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Singularly perturbed integro-differential equations with degenerate Hammerstein's kernel

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Singularly perturbed integro-differential equations with degenerate kernels are considered. It is shown that in the linear case these problems are always uniquely solvable with continuous coefficients, while nonlinear problems either have no real solutions at all or have several of them. For linear problems, the results of Bobojanova are refined; in particular, necessary and sufficient conditions are given for the existence of a finite limit of their solutions as the small parameter tends to zero and sufficient conditions under which the passage to the limit to the solution of the degenerate equation is possible.

Keywords: singularly perturbed, Hammerstein's equation, degenerate kernel, Fredholm's equations, analytic function, Laurent's series, passage to the limit, the Maple program.

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Introduction

Many applied problems lead to nonlinear Hammerstein's equations of the form

$$\varepsilon\frac{dy}{dt}=\int_0^1 K\left(t,s\right)f\left(s,y\left(s,\varepsilon\right)\right)ds,\; y\left(0,\varepsilon\right)=y^0.$$

In the general case, it is impossible to obtain its solution in explicit form. However, if $K\left(t,s\right)$ is represented as a sum of products of functions with separated variables, then the study of this equation can be reduced to an algebraic system of equations. We will not consider the general case, but will show how this issue can be solved for a singularly perturbed equation of the form

$$\begin{gathered}\varepsilon\tfrac{dy(t)}{dt}=\int_0^1 a_1\left(t\right)b_1\left(s\right)f\left(y\left(s,\varepsilon\right),s\right)\,ds+\\ +\int_0^1 a_2\left(t\right)b_2\left(s\right)f\left(y\left(s,\varepsilon\right),s\right)ds,\quad y\left(0,\varepsilon\right)=y^0.\end{gathered}\tag{1}$$

Here $f\left(y,s\right)$ is a known continuous nonlinear function, $a_j\left(t\right),b_j\left(t\right)$ are known continuous functions on the segment $[0,1]$, $y=y\left(t,\varepsilon\right)$ is an unknown scalar function, $\varepsilon>0$ is a small parameter (the segment $[0,1]$ is taken to simplify the calculations; instead, you can take any segment $[0,T]$). Linear version of this problem:

$$\begin{gathered}\varepsilon\tfrac{dy}{dt}=\int_0^1 a_1\left(t\right)b_1\left(s\right)y\left(s,\varepsilon\right)\,ds+\int_0^1 a_2\left(t\right)b_2\left(s\right)y\left(s,\varepsilon\right)\,ds+\\ +h\left(t\right),\;y\left(0,\varepsilon\right)=y^0\end{gathered}\tag{2}$$

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was considered in [1]. Before examining the nonlinear equation (1), we present the results of this work. For a complete understanding, let us recall the scheme for solving equation (2) indicated in [1].

1 Linear singularly perturbed Fredholm's equations

Integrating (2) over t, assuming that it has a continuous solution, we obtain the equivalent problem

$$\varepsilon y(t,\varepsilon)=\int\limits_0^t a_1(\theta)\,d\theta\int\limits_0^1 b_1(s)\,y(s,\varepsilon)\,ds+$$

$$+\int\limits_0^t a_2(\theta)\,d\theta\int\limits_0^1 b_2(s)\,y(s,\varepsilon)\,ds+\int\limits_0^t h(\theta)\,d\theta+\varepsilon y^0.$$

Using the notation $\int\limits_0^t a_j(\theta)\,d\theta=q_j(t)\,,\ \int\limits_0^t h(\theta)\,d\theta+\varepsilon y^0=h_1(t,\varepsilon)\,,$ we reduce the last equation to the integral equation

$$\varepsilon y(t,\varepsilon)=q_1(t)\int\limits_0^1 b_1(s)\,y(s,\varepsilon)\ \,ds+q_2(t)\int\limits_0^1 b_2(s)\,y(s,\varepsilon)\ \,ds+h_1(t,\varepsilon)\tag{3}$$

with a degenerate kernel and solve it using a well-known scheme (see, for example, [2]). Enter constants

$$w_1=\int\limits_0^1 b_1(s)\,y(s,\varepsilon)\ \,ds,\ \ w_2=\int\limits_0^1 b_2(s)\,y(s,\varepsilon)\,ds.\tag{4}$$

Then the solution to equation (3) will be written in the form

$$y(t,\varepsilon)=\frac{1}{\varepsilon}\left(q_1(t)w_1+q_2(t)\,w_2+h_1(t,\varepsilon)\right).\tag{5}$$

Substituting this into (4), we obtain a system of algebraic equations

$$\left\{\begin{array}{l}\varepsilon w_1=\int\limits_0^1 b_1(s)\left(\left(q_1(s)\,w_1+q_2(s)\,w_2\right)\right)\ \,ds+\int\limits_0^1 b_1(s)\,h_1(s,\varepsilon)\,ds,\\[2ex] \varepsilon w_2=\int\limits_0^1 b_2(s)\left(\left(q_1(s)\,w_1+q_2(s)\,w_2\right)\right)\ \,ds+\int\limits_0^1 b_2(s)\,h_1(s,\varepsilon)\,ds\end{array}\right.\Leftrightarrow$$

$$\Leftrightarrow\left\{\begin{array}{l}\varepsilon w_1=c_{11}w_1+c_{12}w_2+H_1(\varepsilon)\,,\\[1ex] \varepsilon w_2=c_{21}w_1+c_{22}w_2+H_2(\varepsilon)\,,\end{array}\right.\tag{6}$$

relative to the unknown constants w_1 and w_2. Here it is indicated:

$$c_{ij}=\int\limits_0^1 b_i(s)\,q_j(s)\,ds,\ \ H_j(\varepsilon)=\int\limits_0^1 b_j(s)\,h_1(s,\varepsilon)\,ds,\ \ i,\,j=1,2.$$

Let $\sigma(C)=\{\lambda_1,\lambda_2\}$ be the spectrum of the matrix $C=(c_{ij})$ (λ_1, λ_2 may coincide). Let's reduce the matrix C to normal form in the space $\mathbb{C}^2$ (see, for example, [3]). The following cases of normal forms of a matrix are possible:

$$1)\ \ J_1=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}(\lambda_1\neq\lambda_2)\,,$$

$$2)\ J_2 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} (\lambda_1 = \lambda_2 = \lambda),$$

$$3)\ J_3 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} (\lambda_1 = \lambda_2 = \lambda),$$

two of which are diagonal, and one have a Jordan's structure. There exists (see, for example, [4]) a transformation matrix $T = T_j$ such that $T^{-1}CT = J_j$, $j = 1, 2, 3$. But then the same matrix T leads to the matrix

$$\varepsilon I - C \equiv \begin{pmatrix} \varepsilon - c_{11} & -c_{12} \\ -c_{21} & \varepsilon - c_{22} \end{pmatrix}$$

of the normal form, i.e. $T^{-1}(\varepsilon I - C)T$ will take one of the following forms:

$$1)\ J_1(\varepsilon) = \begin{pmatrix} \varepsilon - \lambda_1 & 0 \\ 0 & \varepsilon - \lambda_2 \end{pmatrix} (\lambda_1 \neq \lambda_2),$$

$$2)\ J_2(\varepsilon) = \begin{pmatrix} \varepsilon - \lambda & 0 \\ 0 & \varepsilon - \lambda \end{pmatrix} (\lambda_1 = \lambda_2 = \lambda),$$

$$3)\ J_3(\varepsilon) = \begin{pmatrix} \varepsilon - \lambda & 1 \\ 0 & \varepsilon - \lambda \end{pmatrix} (\lambda_1 = \lambda_2 = \lambda).$$

In this case, the solution of the system (6) will be written in one of the following forms:

$$w = w(\varepsilon) = \left(TJ_j^{-1}(\varepsilon)T^{-1}\right) H(\varepsilon),\ j = 1, 2, 3. \tag{7}$$

Let us first assume that $\det C \neq 0$. Then the eigenvalues $\lambda_j \neq 0$. We have in the case $j = 1$:

$$w = \left[T \begin{pmatrix} (\varepsilon - \lambda_1)^{-1} & 0 \\ 0 & (\varepsilon - \lambda_2)^{-1} \end{pmatrix} T^{-1}\right] H(\varepsilon). \tag{8}$$

Since $(\varepsilon - \lambda_j)^{-1} = -\frac{1}{\lambda_j}\frac{1}{1-\frac{\varepsilon}{\lambda_j}} = -\frac{1}{\lambda_j}\sum\limits_{k=0}^{\infty}\left(\frac{\varepsilon}{\lambda_j}\right)^k$ is the analytic function with respect to ε, and the inhomogeneity $H(\varepsilon) = \{h_1(\varepsilon), h_2(\varepsilon)\}$ linearly depends on ε, then $w(\varepsilon)$ is an analytic function with respect to ε, and the solution (5) of the problem (2) will have a first-order pole with respect to ε.

In the case $j = 2$ expression (7) for w has the form

$$w = \left(TJ_2^{-1}(\varepsilon)T^{-1}\right) H(\varepsilon) = T \begin{pmatrix} \frac{1}{\varepsilon-\lambda} & 0 \\ 0 & \frac{1}{\varepsilon-\lambda} \end{pmatrix} T^{-1}H(\varepsilon) =$$

$$= -\frac{1}{\lambda} T \begin{pmatrix} \frac{1}{1-\frac{\varepsilon}{\lambda}} & 0 \\ 0 & \frac{1}{1-\frac{\varepsilon}{\lambda}} \end{pmatrix} T^{-1}H(\varepsilon),$$

i.e. the vector $w = w(\varepsilon)$ is again an analytic function with respect to ε, and therefore the solution (5) of the problem (2) will have a pole of first order with respect to ε.

In the case $j = 3$ the vector w :

$$w = \left(TJ_3^{-1}(\varepsilon)T^{-1}\right) H(\varepsilon) = T \begin{pmatrix} \varepsilon - \lambda & 1 \\ 0 & \varepsilon - \lambda \end{pmatrix}^{-1} T^{-1}H(\varepsilon) =$$

$$= T \begin{bmatrix} \frac{1}{\varepsilon-\lambda} & -\frac{1}{(\varepsilon-\lambda)^2} \\ 0 & \frac{1}{\varepsilon-\lambda} \end{bmatrix} T^{-1}H(\varepsilon) = T \begin{bmatrix} -\frac{1}{\lambda}\frac{1}{1-\frac{\varepsilon}{\lambda}} & -\frac{1}{\lambda^2}\frac{1}{\left(1-\frac{\varepsilon}{\lambda}\right)^2} \\ 0 & -\frac{1}{\lambda}\frac{1}{1-\frac{\varepsilon}{\lambda}} \end{bmatrix} T^{-1}H(\varepsilon)$$

is again an analytic function with respect to ε, and therefore the solution (5) of the problem (2) will have a pole of first order with respect to ε.

Let $\det C = 0$. Three cases have to be considered here:

$$a)\ \lambda_1 = \lambda_2 = 0, \quad b)\ \lambda_1 = 0,\ \lambda_2 \neq 0, \quad c)\ \lambda_1 \neq 0,\ \lambda_2 = 0.$$

In the case a), expression (8) for w takes the form

$$w = \left(T \begin{pmatrix} \frac{1}{\varepsilon} & 0 \\ 0 & \frac{1}{\varepsilon} \end{pmatrix} T^{-1} \right) H(\varepsilon) = \frac{1}{\varepsilon} H(\varepsilon)$$

if $C = 0$, and the form

$$w = \left(T \begin{pmatrix} \frac{1}{\varepsilon} & -\frac{1}{\varepsilon^2} \\ 0 & \frac{1}{\varepsilon} \end{pmatrix} T^{-1} \right) H(\varepsilon) = \frac{1}{\varepsilon^2} T \begin{pmatrix} \varepsilon & -1 \\ 0 & \varepsilon \end{pmatrix} H(\varepsilon)$$

if the matrix C is similar to a Jordan's cell $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In this case the solution (5) of the problem (2) will have a second-order pole with respect to ε and a third-order pole with respect to ε, if the matrix C is similar to a jordan's cell $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

In the case b), the expression (7) takes the form

$$w = T \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & (\varepsilon - \lambda_2)^{-1} \end{pmatrix} T^{-1} H(\varepsilon) = \frac{1}{\varepsilon} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{\lambda_2} \frac{\varepsilon}{1 - \frac{\varepsilon}{\lambda_2}} \end{bmatrix} H(\varepsilon),$$

therefore the solution (5) of the problem (2) will have a second-order pole with respect to ε. In the case c), we also obtain that the solution (5) of the problem (2) has a second-order pole with respect to ε.

Let us write the results obtained in the form of a theorem.

Theorem 1. Let the functions $a_j(t), b_j(t), h(t)$ in the equation (2) be continuous on the segment $[0,1]$. Then the following statements are true.

1. If $\det C \neq 0$, then the solution $y(t,\varepsilon)$ of the problem (2) exists in the class $C^1[0,1]$, is unique in this class and is represented as a Laurent's series $y(t,\varepsilon) = \sum\limits_{k=-1}^{\infty} \varepsilon^k y_k(t)$.

2. If $\det C = 0$ and $\sigma(C) = \{\lambda_1, \lambda_2\}$, then the following statements hold:

a) when $\lambda_1 = \lambda_2 = 0$ the solution $y(t,\varepsilon)$ of the problem (2) exists in the class $C^1[0,1]$, is unique in this class and is represented as a Laurent's series $y(t,\varepsilon) = \sum\limits_{k=-2}^{\infty} \varepsilon^k y_k(t)$, if $C = 0$, and in the form of Laurent's series $y(t,\varepsilon) = \sum\limits_{k=-3}^{\infty} \varepsilon^k y_k(t)$, if C is similar to a Jordan cell $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

b) for $\lambda_1 = 0$, $\lambda_2 \neq 0$ or $\lambda_1 \neq 0$, $\lambda_2 = 0$ the solution $y(t,\varepsilon)$ of the problem (2) exists in the class $C^1[0,1]$, is unique in this class and is represented as a Laurent's series $y(t,\varepsilon) = \sum\limits_{k=-2}^{\infty} \varepsilon^k y_k(t)$.

From this theorem it follows that in the general case the solution $y(t,\varepsilon)$ tends to infinity as $t > 0$ and $\varepsilon \to +0$. Only in exceptional cases $y(t,\varepsilon)$ may tend to a finite limit. For example, if $\det C \neq 0$, then for the existence of a finite limit it is necessary to require that $y_{-1}(t) \equiv 0$. This condition must be expressed through the initial data of the problem (2). This was done in [1], but it is quite cumbersome and we do not present it. In the case of one term in (2), i.e. in the case $a_2(t) \equiv 0$ or $b_2(t) \equiv 0$ condition $y_{-1}(t) \equiv 0$ becomes more visible. Let's show it.

Noting $a_1(t) = a(t)$, $b_1(t) = b(t)$, we rewrite equation (2) in the form

$$\varepsilon \frac{dy}{dt} = \int_0^1 a(t)\, b(s)\, y(s,\varepsilon)\; ds + h(t),\; y(0,\varepsilon) = y^0. \tag{9}$$

Applying the procedure described above to (9), we obtain the following solution:

$$y(t,\varepsilon) = \frac{1}{\varepsilon}\left[\frac{\int_0^t a(x)dx \int_0^1 b(s)\left(\int_0^s h(\theta)d\theta + \varepsilon y^0\right)ds}{\varepsilon - \int_0^1 b(s)\left(\int_0^s a(x)dx\right)ds} + \int_0^t h(\theta)\, d\theta + y^0\varepsilon\right]. \tag{10}$$

Summing up the expression in square brackets, we write the solution in the form

$$y(t,\varepsilon) = \frac{\varepsilon^{-1}}{\varepsilon - \int_0^1 b(s)\left(\int_0^s a(x)dx\right)ds}\left[-\varepsilon y^0 \int_0^1 b(s)\left(\int_0^s a(x)\, dx\right) ds + \right.$$

$$+\varepsilon^2 y^0 + \int_0^t a(s)\, ds \int_0^1 b(s)\left(\int_0^s h(\theta)\, d\theta + \varepsilon y^0\right) ds -$$

$$\left. - \int_0^t h(s)\, ds \int_0^1 b(s)\left(\int_0^s a(x)\, dx\right) ds + \varepsilon \int_0^t h(s)\, ds\right].$$

The free term on ε in the square bracket does not allow one to go to the final limit as $\varepsilon \to +0$, therefore it must be removed. Let's calculate it:

$$\left(\int_0^t a(s)\, ds\right)\left(\int_0^1 b(s)\left(\int_0^s h(\theta)\, d\theta\right) ds\right) - \left(\int_0^t h(s)\, ds\right)\left(\int_0^1 b(s)\left(\int_0^s a(x)\, dx\right) ds\right).$$

This means that if for any $t \in [0,1]$ the condition

$$\left(\int_0^t a(s)\, ds\right)\left(\int_0^1 b(s)\left(\int_0^s h(\theta)\, d\theta\right) ds\right) \equiv$$

$$\equiv \left(\int_0^t h(s)\, ds\right)\left(\int_0^1 b(s)\left(\int_0^s a(x)\, dx\right) ds\right), \tag{$*$}$$

is satisfied, then there is a finite limit $y(t,\varepsilon) \to \bar{y}(t)$ as $\varepsilon \to +0$. This condition is necessary and sufficient for the existence of a finite limit $\lim\limits_{\varepsilon \to +0} y(t,\varepsilon) = \bar{y}(t)$.

Note that the condition $(*)$ is automatically satisfied if $a(t) \equiv h(t)$. It is curious that in this case the limit $\bar{y}(t)$ will coincide with the solution of the equation $\int_0^1 b(s)\, \bar{y}(s)\; ds + 1 = 0$ degenerate with respect to (9). Let us prove this.

Let $h(t) \equiv a(t)$. Then the condition $(*)$ is satisfied and the solution of the problem (2) will be written in the form

$$y(t,\varepsilon) = -\frac{-y^0\left(\int_0^1 b(s)\left(\int_0^s a(x)dx\right)ds\right) + \varepsilon y^0 + \left(\int_0^t a(s)ds\right)\left(\int_0^1 b(s) y^0 ds\right) + \left(\int_0^t h(s)ds\right)}{\left(-\varepsilon + \int_0^1 b(s)\left(\int_0^s a(x)\; dx\right)ds\right)} =$$

$$= \frac{y^0\left(\int_0^1 b(s)\left(\int_0^s a(x)dx\right)\; ds\right) - \varepsilon y^0 - \left(\int_0^t a(s)ds\right)\left(\int_0^1 b(s) y^0\; ds\right) - \left(\int_0^t a(s)ds\right)}{\left(-\varepsilon + \int_0^1 b(s)\left(\int_0^s a(x)dx\right)ds\right)}.$$

Passing here to the limit when $\varepsilon \to +0$, we obtain

$$\bar{y}\,(t) = \frac{1}{\int\limits_0^1 b(s)\left(\int\limits_0^s a(x)\;dx\right)\;ds}\left[y^0\;\left(\int\limits_0^1 b\,(s)\left(\int\limits_0^s a\,(x)\;\;dx\right)\;ds\right)-\right.$$
$$\left.-\left(\int\limits_0^t a\,(s)\;\;ds\right)\left(\int\limits_0^1 b\,(s)\,y^0\;ds\right)-\left(\int\limits_0^t a\,(s)\;\;ds\right)\right]. \tag{11}$$

Let us show that $\bar{y}\,(t)$ is the solution to the degenerate equation

$$\int\limits_0^1 b\,(s)\cdot\bar{y}\,(s)\,ds + 1 = 0. \tag{12}$$

Substituting (11) into the left side of the equation (12), we will have

$$\int\limits_0^1 \frac{1}{\int\limits_0^1 b(s)\left(\int\limits_0^s a(x)dx\right)\;ds}\left[b(s)\left(-\left(\int\limits_0^s a\,(s)\,ds\right)y^0\left(\int\limits_0^1 b\,(s)\,ds\right)+\right.\right.$$
$$\left.\left.+y^0\left(\int\limits_0^1 b\,(s)\left(\int\limits_0^s a\,(x)\,dx\right)ds\right)-\left(\int\limits_0^s a\,(s)\,ds\right)\right)\right]+1.$$

We must show that

$$\int\limits_0^1 b\,(s)\left(-\left(\int\limits_0^s a\,(x)\;\;dx\right)y^0\left(\int\limits_0^1 b\,(s)\,ds\right)+\right.$$
$$\left.+\,y^0\left(\int\limits_0^1 b\,(s)\left(\int\limits_0^s a\,(x)\;\;dx\right)ds\right)-\overline{\overline{\int\limits_0^s a\,(x)\,dx}}\right)ds-$$
$$-\int\limits_0^1 b\,(s)\,\overline{\overline{\left(\int\limits_0^s a\,(x)\;\;dx\right)}}ds\;\;\equiv 0.$$

Removing the terms underlined above and then canceling both sides by y^0, we arrive at the identity

$$\int\limits_0^1 b\,(s)\left(-\left(\int\limits_0^s a\,(x)\,dx\right)\left(\int\limits_0^1 b\,(s)\,ds\right)+\left(\int\limits_0^1 b\,(s)\left(\int\limits_0^s a\,(x)\,dx\right)\;ds\right)\right)\;ds\equiv 0.$$

The proof of this identity for arbitrary functions $a\,(t)$ and $b\,(t)$, continuous on an interval $[0,1]$, is problematic. However, in the case of polynomials $a\,(t)$ and $b\,(t)$, it can be proved by induction on the powers of the polynomials.

The following results were obtained.

Theorem 2. Let the functions $a\,(t)\,,b\,(t)\,,h\,(t)$ in equation (9) be continuous on the segment $[0,1]$. Then:

1) equation (9) has a unique solution $y\,(t,\varepsilon)\in C^1\,[0,1]$ in the form (10), which for arbitrary $a\,(t)$ and $h\,(t)\in C\,[0,1]$ has a first-order pole with respect to ε if $C=\int\limits_0^1 b\,(s)\cdot\int\limits_0^s a\,(\theta)\,d\theta\neq 0$, and a second-order pole with respect to ε if $C=0$;

2) in order for $y(t,\varepsilon)$ to be analytical in ε (for sufficiently small $\varepsilon > 0$), it is necessary and sufficient that the identity $(*)$ holds;

3) for $a(t) \equiv h(t)$ *, the exact solution $y(t,\varepsilon)$ of the equation (2) uniformly (for $t \in [0,1]$) tends to the solution (11) of the degenerate equation (12) when $\varepsilon \to +0$.

Remark 1. In work [1] statement 3) of this theorem was not given. Here it is proved for the first time.

Remark 2. It follows from Theorems 1 and 2 *that there is no boundary layer* in the solutions of problem (2).

Let's look at examples.

Example 1. Consider the problem

$$\varepsilon\frac{dy}{d\,t} = 5\,t^2\int\limits_0^1 \left(4s^2-5s\right) y(s,\varepsilon)\,ds + 2t-1,\; y(0,\varepsilon) = y^0. \tag{13}$$

Substituting $a(t) = 5t^2, b(t) = 4t^2 - 5t,\ h(t) = 2t-1$ into formula (10), we find a solution to this problem in the form

$$y(t,\varepsilon) = y^0 - \frac{t}{\varepsilon} + \frac{t^2}{\varepsilon} - \frac{1}{4}\frac{t^3\left(70y^0\varepsilon - 13\right)}{\varepsilon(9\varepsilon+5)}.$$

The condition $(*)$ that has the form $\frac{13}{36}t^3 \equiv -\frac{5}{9}t^2 + \frac{5}{9}t$, is not satisfied, and therefore the solution to problem (13) has a first-order pole in ε.

Example 2. Now consider the problem

$$\varepsilon\frac{dy}{d\,t} = 3(t-1)^2\int\limits_0^1 \left(2s - \frac{6}{5}\right) y(s,\varepsilon)\;ds + 5\,t+1,\; y(0,\varepsilon) = y^0.$$

Here: $a(t) = 3(t-1)^2$, $b(t) = 2t - \frac{6}{5}$, $h(t) = 5\,t+1$, $C = \int\limits_0^1 b(s)\cdot\left(\int\limits_0^s a(x)\,dx\right)ds = 0$ and the condition $(*)$ is not met. Calculating the solution using formula (10), we obtain the following solution:

$$y(t,\varepsilon) = \frac{y^0}{60}\cdot\frac{-12t^3+36t^2+60\varepsilon-36t}{\varepsilon} + \frac{150\varepsilon t^2+19t^3+60\varepsilon t-57t^2+57t}{60\varepsilon^2}.$$

It can be seen that the solution has a pole of second order in ε.

Example 3. Consider another problem

$$\varepsilon\frac{dy}{dt} = \left(2-5\,t^2\right)\int\limits_0^1 s^3 y(s,\varepsilon)\,ds + \left(2-5t^2\right), y(0) = y^0. \tag{14}$$

Here $a(t) \equiv h(t) = \left(2-5t^2\right)$, it means that the condition $(*)$ is fulfilled in an obvious way and therefore there is a finite limit $\lim\limits_{\varepsilon\to+0} y(t,\varepsilon) = \bar{y}(t)$. Let's make sure of this. Solving problem (14) using the above method, we obtain the following solution:

$$y(t,\varepsilon) = \frac{1}{4}\frac{-175y^0t^3 - 700t^3 + 420\varepsilon y^0 + 210y^0t - 68y^0 + 840t}{-17+105\,\varepsilon}.$$

* In this case the identity $(*)$ is obvious.

We see that the solution is analytic with respect to ε for sufficiently small values $\varepsilon > 0$, and there is a uniform passage to the limit

$$y\left(t,\varepsilon\right) \to \bar{y}\left(t\right) = \frac{175}{68}y^0t^3 + \frac{175}{17}t^3 - \frac{105}{34}ty^0 + y^0 - \frac{210}{17}t \;\; \left(\varepsilon \to +0\right).$$

Substituting $\bar{y}\left(t\right)$ into the right side of the degenerate equation $0 = \int\limits_0^1 s^3 \times \times \bar{y}\left(s\right)ds + 1$, we have

$$\int\limits_0^1 s^3\left(\frac{175}{68}y^0s^3 + \frac{175}{17}s^3 - \frac{105}{34}sy^0 + y^0 - \frac{210}{17}s\right)ds + 1 \equiv 0.$$

Thus, the function $\bar{y}\left(t\right)$ is the solution of a degenerate equation, which is consistent with statement 3) of Theorem 2.

2 Nonlinear singularly perturbed Hammerstein equations

Let's move on to studying the nonlinear equation (1). In the works known to us [5–7] more general linear, nonlinear differential and integro-differential equations are considered and systems are than in our work. However, they are devoted to the construction of asymptotic solutions and the study phenomena of initial and boundary jumps. Assuming that there is a continuous solution of this equation, integrating it by t over the segment $[0, t]$, we obtain the integral equation

$$\begin{gathered}\varepsilon y\left(t,\varepsilon\right) = q_1\left(t\right)\int\limits_0^1 b_1\left(s\right)f\left(y\left(s,\varepsilon\right),s\right)ds + \\ +q_2\left(t\right)\int\limits_0^1 b_2\left(s\right)f\left(y\left(s,\varepsilon\right),s\right)ds + \varepsilon y^0,\end{gathered} \tag{3*}$$

where the notations $q_j\left(t\right) = \int\limits_0^t a_j\left(\theta\right)d\theta$, $j = 1, 2$, are introduced. Let us introduce constants

$$w_1 = \int\limits_0^1 b_1\left(s\right)f\left(y\left(s,\varepsilon\right),s\right)\,ds,\;\; w_2 = \int\limits_0^1 b_2\left(s\right)f\left(y\left(s,\varepsilon\right),s\right)\,ds. \tag{15}$$

Then the solution of the equation (3*) will be written in the form

$$y\left(t,\varepsilon\right) = \frac{1}{\varepsilon}\left(q_1(t)w_1 + q_2\left(t\right)w_2 + \varepsilon y^0\right). \tag{16}$$

Substituting (16) into (15), we obtain an algebraic system of equations

$$\begin{gathered}w_1 = \int\limits_0^1 b_1\left(s\right)f\left(\tfrac{1}{\varepsilon}\left(q_1\left(s\right)w_1 + q_2\left(s\right)w_2 + \varepsilon y^0\right),s\right)ds, \\ w_2 = \int\limits_0^1 b_2\left(s\right)f\left(\tfrac{1}{\varepsilon}\left(q_1\left(s\right)w_1 + q_2\left(s\right)w_2 + \varepsilon y^0\right),s\right)ds.\end{gathered} \tag{17}$$

If the function $f\left(y,s\right)$ is known, then (17) is a nonlinear algebraic system of equations, the solvability of which relative tow_1 and w_2 is not guaranteed by anything. Therefore, it is unlikely that in the general case it will be possible to formulate the conditions for the solvability of the system (17) in terms of the initial data. In a specific case, when all the functions included in equation (1) are given, nothing can be also said about solvability. In this case, difficulties arise in calculating the integrals included in (17). Let's try to solve system (17) using the Maple program. We present the corresponding algorithm.

Restart:

Set the initial data

$$f := f(z,t)\,;\ q_1 := q_1(t)\,;\ q_2 := q_2(t)\,;\ b_1 := b_1(t)\,;\ b_2 := b_2(t)\,.$$

We write system (17) for given data

$$w_1 = \int\limits_0^1 b_1(s)\, f\left(\tfrac{1}{\varepsilon}\left(q_1(s)\,w_1 + q_2(s)\,w_2\ + \varepsilon y^0\right), s\right) ds,$$

$$w_2 = \int\limits_0^1 b_2(s)\, f\left(\tfrac{1}{\varepsilon}\left(q_1(s)\,w_1 + q_2(s)\,w_2\ + \varepsilon y^0\right), s\right) ds.$$

A system of algebraic equations is obtained. We solve it using the *solve* operator. If we manage to find the constants $w_1 = w_1^0,\ w_2 = w_2^0$, then the solution of the equation (1) is obtained as follows:

$$y(t,\varepsilon) = \frac{1}{\varepsilon}\left(q_1(t)\,w_1 + q_2(t)\,w_2\ + \varepsilon y^0\right);$$

$$subs\left(\left\{c_1 = c_1^0, c_2 = c_2^0\right\}, y(t,\varepsilon)\right).$$

Let us demonstrate the implementation of this procedure using specific examples.

Example 4. Solve the Cauchy's problem

$$\varepsilon\frac{d}{d\,t}\,y(t,\varepsilon) = 3t^2\int\limits_0^1 sy^2(s,\varepsilon)\,ds,\ \ y(0,\varepsilon) = y^0. \tag{18}$$

Here: $q_1(t) = \frac{t^3}{3}$, $q_2(t) = 0$, $b_1(t) = t$, $b_2(t) = 0$. Applying the algorithm described above, we obtain the following solution to problem (18):

$$y(t,\varepsilon) = t^3\left(4\varepsilon - \frac{8}{5}y^0 \pm \frac{2}{5}\sqrt{100\varepsilon^2 - 80\varepsilon y^0 - 9\left(y^0\right)^2}\right) + y^0.$$

From this it is clear that for sufficiently small $\varepsilon > 0$ and $y^0 \neq 0$ equation (18) has no real solutions and only for $y^0 = 0$ it has two real solutions $y(t,\varepsilon) = t^3(4\varepsilon \pm 4\varepsilon)$, uniformly tending to zero as $\varepsilon \to +0$.

Example 5. Now consider the problem

$$\varepsilon\frac{d}{d\,t}\,y(t,\varepsilon) = 2t\int\limits_0^1 sy^3(s)\ ds,\ y(0,\varepsilon) = m. \tag{19}$$

Here, instead of quadratic nonlinearity, we took cubic nonlinearity $f(y) = y^3$. Using the Maple program algorithm described above, we find that problem (19) has only one real solution $y(t) = \frac{t^2}{\varepsilon}w + m$, where the constant w has the form

$$w = \left[\frac{1}{3}\left(-10m^3 - 144\varepsilon m + 6\sqrt{3m^6 + 72\varepsilon m^4 + 672\varepsilon^2 m^2 - 384\varepsilon^3}\right)^{1/3} -\right.$$

$$-\frac{3\left(\frac{2}{9}m^2-\frac{8}{3}\varepsilon\right)}{\left(-10m^3-144\varepsilon m+6\sqrt{3m^6+72\varepsilon m^4+672\varepsilon^2m^2-384\varepsilon^3}\right)^{1/3}}-\frac{4}{3}m\Bigg]\cdot\varepsilon.$$

When $\varepsilon \to +0$ the solution $y\left(t,\varepsilon\right)$ has a finite limit

$$\bar{y}\left(t\right)=t^2\left(\frac{1}{3\left(6\sqrt{3}\left|m\right|^3-10m^3\right)^{1/3}}-\frac{2m^2}{3\left(6\sqrt{3}\left|m\right|^3-10m^3\right)^{1/3}}-\frac{4}{3}m\right)+m.$$

For different signs of the initial condition m, the solution tends to different limits.

Remark 3. The results of studies for linear singularly perturbed problems are presented in the works [8–24].

Conclusion

The properties of nonlinear singularly perturbed problems of type (1) differ significantly from the properties of linear problems of type (2); linear problems are always uniquely solvable in the class $C^1\left[0,1\right]$ with continuous initial data, and nonlinear problems may not have real solutions at all or have several of them.

Author Contributions

M.A. Bobodzhanova collected and analyzed data, implemented the program on Maple, B.T. Kalimbetov assisted in collecting and analyzing data, supervised the preparation of the manuscript. V.F. Safonov was the main executor of the research grant and supervised the research process. All authors participated in the revision of the manuscript and approved the final submission.

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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