, \ \kappa = 1, \ldots, s, \ r \leq s.
\end{aligned}
\tag{15}
$$

If the rows of the matrix (p_k^ρ) are linearly independent, then the expressions of the control actions u_k are determined by solving the system of linear equations (15):

$$u_\kappa = c_0 \delta_\kappa \left[p^1 \ldots p^r c^{r+1} \ldots c^{s-1} \right] + \delta_{\beta\kappa} p^{\alpha\beta} \omega_{\alpha\rho} h^\rho,$$

c_0 is an arbitrary value, $c^\rho = (c^\rho 1, \ldots, c^\rho s)$ is an arbitrary vector, $\delta_\kappa = (\delta_\kappa^1, \ldots, \delta_\kappa^s)$,

$$
\begin{aligned}
&p^\rho = (p^{\rho 1}, \ldots, p^{\rho s}), \ \delta_{\beta\beta} = 1, \ \delta_{\beta\kappa} = 0, \ \beta \neq \kappa, \\
&\omega^{\rho\alpha} = p^{\rho\kappa} \delta_{\kappa\beta} p^{\alpha\beta}, \\
&\omega_{\alpha\rho} \omega^{\rho\gamma} = \delta_\gamma^\alpha, \ \delta_\alpha^\alpha = 1, \ \delta_\gamma^\alpha = 0, \ \alpha \neq \gamma, \ \alpha, \rho, \gamma = 1, \ldots, r, \ \beta, \kappa = 1, \ldots, s.
\end{aligned}
$$

As a result of substituting the resulting expressions into the right-hand sides of the equations, the closed system of equations (1) is written in the following form:

$$
\begin{aligned}
&\frac{dq^i}{dt} = a^i(q, v, t), \ \frac{dv^i}{dt} = b^i(q, v, t) + b^{i\kappa}(q, v, t) u_\kappa(q, v, t), \\
&u_\kappa(q, v, t) = u_{\kappa 0}(q, v, t) + u_{\kappa 1}(y, z, q, v, t), \\
&u_{\kappa 0}(q, v, t) = c_0 \delta_\kappa \left[p^1 \ldots p^r c^{r+1} \ldots c^{s-1} \right] - \delta_{\kappa\beta} p^{\alpha\beta} \omega_{\alpha\rho} \left(\frac{\partial \varphi^\rho}{\partial v^i} b^i(q, v, t) + \frac{\partial \varphi^\rho}{\partial q^i} a^i + \frac{\partial \varphi^\rho}{\partial t} \right), \\
&u_{\kappa 1}(y, z, q, v, t) = \delta_{\kappa\beta} p^{\alpha\beta} \omega_{\alpha\rho} Z^\rho(y, z, q, v, t), \\
&y = f(q, t), \ z = \varphi(q, v, t).
\end{aligned}
\tag{16}
$$

The system of equations (16) has partial integrals determined by the constraint equations (3).

<center>*4 Stability of the integral manifold*</center>

Using notation

$$
\begin{aligned}
&x = \left(x^1, \ldots, x^{2n}\right), \ x^i = q^i, \ x^{n+i} = v^i \\
&\eta = \left(\eta^1, \ldots, \eta^{m+r}\right), \ \eta^\mu = y^\mu, \ \eta^{m+\rho} = z^\rho, \\
&g^\sigma(x,t) = 0, \ \sigma = 1, \ldots, m+r, \\
&g^\mu = f^\mu, \ g^\rho = \varphi^\rho, \ \mu = 1, \ldots, m, \ \rho = 1, \ldots, r,
\end{aligned}
\tag{17}
$$

let us rewrite the system of equations (12), (13), (16) in a compact form:

$$
\eta^\sigma = g^\sigma(x,t),
\tag{18}
$$

$$
\tfrac{dx^s}{dt} = X^s(\eta, x, t), \quad s = 1, \ldots, 2n,
\tag{19}
$$

$$
\tfrac{d\eta^\sigma}{dt} = \Upsilon^\sigma(\eta, x, t), \quad \sigma = 1, \ldots, m+r,
\tag{20}
$$

$$
\begin{aligned}
&X^i(y, x, t) = x^{n+i}, \ X^{n+i}(\eta, x, t) = X_0^{n+i}(x, t) + X_1^{n+i}(\eta, x, t), \\
&X_0^{n+i}(x, t) = b^i(x, t) + b^{i\kappa}(x, t) u_{\kappa 0}(x, t), \ X_1^{n+i}(\eta, x, t) = b^{i\kappa}(x, t) u_{\kappa 1}(\eta, x, t), \\
&\Upsilon^\mu(\eta, x, t) = y^{m+\mu}, \quad \Upsilon^{m+\rho}(\eta, x, t) = Z^\rho(\eta, x, t).
\end{aligned}
$$

Setting $x^s(t_0) = x_0^s, \eta^\sigma(t_0) = \eta_0^\sigma \equiv g^\sigma(x_0, t_0)$, we determine the stability conditions [24] of the integral manifold of system (18), given by equalities (17).

Definition 1. The integral manifold of the system of equations (19), defined by the equality $\eta(x,t) = 0$, is stable if for any ϵ there exists a δ such that for all initial conditions $x(t_0) = x_0$ corresponding to the inequalities $|\eta_0| \leq \delta$, the value $\eta = \eta(t)$ will satisfy the condition $|\eta(t)| \leq \epsilon$ for all $t > t_0$.

The stability of a trivial solution to the system of equations (20) depends on the choice of functions $\Upsilon^{(m+\rho)}(\eta, x, t)$. Stability conditions can be obtained using Lyapunov functions. If the functions $\Upsilon^{(m+\rho)}$ are represented by a linear combination of constraint perturbations, then the system of equations (20) turns out to be linear:

$$
\frac{d\eta^\sigma}{dt} = h_\alpha^\sigma(x, t)\, \eta^\alpha, \quad \sigma, \alpha = 1, \ldots, m+r.
\tag{21}
$$

To study the stability of the trivial solution of system (20), we take as the Lyapunov function a positive definite quadratic form with constant coefficients $V = 0.5 c_{\sigma\alpha} \eta^\sigma \eta^\alpha$. Then there are constants c_1, c_2 corresponding to the constraints $c_1|\eta|^2 \leq V \leq c_2|\eta|^2$. If the derivative of function V, calculated by virtue of the equations of system (21),

$$
\frac{dV}{dt} = p_{\sigma\alpha}(x, t)\, \eta^\sigma \eta^\alpha, \quad p_{\sigma\alpha}(x, t) = c_{\sigma\zeta} h_\alpha^\zeta(x, t), \quad \sigma, \alpha, \zeta = 1, \ldots, m+r,
$$

will be limited:

$$
\frac{dV}{dt} \leq -a|\eta|^2,
$$

then the inequality will be satisfied $|\eta|^2 \leq \frac{c_2}{c_1}|\eta_0^2|e^{t-t_0}$, $\lambda = \frac{2a}{c_2}$, and the integral manifold (17) of the system of equations (19) will be stably exponential. If the coefficients h_α^σ of the equations of system (21) are constant, then the stability of the trivial solution is determined by the roots of the characteristic equation.

<center>*5 Constraint stabilization of in the numerical solution of dynamic equations*</center>

The asymptotic stability of the trivial solution of system (20) is not enough to limit deviations from the constraint equations when numerically solving the dynamic equations

$$\frac{dx^s}{dt} = X^s \left(g\left(x,t\right), x, t\right), \quad x^s\left(t_0\right) = x_0^s. \tag{22}$$

The requirement to stabilize the constraints imposes additional conditions [26] on the right-hand sides of the constraint perturbation equations (19), which are determined by the value of the limitation of deviations from the constraint equations and the choice of the numerical method for solving system (22) [25–28]. Let $|\eta_0| \le \epsilon$ and let the difference scheme be used to solve system (22)

$$x_{l+1}^s = x_l^s + (\Delta x^s)_l, \ (\Delta x^s)_l = \tau X^s (x_l, t_l), \ \tau = t_{l+1} - t_l, \ l = 1, \ldots, N. \tag{23}$$

Let us represent the functions $\eta_{l+1}^\sigma = g^\sigma(x_{l+1}, t_{l+1})$ by series expansions in powers of τ:

$$g^\sigma \left(x_{l+1}, t_{l+1}\right) = g^\sigma \left(x_l, t_l\right) + \left(\frac{\partial g^\sigma}{\partial x^s}\right)_l (\Delta x^s)_l + \tau \left(\frac{\partial g^\sigma}{\partial t}\right)_l + \frac{\tau^2}{2} \overset{\sigma(2)}{\widetilde{g}}_l, \tag{24}$$

or taking into account equalities (23), (24):

$$\eta_{l+1}^\sigma = \eta_l^\sigma + \tau \Upsilon_l^\sigma \left(y, x, t\right) + \frac{\tau^2}{2} \overset{\sigma(2)}{\widetilde{g}}_l.$$

After expanding the function $\Upsilon_l^\sigma = \Upsilon_l^\sigma(\eta, x, t)$ into a series in powers of magnitude η_l^α, the last equality will be rewritten in the following form:

$$\eta_{l+1}^\sigma = \eta_l^\sigma + \tau k_\alpha^\sigma \left(x_l, t_l\right) \eta_l^\alpha + \frac{\tau^2}{2} \overset{\sigma(2)}{\widetilde{\Upsilon}}_l + \frac{\tau^2}{2} \overset{\sigma(2)}{\widetilde{g}}_l. \tag{25}$$

From equalities (25) the following estimates follow:

$$|\eta_{l+1}^\sigma| \le |\delta_\sigma^\alpha + \tau k_\alpha^\sigma \left(x_l, t_l\right) \eta_l^\alpha| + \frac{\tau^2}{2} \overset{\sigma(2)}{\widetilde{\Upsilon}}_l + \frac{\tau^2}{2} \overset{\sigma(2)}{\widetilde{g}}_l, \ \delta_\alpha^\sigma = 0, \ \sigma \ne \alpha, \ \delta_\sigma^\sigma = 1,$$

and statement.

Theorem. If the inequality $|\eta_0| \le \epsilon$ is satisfied and the functions $\Upsilon^{m+\rho}(\eta, x, t)$, $\eta^\mu = f^\mu(x, t)$, $\eta^{m+\rho}(t) = \varphi^\rho(x, t)$ for all values of x, t corresponding to the solution of system (22), satisfy the conditions $1 + \tau \kappa x$, $t \le \theta \le 1$, $\frac{\tau^2}{2} \Upsilon^2 + g^2 \le 1 - \Theta \epsilon$, then for all $l = 1, \ldots, N$ the inequalities $|\eta_l| \le \epsilon$ will be satisfied.

Example. Determine the control function $u = u(q^1, q^2)$ for the system

$$\frac{dq^1}{dt} = -4q^2 - q^1 b\left(q^1, q^2\right) u, \quad \frac{dq^2}{dt} = q^1 - 4q^2 b\left(q^1, q^2\right) u,$$
$$b\left(q^1, q^2\right) = \frac{1}{\left(q^1\right)^2 + \left(q^2\right)^2}, \quad q^1\left(0\right) = 2, \quad q^2\left(0\right) = 0, \tag{26}$$

ensuring the existence of the partial integral $y = 0.5(q^1)^2 + 2(q^2)^2 - 2 = 0$ and its stabilization when solving system (26) numerically using the Euler method with a step $\tau = 0.001$. Constraint perturbation equation

$$\frac{dy}{dt} = -ky, \quad k > 0,$$

has an asymptotically stable trivial solution $y = 0$. Control $u = k((q^1)^2 + 4(q^2)^2 - 4)$ ensures the fulfillment of the constraint equation $0.5(q^1)^2 + 2(q^2)^2 - 2 = 0$ with an accuracy of $\epsilon = 0.001$ at values of k that satisfy the restrictions: $200 < k < 1800$. Figure 2 shows graphs of changes in calculation error values corresponding to the values $k = 50; 300; 2050$.
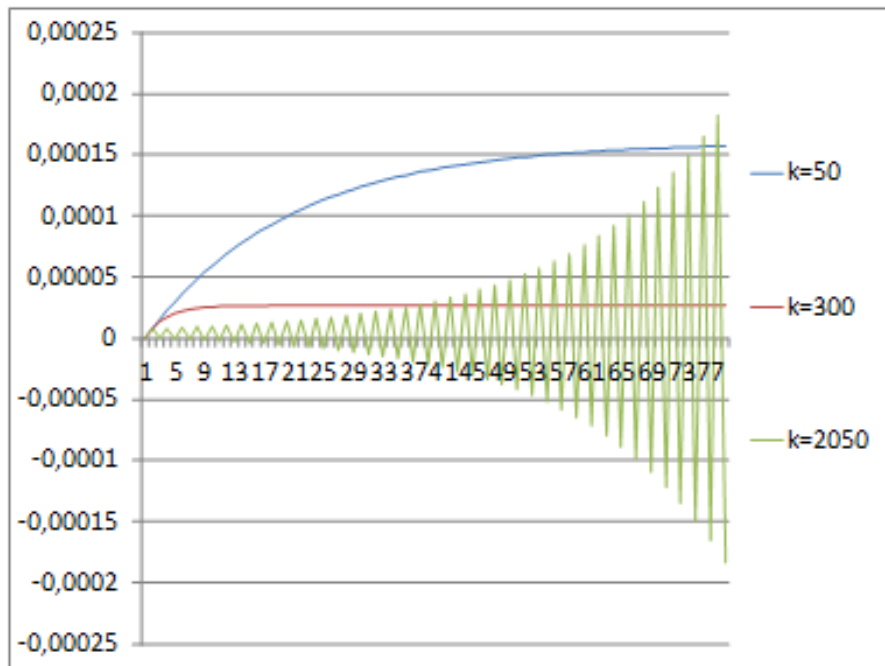
Figure 2. Deviations values on time

6 Conclusion

Methods of constraint stabilization, based on the construction of systems of differential equations with asymptotically stable partial integrals, represent effective ways to model solutions to problems of determining the reactions of constraints and controlling the dynamics of systems for various purposes.

Author Contributions

All authors contributed equally to this work

Conflict of Interest

The authors declare no conflict of interest.

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Динамика процестерін модельдеу және байланыстарды тұрақтандыруды ескере отырып, жүйені басқару синтезі

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Әр түрлі физикалық сипаттағы элементтерден тұратын техникалық басқару жүйелерінің динамикасын, өндіріс пен экономикалық объектілерді жоспарлау және басқару міндеттерін сипаттау үшін классикалық механиканың теңдеулері мен әдістері қолданылады. Анықталмаған факторлары бар белгілі динамикалық теңдеулерді тікелей пайдалану сандық шешімдегі байланыс теңдеулерінен ауытқулардың артуына әкеледі. Басылымдардан белгілі байланыстарды тұрақтандырудың кең таралған әдістері әрдайым тиімді бола бермейді. Жалпы есептің қойылуында байланыстарды тұрақтандыру есебі динамиканың кері есебі ретінде қарастырылған және голономикалық байланыстар мен дифференциалдық байланыстар тұйық жүйе динамикасы теңдеулерінің дербес интегралдары болып табылатын Лагранж факторларын немесе басқару әсерлерін анықтауды талап етеді. Байланыс теңдеулерімен анықталған интегралдық көпбейнеліктердің тұрақтылығы және динамикалық теңдеулерді сандық шешуде байланыстарды тұрақтандыру шарттары тұжырымдалған.

Кілт сөздер: байланыстарды тұрақтандыру, сандық әдістер, голономиялық емес байланыс, Гельмгольц шарттары.

Моделирование процессов динамики и синтез управления системой с учетом стабилизации связей

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Для описания динамики технических систем управления, содержащих элементы различной физической природы, задач планирования и управления производством и экономическими объектами используются уравнения и методы классической механики. Непосредственное применение известных уравнений динамики с неопределенными множителями приводит к возрастанию отклонений от уравнений связей при численном решении. Распространенные методы стабилизации связей, известные по публикациям, оказываются не всегда эффективными. В общей постановке задача стабилизации связей рассмотрена как обратная задача динамики, и она требует определения множителей Лагранжа или управляющих воздействий, при которых голономные связи и дифференциальные связи являются частными интегралами уравнений динамики замкнутой системы. Сформулированы условия устойчивости интегрального многообразия, определяемого уравнениями связей, и стабилизации связей при численном решении уравнений динамики.

Ключевые слова: стабилизация связей, численные методы, неголономная связь, условия Гельмгольца.

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