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On some linear two-point inverse problem for a multidimensional heat conduction equation with semi-nonlocal boundary conditions

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It is known that V.A. Ilyin and E.I. Moiseev studied generalized nonlocal boundary value problems for the Sturm-Liouville equation, the nonlocal boundary conditions specified at the interior points of the interval under consideration. For such problems, uniqueness and existence theorems for a solution to the problem were proven. There are many difficulties in studying these generalized nonlocal boundary value problems for partial differential equations, especially in obtaining a priori estimates. Therefore, it is necessary to use new methods for solving generalized nonlocal problems (forward problems). As we know, it is not difficult to establish a connection between forward and inverse problems. Therefore, when solving generalized nonlocal boundary value problems for partial differential equations, reducing them to multipoint inverse problems is necessary. The first results in the direction belong to S.Z. Dzhamalov. In his works, he proposed and investigated multipoint inverse problems for some equations of mathematical physics. In this article, the authors studied the correctness of one linear two-point inverse problem for the multidimensional heat conduction equation. Using the methods of a priori estimates, Galerkin's method, a sequence of approximations and contracting mappings, the unique solvability of the generalized solution of the linear two-point inverse problem for the multidimensional heat equation was proved.

Keywords: multidimensional heat conduction equation, linear two-point inverse problem, unique solvability of a generalized solution, methods of a priori estimates, Galerkin's method, sequences of approximations and contracting mappings.

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Introduction

Due to the significant increase in the capabilities of computer technology over the past decades, complex mathematical models that take into account a more significant number of physical factors are beginning to be used in applied mathematics. In [1–4], mathematical models that arise in the study of several applied problems and lead to the consideration of nonlocal boundary value problems were first proposed. As is known, it is not difficult to establish a connection between nonlocal boundary value problems and multipoint inverse problems [3–6]. In this regard, it should be especially noted that heat propagation processes are closely related precisely to multipoint inverse problems for parabolic equations [4]. For parabolic equations, particularly heat equations, the difference between inverse problems was studied in [7–19].

To this end, in this work, using the results of [5, 6], we study the unique solvability of a particular linear two-point inverse problem (LTIP) for a multidimensional heat equation.

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Let Ω be a simply connected domain in space R^n with sufficiently smooth boundary $\partial\Omega$. Consider the multidimensional heat conduction equation in domain $G = \Omega \times (0,T) \times (0,l) = Q \times (0,l) \subset R^{n+2}$:

$$Lu = u_t - \Delta_x u - u_{yy} + c(x,t)u = g(x,t,y) + \sum_{i=1}^{2} h_i(x,t) f_i(x,t,y), \tag{1}$$

where $\Delta_x u = \sum\limits_{m=1}^{n} u_{x_m x_m}$ is the Laplace operator with regard to variables x, here $c(x,t)$, $g(x,t,y)$ and $f_i(x,t,y)$ $i = 1,2$ are given functions, and $h_1(x,t)$, $h_2(x,t)$ are the unknown functions.

1 *Linear two-point inverse problem*

It is required to find functions $\{u(x,t,y),\, h_1(x,t),\, h_2(x,t)\}$, that satisfy equation (1) in domain G, such that function $u(x,t,y)$ satisfies the following semi-nonlocal boundary conditions:

$$\gamma u\,|_{t=0} = u\,|_{t=T}, \tag{2}$$

$$u\,|_{\partial\Omega} = 0, \tag{3}$$

$$u\,|_{y=0} = u\,|_{y=l} = 0, \tag{4}$$

where γ is some constant nonzero number, the value of which will be specified below.

In addition, the solution to problem (1)–(4) satisfies the following auxiliary conditions:

$$u(x,t,\ell_j) = \varphi_j\,(x,t), \tag{5}$$

where $\ell_j \in (0,\ell)$, $j = 1,2$ are such that $0 < \ell_1 < \ell_2 < \ell < +\infty$, and functions $u(x,t,y)$ and $h_i(x,t)$, $i = 1,2$ belong to the following class:

$$U = \left\{(u, h_i,\, i = 1,2);\; u \in W_2^{2,1}(G),\; D_y^3(u_t, u_x, u_{xx}) \in L_2(G),\; h_i \in W_2^{2,1}(Q)\right\},$$

here $W_2^{2,1}(G)$ is the Sobolev space with norm

$$\|u\|^2_{W_2^{2,1}(G)} = \int_G \left(u_{xx}^2 + u_{yy}^2 + u_{xy}^2\right) dxdtdy + \int_G \left(u_x^2 + u_t^2 + u_y^2 + u^2\right) dxdtdy.$$

Let us introduce the following notation.
Let $g_j(x,t) = g(x,t,\ell_j), f_{ij}(x,t) = f_i(x,t,\ell_j)$, $\forall i,j = 1,2$.
$\mathfrak{F}^2 = \max\{\|f_{11}\|^2_{C(Q)}, \|f_{12}\|^2_{C(Q)}, \|f_{21}\|^2_{C(Q)}, \|f_{22}\|^2_{C(Q)}\}$.
Then we define a square matrix of the second order by $\mathbb{F} = \{f_{ij}\}_{i,j=1}^2$, i.e., $\mathbb{F} = \begin{pmatrix} f_{11} & f_{21} \\ f_{12} & f_{22} \end{pmatrix}$, and we denote its determinant by $H = \det \mathbb{F} = \begin{vmatrix} f_{11} & f_{21} \\ f_{12} & f_{22} \end{vmatrix}$.

Definition 1. Function $u(x,t,y) \in U$ that satisfies equation (1) almost everywhere in domain G with conditions (2)–(5), is called a generalized solution to problem (1)–(5).

Let all the coefficients of equation (1) be sufficiently smooth functions in domain Q and let the following conditions be satisfied regarding the coefficients on the right-hand sides of equation (1) and the given function $\varphi_j(x,t)$, $j = 1,2$.

Condition 1:
Periodicity: $c(x,0) = c(x,T)$, for all $x \in \bar{\Omega}$.

Nonlocal conditions: $\gamma g(x,0,y) = g(x,T,y)$, $\gamma f_j(x,0,y) = f_j(x,T,y)$, $j = 1,2$.
Smoothness: $g_j(x,t) = g(x,t,\ell_j) \in C^{0,1}_{x,t}(Q)$, $f_{ij}(x,t) = f_i(x,t,\ell_j) \in C^{0,1}_{x,t}(Q)$, $i,j = 1,2$;
$H = |\det \mathbb{F}| \geq \eta > 0$, $(1+D_y^3)g \in W_2^1(G)$, $(1+D_y^3)f_i \in W_2^2(G)$, $i = 1,2$.

Condition 2:
$\varphi_j(x,t) \in W_2^{2,1}(Q)$; $\gamma\varphi_j|_{t=0} = \varphi_j|_{t=T}$; $\varphi_j|_{\partial\Omega} = 0$, $j = 1,2$;
here $W_2^{2,1}(Q)$ is the Sobolev space with norm $\|u\|^2_{W_2^{2,1}(Q)} = \int\limits_Q \left(u_{xx}^2 + u_x^2 + u_t^2 + u^2\right)dxdt$.

2 *Unique solvability to problem* (1)–(5)

Theorem 1. Let the above conditions 1 and 2 be satisfied for the coefficients of equation (1), in addition, let $\lambda c(x,t) - c_t(x,t) \geq \delta_1 > 0$ for all $(x,t) \in \overline{Q}$, where $\lambda = \frac{2}{T}\ln|\gamma| > 0$, $|\gamma| > 1$ and let there exist a small positive number σ such that the following estimates hold: $\delta_0 - 10\sigma^{-1} \geq \delta > 0$, $q \equiv M \cdot \sum\limits_{i=1}^{2} \left\|(1+D_y^3)f_i\right\|_{W_2^2(G)} < 1$, (where $\delta_0 = \min\left\{2, \lambda, \delta_1 + \left(\frac{\pi}{\ell}\right)^2\right\}$, $M = 4\sigma\,\eta^{-2}\mathfrak{F}^2\ c_1c_2$; where $c_1 = \sum\limits_{k=1}^{\infty} \frac{\mu_k^4}{\left(1+\mu_k^2\right)^3}$, $\mu_k = \frac{k\pi}{\ell}$, c_2 is a constant number determined from the Sobolev embedding theorem). Then, there is a unique solution to problem (1)–(5) from the specified class U.

We first use the Fourier method to prove the solvability of problem (1)–(5). Namely, the solution to problem (1)–(5) is sought in the following form:

$$u(x,t,y) = \sum_{k=1}^{\infty} u_k(x,t)Y_k(y),$$

where functions $Y_k(y) = \left\{\sqrt{\frac{2}{\ell}}\sin\mu_k y\right\}$, $\mu_k = \frac{\pi k}{\ell}$, $k = 1,2,3,...$ are solutions of the Sturm- Liouville spectral problem with Dirichlet conditions. It is known that the system of eigenfunctions $\{Y_k(y)\}$ is complete in space $L_2(0,\ell)$ and forms an orthonormal basis in it [7–10].

In order to determine unknown functions, some construction formalities must be performed.

Let us consider the traces of equation (1) for $y = \ell_j$, $j = 1,2$.

$$\begin{gathered} Lu(x,t,\ell_j) = u_t(x,t,\ell_j) - \Delta_x u(x,t,\ell_j) - u_{yy}(x,t,\ell_j) + \\ +c(x,t)u(x,t,\ell_j) = g(x,t,\ell_j) + h_1(x,t)f_{1j}(x,t) + h_2(x,t)f_{1j}(x,t). \end{gathered} \tag{6}$$

Now, considering condition (5), $H = |\det \mathbb{F}| \geq \eta > 0$, and the corresponding notation, we define the formally unknown functions $h_j(x,t)$, $j = 1,2$ from the equation (6) in the following form:

$$h_1(x,t) = \frac{1}{H}[\Phi_1(x,t)f_{22}(x,t) - \Phi_2(x,t)f_{21}(x,t)],$$

$$h_2(x,t) = \frac{1}{H}[\Phi_2(x,t)f_{11}(x,t) - \Phi_1(x,t)f_{12}(x,t)],$$

here

$$\begin{gathered} \Phi_j(x,t) = \varphi_{jt}(x,t) - \Delta_x\varphi_j(x,t) + c(x,t)\varphi_j(x,t) - g_j(x,t) + \sum_{k=1}^{\infty}\mu_k^2 u_k(x,t)\sin\mu_k\ell_j = \\ = L_0\varphi_j(x,t) - g_j(x,t) + \sum_{k=1}^{\infty}\mu_k^2 u_k(x,t)\sin\mu_k\ell_j, \end{gathered}$$

$$L_0\varphi_j \equiv \varphi_{jt}(x,t) - \Delta_x\varphi_j(x,t) + c(x,t)\varphi_j(x,t),\ j = 1,2,$$

where functions $u_k(x,t)$ are defined in domain $Q = \Omega \times (0,T)$ as a solution to the following infinite system of loaded heat equations [3], [11]:

$$\begin{gathered} Lu_k = u_{kt} - \Delta_x u_k + (c(x,t) + \mu_k^2)u_k = g_k + \\ + \frac{f_{1k}}{H}[f_{22}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin \mu_m \ell_1) - f_{21}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin \mu_m \ell_2)] + \\ \frac{f_{2k}}{H}[f_{11}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin \mu_m \ell_2) - f_{12}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin \mu_m \ell_1)] \end{gathered} \tag{7}$$

with semi-nonlocal boundary conditions

$$\gamma u_k \left|_{t=0}\right. = u_k \left|_{t=T}\right., \tag{8}$$

$$u_k \left|_{\partial\Omega}\right. = 0, \tag{9}$$

where $f_1(x,t,y) = \sum\limits_{k=1}^{\infty} f_{1k}(x,t) \sin \mu_k y, \quad f_1(x,t,\ell_1) = f_{11}(x,t) = \sum\limits_{k=1}^{\infty} f_{1k}(x,t) \sin \mu_k \ell_1,$

$f_2(x,t,y) = \sum\limits_{k=1}^{\infty} f_{2k}(x,t) \sin \mu_k y, \quad f_2(x,t,\ell_1) = f_{21}(x,t) = \sum\limits_{k=1}^{\infty} f_{2k}(x,t) \sin \mu_k \ell_1,$

$f_{ik} = \sqrt{\frac{2}{\ell}} \int\limits_0^{\ell} f_i \sin \mu_k y dy$, for any $i = 1,2$; $g_k = \sqrt{\frac{2}{\ell}} \int\limits_0^{\ell} g \sin \mu_k y dy, \; k = 1,2,3,\ldots$

Proof. Let us prove the theorem 1 step by step. First, we show that function $u(x,t,y)$ for any $j = 1,2$ satisfies condition (5) i.e. $u|_{y=\ell_j} = u(x,t,\ell_j) = \varphi_j(x,t)$.

Let us prove the fulfilment of these conditions using inverse assumptions. Let there be function $\vartheta_j(x,t)$ satisfying condition (5): $\vartheta_j(x,t)$, such that $u|_{y=\ell_j} = \vartheta_j(x,t) \neq \varphi_j(x,t)$, i.e.,
$u|_{y=\ell_j} = \sum\limits_{k=0}^{\infty} u_k(x,t) \sin \mu_k \ell_j = \vartheta_j(x,t) \neq \varphi_j(x,t)$.

Then for functions $z_j(x,t) = \vartheta_j(x,t) - \varphi_j(x,t)$ in domain Q, considering conditions (8)-(9), multiplying equation (7) by $\sin \mu_k \ell_j$ and summing over k from 1 to ∞, we obtain the following loaded equations:

$$\begin{gathered} \sum_{k=1}^{\infty} u_{kt} \sin \mu_k \ell_j - \sum_{k=1}^{\infty} \Delta_x u_k \sin \mu_k \ell_j + \sum_{k=1}^{\infty} (c + \mu_k^2) u_k \sin \mu_k \ell_j = \sum_{k=1}^{\infty} g_k \sin \mu_k \ell_j + \\ + \frac{\sum_{k=1}^{\infty} f_{1k} \sin \mu_k \ell_j}{H}[\Phi_1 f_{22} - \Phi_2 f_{21}] + \frac{\sum_{k=1}^{\infty} f_{2k} \sin \mu_k \ell_j}{H}[\Phi_2 f_{11} - \Phi_1 f_{12}] = \sum_{k=1}^{\infty} g_k \sin \mu_k \ell_j + \\ + \frac{\sum_{k=1}^{\infty} f_{1k} \sin \mu_k \ell_j}{H}[f_{22}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin \mu_m \ell_1) - f_{21}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin \mu_m \ell_2)] + \\ + \frac{\sum_{k=1}^{\infty} f_{2k} \sin \mu_k \ell_j}{H}[f_{11}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin \mu_m \ell_2) - f_{12}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin \mu_m \ell_1)] = \\ = g_j + \frac{f_{1j}}{H}[f_{22}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin \mu_m \ell_1) - f_{21}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin \mu_m \ell_2)] + \\ + \frac{f_{2j}}{H}[f_{11}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin \mu_m \ell_2) - f_{12}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin \mu_m \ell_1)]. \end{gathered} \tag{10}$$

We consider each case separately to make it easier to understand the formula (10). First, we consider the case for $j = 1$. Then, from formula (10), we obtain:

$$\begin{aligned}
&\vartheta_{1t} - \Delta_x\vartheta_1 + c(x,t)\vartheta_1 + \sum_{k=1}^{\infty} \mu_k^2 u_k \sin\mu_k\ell_1 = g_1 + \\
&+\tfrac{f_{11}}{H}[f_{22}(L_0\varphi_1 - g_1 + \sum_{k=1}^{\infty}\mu_k^2 u_k \sin\mu_k\ell_1) - f_{21}(L_0\varphi_2 - g_2 + \sum_{k=1}^{\infty}\mu_k^2 u_k \sin\mu_k\ell_2)] + \\
&+\tfrac{f_{21}}{H}[f_{11}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty}\mu_k^2 u_k \sin\mu_k\ell_2) - f_{12}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty}\mu_k^2 u_k \sin\mu_k\ell_1)] = \\
&= g_1 + \tfrac{L_0\varphi_1 - g_1}{H}[f_{11}f_{22} - f_{12}f_{21}] + \frac{\sum_{k=1}^{\infty}\mu_k^2 u_k \sin\mu_k\ell_1}{H}[f_{11}f_{22} - f_{12}f_{21}] + \\
&\tfrac{L_0\varphi_2 - g_2}{H}[f_{21}f_{11} - f_{21}f_{11}] + \frac{\sum_{k=1}^{\infty}\mu_k^2 u_k \sin\mu_k\ell_2}{H}[f_{21}f_{11} - f_{21}f_{11}] = \\
&= g_1 + L_0\varphi_1 - g_1 + \sum_{k=1}^{\infty}\mu_k^2 u_k \sin\mu_k\ell_1 = L_0\varphi_1 + \sum_{k=1}^{\infty}\mu_k^2 u_k \sin\mu_k\ell_1.
\end{aligned} \tag{11}$$

Then from formulas (7)–(11) for function $z_1(x,t) = \vartheta_1(x,t) - \varphi_1(x,t) \Rightarrow \vartheta_1 = z_1 + \varphi_1$ in domain Q, we obtain the following identity

$$L_0(z_1 + \varphi_1) + \sum_{k=1}^{\infty}\mu_k^2 u_k \sin\mu_k\ell_1 = L_0\varphi_1 + \sum_{k=1}^{\infty}\mu_k^2 u_k \sin\mu_k\ell_1.$$

Hence, we obtain the following problem:

$$L_0 z_1 = z_{1t} - z_{1xx} + c(x,t)z_1 = 0, \tag{12}$$

$$\gamma\, z_1|_{t=0} = z_1|_{t=T}, \tag{13}$$

$$z_1\,|_{\partial\Omega} = 0. \tag{14}$$

Now we will prove the uniqueness of the solution to problem (12)–(14) using the method of energy integrals [3], [4], [8]. To do this, consider identity $2(L_0 z_1, e^{-\lambda t} z_{1\,t}) = 0$ and, integrating identity (12) by parts, considering conditions of Theorem 1 and boundary conditions (13), (14) for $|\gamma| > 1$, we obtain the inequality $\|z_j\|_{W_2^1(Q)} \le 0$, which implies that $z_1(x,t) = 0$.

So, problem (12)–(14) has a unique solution, i.e. $\vartheta_1(x,t) \equiv \varphi_1(x,t)$. From this, we obtain that problem (1)–(4) satisfies condition (5) for $j = 1$, i.e. $u(x,t,\ell_1) = \varphi_1(x,t)$. $u(x,t,\ell_2) = \varphi_2(x,t)$ is proved similarly for $j = 2$.

Now we will prove the solvability of problem (7)–(9) using the methods of a priori estimates, Galerkin's, and successive approximations [3], [8], namely, in domain Q, we consider a family of infinite loaded heat conduction equations:

$$\begin{aligned}
&Lu_k^{(l)} = u_{kt}^{(l)} - \Delta_x u_k^{(l)} + (c(x,t) + \mu_k^2)u_k^{(l)} = g_k + \\
&+\frac{f_{1k}}{H}[f_{22}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty}\mu_m^2 u_m^{(l-1)} \sin\mu_m\ell_1) - f_{21}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty}\mu_m^2 u_m^{(l-1)} \sin\mu_m\ell_2)] + \\
&+\frac{f_{2k}}{H}[f_{11}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty}\mu_m^2 u_m^{(l-1)} \sin\mu_m\ell_2) - f_{12}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty}\mu_m^2 u_m^{(l-1)} u_m \sin\mu_m\ell_1)] = F(u_k^{(l-1)})
\end{aligned} \tag{15}$$

with semi-nonlocal boundary conditions

$$\gamma u_k^{(l)}\,|_{t=0} = u_k^{(l)}\,|_{t=T}, \tag{16}$$

$$u_k^{(l)}\,|_{\partial\Omega}\, = 0, \tag{17}$$

where $l \in N \cup \{0\}$, N is the set of natural numbers. In the future, to prove the unique solvability of problem (15)–(17), we need the following notation and lemmas.

Let us define the space of vector functions

$$W_{p,q}(Q) = \{\,\vartheta_k |\, \vartheta_k \in W_{2,x,t}^{p,q}(Q),\ k \in N;\ p,q = 0,1,2\}$$

with norm

$$\langle \vartheta_k \rangle_{p,q}^2 = \sqrt{\frac{2}{\ell}} \sum_{k=1}^{\infty} \left(1+\mu_k^2\right)^3 \|\vartheta_k\|_{W_{2,x,t}^{p,q}(Q)}^2\,, \tag{18}$$

where $W_{2,x,t}^{p,q}(Q)$ may be one of the following Sobolev spaces

$$W_{2,x,t}^{2,2}(Q) \equiv W_2^{2,2}(Q) \equiv W_2^2(Q);\ W_{2,x,t}^{2,1}(Q) \equiv W_2^{2,1}(Q);\ W_{2,x,t}^{1,1}(Q) \equiv W_2^1(Q);\ W_{2,x,t}^{0,0}(Q) = W_2^0 = L_2(Q).$$

The norm in space $W_{2,1}(Q)$ is defined as follows

$$\langle \vartheta_k \rangle_{2,1}^2 = \sqrt{\frac{2}{\ell}} \sum_{k=1}^{\infty} \left(1+\mu_k^2\right)^3 \|\vartheta_k\|_{W_2^{2,1}(Q)}^2\,,$$

and the norm in space $W_{0,0}(Q)$ is defined as follows

$$\langle \vartheta_k \rangle_{0,0}^2 = \sqrt{\frac{2}{\ell}} \sum_{k=1}^{\infty} \left(1+\mu_k^2\right)^3 \|\vartheta_k\|_{L_2(Q)}^2\,.$$

It is obvious that the space $W_{p,q}(Q)$ with a certain norm (18) is a Banach space [3], [8]. From the definition of spaces $W_{p,q}(Q)$ it follows that $W_{2,2}(Q) \subset W_{2,1}(Q) \subset W_{1,1}(Q) \subset W_{0,0}(Q)$.

Now let us denote the class of vector functions $\{\vartheta_k(x,t)\}_{k=1}^{\infty}$ such that $\{\vartheta_k(x,t)\}_{k=1}^{\infty} \in W_{2,1}(Q)$, satisfying the corresponding conditions (16), (17) by $W(Q)$.

Definition 2. The solution to problem (15)–(17) is called vector function $\{\vartheta_k(x,t)\}_{k=1}^{\infty} \in W(Q)$ that satisfies equation (15) almost everywhere in domain Q.

Lemma 1. Let all the conditions of the theorem be satisfied. Then, to solve problem (15)–(17), the following estimates are valid:

I) $\left\langle u_k^{(l)} \right\rangle_{1,1}^2 \le const(\hat{k},\ \hat{l}) < +\infty$;

II) $\left\langle u_k^{(l)} \right\rangle_{2,1}^2 \le const(\hat{k},\ \hat{l}) < +\infty$.

Here and below, we will use the symbol $const(\hat{k},\ \hat{l})$ to denote the constant independent on parameters k, l.

Proof. Consider the following identity

$$2(Lu_k^{(l)}, e^{-\lambda t} u_{kt}^{(l)})_0 = 2(F(u_k^{(l-1)}), e^{-\lambda t} u_{kt}^{(l)})_0, \tag{19}$$

where constant $\lambda > 0$ will be chosen later.

Considering the conditions of the theorem, integrating identity (19) by parts and applying Cauchy's inequality with σ [8], it is easy to obtain the lower bound of the following inequality

$$\begin{aligned} 2\int_Q Lu_k^{(l)} \cdot e^{-\lambda t} \cdot u_{kt}^{(l)} dxdt \ge \int_Q e^{-\lambda t} \{2 \cdot u_{kt}^{2(l)} + \lambda \cdot u_{kx}^{2(l)} + (\lambda c - c_t + \lambda \mu_k^2) \cdot u_k^{2(l)}\} dxdt - \\ - \int_{\partial Q} e^{-\lambda t} \{2u_{kt}^{(l)} u_{kx}^{(l)} e_x - 2u_{kx}^{2(l)} e_t - (c+\mu_k^2) u_k^{2(l)} e_t\} ds, \end{aligned} \tag{20}$$

where $\overrightarrow{e} = ((e_x, e_t); (e_x = (\overrightarrow{e}, x); e_t = (\overrightarrow{e}, t))$ is the unit vector of the internal normal to boundary ∂Q. The conditions of Theorem 1 ensure that the integral over domain Q is not negative. Considering the semi-nonlocal boundary conditions (16), (17) and conditions of Theorem 1, with the choice of $\gamma^2 = e^{\lambda T}$, we obtain the conversion of the boundary integrals to zero. Thus, from inequality (20), we obtain the lower bound of the following inequality

$$\begin{gathered} 2\int\limits_Q Lu_k^{(l)} \cdot e^{-\lambda t} \cdot u_{kt}^{(l)} dxdt \geq \\ \geq \int\limits_Q e^{-\lambda t} \left\{ 2 \cdot u_{kt}^{2(l)} + \lambda \cdot u_{kx}^{2(l)} + \left(\delta_1 + \lambda \left(\frac{\pi}{\ell}\right)^2\right) \cdot u_k^{2(l)} \right\} dxdt \geq \delta_0 \left\| u_k^{(l)} \right\|^2_{W_2^{1,1}(Q)}, \end{gathered} \tag{21}$$

where $\delta_0 = \min\left\{2,\ \lambda, \delta_1 + \left(\frac{\pi}{\ell}\right)^2\right\}$, $\lambda c - c_t \geq \delta_1 > 0$.

Applying Cauchy's inequality with σ to identity (19), we obtain the upper bound

$$\begin{gathered} \left| 2(F(u_k^{(l-1)}), e^{-\lambda t} u_{kt}^{(l)})_0 \right| \leq \left| 2 \left(g_k + \frac{f_{1k}}{H} [f_{22}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty} \mu_m^2 u_m^{(l-1)} \sin \mu_m \ell_1) - \right.\right. \\ -f_{21}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty} \mu_m^2 u_m^{(l-1)} \sin \mu_m \ell_2)] + \frac{f_{2k}}{H} [f_{11}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty} \mu_m^2 u_m^{(l-1)} \sin \mu_m \ell_2) - \\ \left.\left. -f_{12}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty} \mu_m^2 u_m^{(l-1)} u_m \sin \mu_m \ell_1)], e^{-\lambda t} u_{kt}^{(l)} \right)_0 \right| \leq 9\sigma^{-1} \left\| u_k^{(l)} \right\|^2_{W_2^1(Q)} + \\ +\sigma \left[\|g_k\|_0^2 + \eta^{-2} \mathfrak{F}^2 \sum_{i,j=1}^{2} \left(T_0 \|\varphi_j\|^2_{W_2^{2,1}(Q)} + \|g_j\|_0^2 \right) \|f_{ik}\|^2_{C(Q)} \right] + \\ +2c_1\eta^{-2}\sigma\, \mathfrak{F}^2 \sum_{i=1}^{2} \|f_{ik}\|^2_{C(Q)} \sum_{m=1}^{\infty} \left(1 + \mu_m^2\right)^3 \left\| u_m^{(l-1)} \right\|^2_{W_2^{1,1}(Q)}, \end{gathered} \tag{22}$$

where $T_0 = \max\{1, \|c\|_{C(Q)}\}$, $\mathfrak{F}^2 = \max\{\|f_{11}\|^2_{C(Q)}, \|f_{12}\|^2_{C(Q)}, \|f_{21}\|^2_{C(Q)}, \|f_{22}\|^2_{C(Q)}\}$.

Combining inequalities (21) and (22), we obtain

$$\begin{gathered} (\delta_0 - 9\sigma^{-1}) \left\| u_k^{(l)} \right\|^2_{W_2^{1,1}(Q)} \leq \sigma \left[\|g_k\|_0^2 + \eta^{-2} \mathfrak{F}^2 \sum_{i,j=1}^{2} \left(T_0 \|\varphi_j\|^2_{W_2^{2,1}(Q)} + \|g_j\|_0^2 \right) \|f_{ik}\|^2_{C(Q)} \right] + \\ +2c_1\eta^{-2}\sigma\, \mathfrak{F}^2 \sum_{i=1}^{2} \|f_{ik}\|^2_{C(Q)} \sum_{m=1}^{\infty} \left(1 + \mu_m^2\right)^3 \left\| u_m^{(l-1)} \right\|^2_{W_2^{1,1}(Q)}. \end{gathered} \tag{23}$$

Applying the Sobolev embedding theorem $\|f_{ik}\|^2_{C(Q)} \leq c_2 \|f_{ik}\|^2_{W_2^2(Q)}$ [8,9] to inequality (23), we obtain

$$\begin{gathered} (\delta_0 - 9\sigma^{-1}) \left\| u_k^{(l)} \right\|^2_{W_2^{1,1}(Q)} \leq \sigma \left[\|g_k\|_0^2 + 2c_2\eta^{-2} \mathfrak{F}^2 \sum_{i,j=1}^{2} \left(T_0 \|\varphi_j\|^2_{W_2^{2,1}(Q)} + \|g_j\|_0^2 \right) \|f_{ik}\|^2_{W_2^2(Q)} \right] + \\ +2c_1c_2\eta^{-2}\sigma\, \mathfrak{F}^2 \sum_{i=1}^{2} \|f_{ik}\|^2_{W_2^2(Q)} \sum_{m=1}^{\infty} \left(1 + \mu_m^2\right)^3 \left\| u_m^{(l-1)} \right\|^2_{W_2^{1,1}(Q)}. \end{gathered} \tag{24}$$

Taking into account the condition of Theorem 1 $\delta_0 - 9\sigma^{-1} > \delta_0 - 10\sigma^{-1} \geq \delta > 0$, dividing inequalities (24) by δ, multiplying inequalities (24) by $\left(1 + \mu_m^2\right)^3$ and summing over k from 1 to ∞, we obtain the first recurrent formula

$$\begin{gathered} \left\langle u_k^{(l)} \right\rangle^2_{1,1} \leq \sigma\delta^{-1} \left[\langle g_k \rangle_0^2 + c_2\eta^{-2} \mathfrak{F}^2 \sum_{i,j=1}^{2} \left(T_0 \|\varphi_j\|^2_{W_2^{2,1}(Q)} + \|g_j\|_0^2 \right) \langle f_{ik} \rangle_2^2 \right] + \\ 2c_1c_2\eta^{-2}\sigma\delta^{-1} \mathfrak{F}^2 \sum_{i=1}^{2} \langle f_{ik} \rangle_2^2 \left\langle u_m^{(l-1)} \right\rangle^2_{1,1}, \end{gathered} \tag{25}$$

where $c_1 = \sum\limits_{k=1}^{\infty} \frac{\mu_k^4}{\left(1+\mu_k^2\right)^3}$, c_2 is the Sobolev embedding coefficient.

Introduce notation $\sigma\delta^{-1}\left[\langle g_k\rangle_0^2 + c_2\eta^{-2}\mathfrak{F}^2 \sum\limits_{i,j=1}^{2}\left(T_0\left\|\varphi_j\right\|^2_{W_2^{2,1}(Q)} + \|g_j\|_0^2\right)\langle f_{ik}\rangle_2^2\right] \equiv A$ and, considering the conditions of Theorem 1 $2c_1c_2\eta^{-2}\sigma\delta^{-1}\mathfrak{F}^2\sum\limits_{i=1}^{2}\langle f_{ik}\rangle_2^2 \le q = M\sum\limits_{i=1}^{2}\langle f_{ik}\rangle_2^2 < 1$, from recurrent formula (25), we obtain the validity of estimate I), i.e. we get the first estimate. Indeed, for this purpose we take function $\{u_k^{(-1)}\} \equiv \{0\}$ as an initial approximation.

Then, for the zero approximation, we obtain

$$\left\langle u_k^{(0)}\right\rangle_{1,1}^2 \le \sigma\delta^{-1}\left[\langle g_k\rangle_0^2 + 2c_2\eta^{-2}\mathfrak{F}^2\sum_{i,j=1}^{2}\left(T_0\left\|\varphi_j\right\|^2_{W_2^{2,1}(Q)} + \|g_j\|_0^2\right)\langle f_{ik}\rangle_2^2\right] \equiv A.$$

Continuing this process, by induction, we obtain the first a priori estimate for any function $u_k^{(l)}$, $\forall l \ge 1$

$$\left\langle u_k^{(l)}\right\rangle_{1,1}^2 \le A\cdot\sum_{n=0}^{l} q^n \le \frac{A}{1-q}.$$

Now, let us prove the validity of the second estimate II). To do this, consider the following identity

$$-2\int\limits_Q e^{-\lambda t}Lu_k^{(l)}\cdot\Delta_x u_k^{(l)}dxdt = -2\int\limits_Q e^{-\lambda t}F(u_k^{(l-1)})\cdot\Delta_x u_k^{(l)}dxdt. \tag{26}$$

Reasoning similarly to the proof of estimate I), based on integration by parts (26), considering the conditions of the theorem and semi-nonlocal boundary conditions (16), (17), we arrive at the following lower bound

$$\begin{gathered}\left|-2\int\limits_Q e^{-\lambda t}Lu_k^{(l)}\cdot\Delta_x u_k^{(l)}dxdt\right| \ge \int\limits_Q\left(2\Delta_x u_k^{2(l)} + (\lambda+\mu_k^2)u_{kx}^{2(l)}\right)dxdt - \sigma^{-1}\int\limits_Q\Delta_x u_k^{2(l)}dxdt - \\ -\sigma\|c\|^2_{C(Q)}\left\|u_k^{(l)}\right\|_0^2 + 2\int\limits_{\partial Q} e^{-\lambda t}[u_{kt}^{(l)}u_{kx}^{(l)}e_x + (u_{kx}^{2(l)} - u_{kt}^{2(l)})e_t + u_k^{(l)}u_{kx}^{(l)}e_x]ds \ge \\ \ge \int\limits_Q\left(2\Delta_x u_k^{2(l)} + \left(\lambda+\left(\tfrac{\pi}{\ell}\right)^2\right)u_{kx}^{2(l)}\right)dxdt - \sigma^{-1}\int\limits_Q\Delta_x u_k^{2(l)}dxdt - \sigma\|c\|^2_{C(Q)}\left\|u_k^{(l)}\right\|_0^2 \ge \\ \ge \delta_0\left\|u_k^{(l)}\right\|^2_{W_2^{2,1}(Q)} - \sigma^{-1}\left\|\Delta_x u_k^{(l)}\right\|_0^2 - \sigma\|c\|^2_{C(Q)}\left\|u_k^{(l)}\right\|_0^2,\end{gathered} \tag{27}$$

where $\delta_0 = \min\left\{2,\,\delta_1,\,\lambda+\left(\frac{\pi}{\ell}\right)^2\right\}$. The conditions of Theorem 1 ensure that the integral over domain Q is not negative. Considering the semi-nonlocal boundary conditions (16), (17) and the conditions of Theorem 1, with the choice of $\gamma^2 = e^{\lambda T}$, we obtain the conversion of the boundary integrals to zero. Thus, from inequalities (21) and (27), we obtain the lower bound of the following inequality

$$\left|-2\int\limits_Q e^{-\lambda t}Lu_k^{(l)}\cdot\Delta_x u_k^{(l)}dxdt\right| \ge \delta_0\left\|u_k^{(l)}\right\|^2_{W_2^{2,1}(Q)} - \sigma^{-1}\left\|\Delta_x u_k^{(l)}\right\|_0^2 - \sigma\|c\|^2_{C(\bar{Q})}\left\|u_k^{(l)}\right\|_0^2. \tag{28}$$

Now, applying the Cauchy inequality with σ to identity (27), we obtain the upper bound of the following inequality

$$\left|-2\int\limits_Q e^{-\lambda t}F(u_k^{(l-1)})\cdot\Delta_x u_k^{(l)}dxdt\right|\le 9\sigma^{-1}\left\|\Delta_x u_k^{(l)}\right\|_0^2+\sigma\left\|F(u_k^{(l-1)})\right\|_0^2\le$$
$$\le 9\sigma^{-1}\left\|\Delta_x u_k^{(l)}\right\|_0^2+\sigma\left\|g_k\right\|_0^2+\sigma\eta^{-2}\mathfrak{F}^2\sum_{i,j=1}^{2}\left(T_0\left\|\varphi_j\right\|^2_{W_2^{2,1}(Q)}+\left\|g_j\right\|_0^2\right)\left\|f_{ik}\right\|^2_{C(Q)}+ \tag{29}$$
$$+2\sigma\eta^{-2}c_1\mathfrak{F}^2\sum_{i=1}^{2}\left\|f_{ik}\right\|^2_{C(Q)}\sum_{m=1}^{\infty}\left(1+\mu_m^2\right)^3\left\|u_m^{(l-1)}\right\|^2_{W_2^{2,1}(Q)}.$$

Combining inequalities (28) and (29), we obtain

$$(\delta_0-10\sigma^{-1})\left\|u_k^{(l)}\right\|^2_{W_2^{2,1}(Q)}\le\sigma\left\|c\right\|^2_{C(\overline{Q})}\left\|u_k^{(l)}\right\|_0^2+\sigma\left\|g_k\right\|_0^2+$$
$$+\sigma\eta^{-2}\mathfrak{F}^2\sum_{i,j=1}^{2}\left(T_0\left\|\varphi_j\right\|^2_{W_2^{2,1}(Q)}+\left\|g_j\right\|_0^2\right)\left\|f_{ik}\right\|^2_{C(Q)}+ \tag{30}$$
$$+2\sigma\eta^{-2}c_1\mathfrak{F}^2\sum_{i=1}^{2}\left\|f_{ik}\right\|^2_{C(Q)}\sum_{m=1}^{\infty}\left(1+\mu_m^2\right)^3\left\|u_m^{(l-1)}\right\|^2_{W_2^{2,1}(Q)}.$$

Applying the Sobolev embedding theorem $\left\|f_{ik}\right\|^2_{C(Q)}\le c_2\left\|f_{ik}\right\|^2_{W_2^2(Q)}$ to inequality (30), we obtain

$$(\delta_0-10\sigma^{-1})\left\|u_k^{(l)}\right\|^2_{W_2^{2,1}(Q)}\le\sigma\delta^{-1}\left\|c\right\|^2_{C(\overline{Q})}\left\|u_k^{(l)}\right\|^2_{W_2^{2,1}(Q)}+$$
$$+\sigma\left\|g_k\right\|_0^2+\sigma\eta^{-2}c_2\mathfrak{F}^2\sum_{i,j=1}^{2}(T_0\left\|\varphi_j\right\|^2_{W_2^{2,1}(Q)}+\left\|g_j\right\|_0^2)\left\|f_{ik}\right\|^2_{W_2^2(Q)}+ \tag{31}$$
$$+2\sigma\eta^{-2}c_1c_2\mathfrak{F}^2\sum_{i=1}^{2}\left\|f_{ik}\right\|^2_{W_2^2(Q)}\sum_{m=1}^{\infty}\left(1+\mu_m^2\right)^3\left\|u_m^{(l-1)}\right\|^2_{W_2^{2,1}(Q)}.$$

Considering the conditions of the theorem and $\delta_0-10\sigma^{-1}\ge\delta>0$, dividing inequalities (31) by δ, multiplying by $\left(1+\mu_m^2\right)^3$ and summing over k from 1 to ∞, we obtain the second recurrent formula

$$\left\langle u_k^{(l)}\right\rangle^2_{2,1}\le 2\sigma\delta^{-1}\left\|c\right\|^2_{C(\overline{Q})}\left[\langle g_k\rangle_0^2+c_2\eta^{-2}\mathfrak{F}^2\sum_{i,j=1}^{2}\left(\left(T_0\left\|\varphi_j\right\|^2_{W_2^{2,1}(Q)}+\left\|g_j\right\|_0^2\right)\langle f_{ik}\rangle_2^2\right)\right]+$$
$$+\sigma\delta^{-1}\left[\langle g_k\rangle_0^2+\eta^{-2}c_2\mathfrak{F}^2\sum_{i,j=1}^{2}(T_0\left\|\varphi_j\right\|^2_{W_2^{2,1}(Q)}+\left\|g_j\right\|_0^2)\langle f_{ik}\rangle_2^2\right]+ \tag{32}$$
$$+2\sigma\delta^{-1}\eta^{-2}c_1c_2\mathfrak{F}^2\sum_{i=1}^{2}\langle f_{ik}\rangle_2^2\left\langle u_k^{(l-1)}\right\rangle^2_{2,1}.$$

From estimate (32), considering (24), we obtain the following recurrent formulas

$$\left\langle u_k^{(l)}\right\rangle^2_{2,1}\le 3\sigma\delta^{-1}\left\|c\right\|^2_{C(\overline{Q})}\left[\langle g_k\rangle_0^2+c_2\eta^{-2}\mathfrak{F}^2\sum_{i,j=1}^{2}\left(\left(T_0\left\|\varphi_j\right\|^2_{W_2^{2,1}(Q)}+\left\|g_j\right\|_0^2\right)\langle f_{ik}\rangle_2^2\right)\right]+ \tag{33}$$
$$+2\sigma\delta^{-1}\eta^{-2}c_1c_2\mathfrak{F}^2\sum_{i=1}^{2}\langle f_{ik}\rangle_2^2\left\langle u_k^{(l-1)}\right\rangle^2_{2,1}.$$

Introducing the following notation

$$3\sigma\delta^{-1}\left\|c\right\|^2_{C(\overline{Q})}\left[\langle g_k\rangle_0^2+c_2\eta^{-2}\mathfrak{F}^2\sum_{i,j=1}^{2}(T_0\left\|\varphi_j\right\|^2_{W_2^{2,1}(Q)}+\left\|g_j\right\|_0^2)\langle f_{ik}\rangle_2^2\right]\equiv A_1$$

and, considering the conditions of Theorem 1 and

$$2\sigma\delta^{-1}\eta^{-2}c_1c_2\mathfrak{F}^2\sum_{i,j=1}^{2}\langle f_{ik}\rangle_2^2 \le q \equiv M\sum_{i,j=1}^{2}\langle f_{ik}\rangle_2^2 < 1$$

from recurrent formula (33), we obtain the validity of estimate II), taking $\{u_k^{(-1)}\} \equiv \{0\}$ as an initial approximation. As a result, for the zero approximation, we obtain

$$\left\langle u_k^{(0)}\right\rangle_{2,1}^2 \le 3\sigma\delta^{-1}\|c\|_{C(\overline{Q})}^2\left[\langle g_k\rangle_0^2 + c_2\eta^{-2}\mathfrak{F}^2\sum_{i,j=1}^{2}\left(T_0\|\varphi_j\|_{W_2^{2,1}(Q)}^2 + \|g_j\|_0^2\right)\langle f_{ik}\rangle_2^2\right] \equiv A_1.$$

Continuing this process, by induction, we obtain the second a priori estimate for any function $u_k^{(l)}$, $\forall l \ge 1$

$$\left\langle u_k^{(l)}\right\rangle_{2,1}^2 \le A_1\cdot\sum_{n=0}^{l}q^n \le \frac{A_1}{1-q}.$$

Similar to the proof of estimate I), estimate II) is easily obtained. Lemma 1 is proven.

Let us now introduce a new function from $W(Q)$ according to formula $\vartheta_k^{(l)} = u_k^{(l)} - u_k^{(l-1)}$, $\forall l = N\cup\{0\},\ k = 1,2,...\ \{u_k^{(-1)}\} \equiv \{0\}$. Then the following Lemma holds for it.

Lemma 2. Let all the conditions of Theorem 1 and Lemma 1 be satisfied. Then the following a priori estimates are valid for functions $\{\vartheta_k^{(l)}\} \in W(Q)$:

III) $\left\langle \vartheta_k^{(l)}\right\rangle_{1,1}^2 \le A\cdot q^{(l)}$;

IV) $\left\langle \vartheta_k^{(l)}\right\rangle_{2,1}^2 \le A_1\cdot q^{(l)}$.

Here and below we will use symbol $const(\hat{k},\ \hat{l})$ to denote the constant independent on parameters k, l.

Proof. From (15)–(17) for function $\{\vartheta_k^{(l)}\} \in W(Q)$, we obtain the following problem

$$\begin{gathered} L\vartheta_k^{(l)} = \vartheta_{kt}^{(l)} - \Delta_x\vartheta_k^{(l)} + (c(x,t)+\mu_k^2)\vartheta_k^{(l)} = \\ = \tfrac{f_{1k}}{H}[f_{22}\sum_{m=1}^{\infty}\mu_m^2\vartheta_m^{(l-1)}\sin\mu_m\ell_1 - f_{21}\sum_{m=1}^{\infty}\mu_m^2\vartheta_m^{(l-1)}\sin\mu_m\ell_2]+ \\ +\tfrac{f_{2k}}{H}[f_{11}\sum_{m=1}^{\infty}\mu_m^2\vartheta_m^{(l-1)}\sin\mu_m\ell_2 - f_{12}\sum_{m=1}^{\infty}\mu_m^2\vartheta_m^{(l-1)}\sin\mu_m\ell_1] = T(\vartheta_k^{(l-1)}) \end{gathered} \tag{34}$$

with semi-nonlocal boundary conditions

$$\gamma\vartheta_k^{(l)}\,|_{t=0} = \vartheta_k^{(l)}\,|_{t=T}, \tag{35}$$

$$\vartheta_k\,|_{\partial\Omega} = 0, \tag{36}$$

where $l = 0,1,2,\ldots$

Therefore, as in the proof of Lemma 1, for the function $\{\vartheta_k^{(l)}\} = \{u_k^{(l)}\} - \{u_k^{(l-1)}\} \in W(Q)$ from (34)–(36), as a proof of Lemma 1, consider the following identity

$$2\Big(L\vartheta_k^{(l)}, e^{-\lambda t}\vartheta_{kt}^{(l)}\Big)_0 = 2\Big(T(\vartheta_k^{(l-1)}), e^{-\lambda t}\vartheta_{kt}^{(l)}\Big)_0. \tag{37}$$

Integrating by parts (37), taking into account the conditions of Theorem 1, we obtain the third recurrent formula

$$\left\langle \vartheta_k^{(l)}\right\rangle_{1,1}^2 \le q\left\langle \vartheta_k^{(l-1)}\right\rangle_{1,1}^2. \tag{38}$$

Repeating the reasoning, similar to the proof of Lemma 1, from (38), we obtain a priori estimate III) for the function $\{\vartheta_k^{(l)}\}$, $k = 1, 2, 3, \ldots$ Estimate IV) is proven similarly. Lemma 2 is proven.

Theorem 2. Let all the conditions of Theorem 1 be satisfied. Then problem (15)–(17) is uniquely solvable in $W(Q)$.

Proof. Let us define the following mapping in space $W(Q)$

$$u_k^{(l)} = L^{-1}F(u_k^{(l-1)}) = \mathcal{F}u_k^{(l-1)}.$$

1. Let us show that operator $\mathcal{F}$ maps space $W(Q)$ into itself. Let $\left\{ u_k^{(l-1)} \right\} \in W(Q)$, then to solve problem (15)–(17) the statement of Lemma 1 is true, i.e. estimate II) is valid for the function $\{u_k^{(l)}\}$, $k = 1, 2, 3, \ldots$ It follows that for any $l = 1, 2, 3 \ldots$ we obtain $\left\{ u_k^{(l)} \right\} \in W(Q)$. Thus, $\mathcal{F} : W(Q\,) \to W(Q)$.

2. Let us show that $\mathcal{F}$ is a contraction operator. Let $\left\{ u_k^{(l)} \right\}, \left\{ u_k^{(l-1)} \right\} \in W(Q)$. Consider new function $\left\{ \vartheta_k^{(l)} \right\} = \left\{ u_k^{(l)} \right\} - \left\{ u_k^{(l-1)} \right\}$, the statement of Lemma 2 is valid for it, i.e. estimate IV) is true for the function $\{\vartheta_k^{(l)}\}$, $k = 1, 2, 3, \ldots$, and

$$\left|\left|\left| \vartheta_k^{(l)} \right|\right|\right|_{2,1}^2 = \left\langle \vartheta_k^{(l)} \right\rangle_{2,1}^2 \le A_1 \cdot q^{(l)} \tag{39}$$

is true.

Now let us establish the fundamentality of sequence $\left\{ u_k^{(l)} \right\} \in W(Q)$. From (34)–(36), the triangle inequality and a priori estimates (39), we obtain

$$\left|\left|\left| u_k^{(l+p+1)} - u_k^{(l)} \right|\right|\right|_{2,1}^2 \le \left|\left|\left| u_k^{(l+p+1)} - u_k^{(l+p)} \right|\right|\right|_{2,1}^2 + \left|\left|\left| u_k^{(l+p)} - u_k^{(l+p-1)} \right|\right|\right|_{2,1}^2 + \ldots + \left|\left|\left| u_k^{(l+1)} - u_k^{(l)} \right|\right|\right|_{2,1}^2 \le$$
$$\le A_1(q^{(l+p+1)} + q^{(l+p)} + \ldots + q^{(l)}) = A_1 q^{(l)}(1 + q + \ldots + q^{(p+1)}) \le \tfrac{A_1 q^{(l)}}{1-q}.$$

This implies the fundamental nature of sequence $\left\{ u_k^{(l)} \right\}$. Thus, $\mathcal{F}$ is a contraction operator according to the well-known principle of contracting mappings [3], [9], problem (15)–(17) has a unique solution belonging to space $W(Q)$. Here $u_k^{(l)} \to u_k$ as $l \to \infty$, and $u_k(x,t)$ is a unique solution to problem (7)–(9) for fixed k.

From the principle of contraction mappings, we conclude that problem (7)–(9) has a unique solution from $W(Q)$. Theorem 2 is proven.

Now we prove Theorem 1. Applying the Parseval–Steklov equality to functions $\{u_k\} \in W(Q)$, we obtain the assertion of the theorem, that is, $u(x,t,y) \in U$ [8,9]. Theorem 1 is proven.

Remark 1. If we take function $\varphi_j(x,t)$ as a solution to the following problem $\varphi_j(x,t) \in W_2^{2,1}(Q)$, $g_j \in W_2^1(Q)$

$$L_0\varphi = \varphi_{jt} - \Delta_x\varphi_j + c(x,t)\varphi = g_j,$$
$$\gamma\varphi_j|_{t=0} = \varphi_j|_{t=T},$$
$$\varphi_j\,|_{\partial\Omega} = 0,$$

then function $\Phi_j(x,t)$ is defined as follows: $\Phi_j(x,t) = L_0\varphi_j - g_j + \sum\limits_{k=1}^{\infty} \mu_k^2 u_k \sin\mu_k\ell_j = \sum\limits_{k=1}^{\infty} \mu_k^2 u_k \sin\mu_k\ell_j$, $j = 1, 2$, and the proof of the theorem is greatly simplified.

Remark 2. For equation (1), LTPIPs with the Cauchy condition are studied similarly; in this case, instead of condition (2), the Cauchy condition $u\,|_{t=0} = u_0(x)$ is proposed.

Conclusion

In this article, the authors studied the correctness of one linear two-point inverse problem for the multidimensional heat conduction equation. Using the methods of a priori estimates, Galerkin's method, and successive approximations and contraction mappings, the theorem of unique solvability of the generalized solution in the specified class of integrable functions is proved.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Жартылай локольдыемес шектік шарттары бар көпөлшемді жылуөткізгіштік теңдеуіне қойылған сызықты екінүктелі кері есептер туралы

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В.А. Ильин және Е.И. Моисеевтер Штурм-Лиувилл теңдеулері үшін жалпылама локальдыемес шектік есептердің шешімінің бар болуын және жалғыздығын дәлелдеген. Дербес туындылы дифференциальдық теңдеулер үшін жалпылама локальдыемес шектік есептерді қарастырғанда априорлық бағаларды алуда көп қиындықтарға тап боламыз. Сондықтан, дербес туындылы дифференциальдық теңдеулерге қойылған локальдыемес шектік есептерді шешу үшін көп нүктелі кері есептерге келтіру қажет. Бұл бағыттағы алғашқы нәтижелер С.З. Джамаловқа тиесілі. Ол өз жұмысында математикалық физиканың көп нүктелі қисықтар сияқты көптеген параметрлерін де зерттеді. Мақалада көп өлшемді жылуөткізгіштік теңдеуіне қойылған сызықты екінүктелі кері есептің қисындылығы қарастырылған. Априорлық бағалау, Галеркин, біртіндеп жуықтау және қысушы бейнелеу әдістерін қолданып, көпөлшемді жылуөткізгіштік теңдеуіне қойылған сызықты екінүктелі кері есептің жалғыз шешімінің бар болуы дәлелденген.

Кілт сөздер: көп өлшемді жылуөткізгіштік теңдеуі, сызықты екінүктелі кері есеп, жалпылама шешімнің жалғыз болуы, априорлық бағалау, Галеркин әдісі, біртіндеп жуықтау және қысушы бейнелеу әдістері.

О некоторой линейной двухточечной обратной задаче для многомерного уравнения теплопроводности с полунелокальными краевыми условиями

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Известно, что В.А. Ильин и Е.И. Моисеев изучали обобщенные нелокальные краевые задачи для уравнения Штурма-Лиувилля, нелокальные краевые условия которого задаются во внутренних точках рассматриваемого интервала. Для таких задач доказаны теоремы единственности и существования решения задачи. Существует много проблем при исследовании этих обобщенных нелокальных краевых задач для дифференциальных уравнений с частными производными, особенно при получении априорных оценок. Поэтому необходимо использовать новые методы для решения обобщенных нелокальных задач (прямых задач). Как нам известно, нетрудно установить связь между прямыми и обратными задачами. Поэтому при решении обобщенных нелокальных краевых задач для дифференциальных уравнений в частных производных необходимо свести их к многоточечным обратным задачам. В этом направлении первые результаты принадлежат С.З. Джамалову. Он в своих работах предложил и исследовал многоточечные обратные задачи для некоторых уравнений математической физики. В настоящей работе исследована корректность одной линейной двухточечной обратной задачи для многомерного уравнения теплопроводности. Методами априорных оценок, Галеркина, последовательности приближений и сжимающихся отображений доказана однозначная разрешимость обобщённого решения одной линейной двухточечной обратной задачи для многомерного уравнения теплопроводности.

Ключевые слова: многомерное уравнение теплопроводности, линейная двухточечная обратная задача, однозначная разрешимость обобщённого решения, методы априорных оценок, Галеркина, последовательности приближений и сжимающихся отображений.

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