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On g_{MT}- and g_β-convexity and the Ostrowski type inequalities

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In this study, motivated by recent results on Ostrowski-type inequalities, we introduce a new identity that serves as a basis for establishing fractional Ostrowski inequalities. Specifically, we focus on functions whose modulus of the first derivatives are g_{MT}-convex and g_β-convex. This approach allows us to extend classical results to more general settings. Several special cases are discussed, recovering known inequalities while highlighting the versatility of our method.

Keywords: g_{MT}- convexity, g_β-convexity, power mean inequality, generalized Riemann-Liouville fractional integrals.

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Introduction

The Ostrowski inequality (see [1]), can be stated as follows:

Theorem 1. Let $\varepsilon : \mathfrak{I} \to \mathbb{R}$, be a differentiable mapping with bounded first derivatives, then

$$\left| \varepsilon(x) - \frac{1}{k-r} \int_r^k \varepsilon(\mathfrak{u}) d\mathfrak{u} \right| \le M(k-r) \left[\frac{1}{4} + \frac{(x-\frac{r+k}{2})^2}{(k-r)^2} \right] \tag{1}$$

holds, where $r, k \in \mathfrak{I}$ with $r < k$.

In recent years, such inequalities were studied extensively by many researchers. Regarding some papers with closed relationship with inequality (1) we refer readers to [2–16], and references cited therein.

In [17], Liu used the so-called MT-convex function defined by Tunç [18,19] and derived the following fractional Ostrowski type inequalities:

Definition 1. [18,19] Let the function $\varepsilon : \mathfrak{I} \subseteq \mathbb{R}_+ \to \mathbb{R}$, if for all η, $\mu \in \mathfrak{I}$ and $t \in [0,1]$

$$\varepsilon(t\eta + (1-t)\mu) \le \frac{\sqrt{t}}{2\sqrt{1-t}} \varepsilon(\eta) + \frac{\sqrt{1-t}}{2\sqrt{t}} \varepsilon(\mu) \tag{2}$$

holds, then ε is said an MT-convex. If (2) holds in the opposite sense, then ε is said MT-concave.

Theorem 2. [17] Let the differentiable function $\varepsilon : [r,k] \to \mathbb{R}$ with $\varepsilon' \in L^1[r,k]$. If $|\varepsilon'|$ is MT-convex on $[r,k]$, where $\alpha > 0$, $0 \le r < k$ and for $x \in [r,k] : |\varepsilon'(x)| \le \mathfrak{M}$, then we have

$$\begin{aligned} &\left| \frac{(x-r)^\alpha + (k-x)^\alpha}{k-r} \varepsilon(x) - \frac{\Gamma(\alpha+1)}{k-r} \left[J^\alpha_{x^-} \varepsilon(r) + J^\alpha_{x^+} \varepsilon(k) \right] \right| \\ \le& \mathfrak{M} \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+1)} \frac{(x-r)^{\alpha+1} + (k-x)^{\alpha+1}}{k-r}. \end{aligned}$$

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Theorem 3. [17] Let the differentiable function $\varepsilon : [r,k] \to \mathbb{R}$ with $\varepsilon' \in L^1[r,k]$. If $|\varepsilon'|^q$ is MT-convex on $[r,k]$, where $\alpha > 0$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $0 \le r < k$ and for $x \in [r,k] : |\varepsilon'(x)| \le \mathfrak{M}$, then we have

$$\left|\tfrac{(x-r)^\alpha + (k-x)^\alpha}{k-r}\varepsilon(x) - \tfrac{\Gamma(\alpha+1)}{k-r}\left[J^\alpha_{x^-}\varepsilon(r) + J^\alpha_{x^+}\varepsilon(k)\right]\right|$$
$$\le \tfrac{\mathfrak{M}}{(1+p\alpha)^{\frac{1}{p}}}\left(\tfrac{\pi}{2}\right)^{\frac{1}{q}}\tfrac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{k-r}.$$

Theorem 4. [17] Let the differentiable function $\varepsilon : [r,k] \to \mathbb{R}$ with $\varepsilon' \in L^1[r,k]$. If $|\varepsilon'|^q$ is MT-convex on $[r,k]$, where $\alpha > 0$, $q > 1$, $0 \le r < k$ and for $x \in [r,k] : |\varepsilon'(x)| \le \mathfrak{M}$, then we have

$$\left|\tfrac{(x-r)^\alpha + (k-x)^\alpha}{k-r}\varepsilon(x) - \tfrac{\Gamma(\alpha+1)}{k-r}\left[J^\alpha_{x^-}\varepsilon(r) + J^\alpha_{x^+}\varepsilon(k)\right]\right|$$
$$\le \tfrac{\mathfrak{M}}{(1+\alpha)^{1-\frac{1}{q}}}\left(\tfrac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+1)}\right)^{\frac{1}{q}}\tfrac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{k-r}.$$

The concept of convex functions and sets are extended to the new class of g-convex function and g-convex sets (see [20]). This class is more general and plays an important role in nonlinear programming problems and optimization theory in which the constraint set the objective function are g-convex.

The objective of this research is to use extended Riemann-Liouville fractional integrals to construct new Ostrowski inequalities for functions whose absolute value of first derivatives is g_{MT}- and g_β-convex. These results generalize those of [8] and give fresh estimates of this kind of disparity.

1 Preliminaries

The following section is devoted to some definitions and remarks.

Definition 2. [21] Let $\varepsilon : \mathfrak{I} \to \mathbb{R}$, if for all η, $\mu \in \mathfrak{I}$ and $\varpi \in [0,1]$

$$\varepsilon(\varpi\eta + (1-\varpi)\mu) \le \varpi\varepsilon(\eta) + (1-\varpi)\varepsilon(\mu)$$

holds, then ε is called a convex function.

Definition 3. [22] Let $\varepsilon : \mathfrak{I} \to \mathbb{R}$, if for all η, $\mu \in \mathfrak{I}$ and $\varpi \in [0,1]$

$$\varepsilon(\varpi\eta + (1-\varpi)\mu) \le \varepsilon(\eta) + \varepsilon(\mu)$$

holds, then ε is an P-convex function.

Definition 4. [23] Let $\varepsilon : \mathfrak{I} \to \mathbb{R}$ be a nonnegative function, if for all η, $\mu \in \mathfrak{I}$, some fixed $s \in (0,1]$ and $\varpi \in [0,1]$

$$\varepsilon(\varpi\eta + (1-\varpi)\mu) \le \varpi^s\varepsilon(\eta) + (1-\varpi)^s\varepsilon(\mu)$$

holds, then ε is an s-convex function.

Definition 5. [24] Let $\varepsilon : \mathfrak{I} \to \mathbb{R}$ be a nonnegative function, if for all η, $\mu \in \mathfrak{I}$ and $\varpi \in (0,1)$

$$\varepsilon(\varpi\eta + (1-\varpi)\mu) \le \varpi(1-\varpi)[\varepsilon(\eta) + \varepsilon(\mu)]$$

holds, then ε is a tgs-convex function.

Definition 6. [25] Let $\varepsilon : \mathfrak{I} \to \mathbb{R}$, if for all η, $\mu \in \mathfrak{I}$, p, $q > -1$ and $\varpi \in (0,1)$

$$\varepsilon(\varpi\eta + (1-\varpi)\mu) \le \varpi^p(1-\varpi)^q\varepsilon(\eta) + \varpi^q(1-\varpi)^p\varepsilon(\mu)$$

holds, then ε is an β-convex function.

Remark 1. For $(p, q) \in \{(0,0),(s,0),(1,1),(1,0)\}$, Definition 6, recapture the P-convexity, s-convexity, tgs-convexity and classical convexity, respectively.

Definition 7. [20] We say that a set $K_g \subseteq \mathbb{R}^n$ is g-convex, if there exists a function $g : \mathbb{R}^n \to \mathbb{R}^n$ and

$$\varpi g(\eta) + (1-\varpi) g(\mu) \in K_g$$

holds $\forall\ \eta,\ \mu \in \mathbb{R}^n : g(\eta),\ g(\mu) \in K_g$ and $\varpi \in [0,1]$.

Definition 8. [20] Let $\varepsilon : \mathbb{R}^n \to \mathbb{R}$, if there exists a function $g : \mathbb{R}^n \to \mathbb{R}^n$ and for all $\eta,\ \mu \in \mathbb{R}^n : g(\eta)$, $g(\mu) \in K_g$ and $\varpi \in [0,1]$

$$\varepsilon(\varpi g(\eta) + (1-\varpi) g(y)) \le \varpi \varepsilon(g(\eta)) + (1-\varpi)\varepsilon(g(\mu))$$

holds, then ε is an g-convex on K_g.

Remark 2. Every convex function ε on a convex set K_g is a g-convex function, where g is the identity map. However, the converse is not true.

Example 1. Let $K_g \subset \mathbb{R}^2$ be given as

$$K_g = \left\{(\eta,\mu) \in \mathbb{R}^2 : (\eta,\mu) = \alpha_1(0,\ 0) + \alpha_2(0,\ 3) + \alpha_3(2,\ 1)\right\}$$

with $\alpha_i > 0$, $\sum\limits_{i=1}^{3} \alpha_i = 1$, and define a mapping $g : \mathbb{R}^2 \to \mathbb{R}^2$ as $g(\eta,\mu) = (0,\ \mu)$.

The function $\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\varepsilon(\eta,\ \mu) = \begin{cases} \eta^3, & \text{if } \ \mu < 1, \\ \eta\mu^3, & \text{if } \ \mu \ge 1 \end{cases}$$

is g-convex on K_g but is not convex.

In [26], Sarikaya defined the so-called g_h-convex functions which are a generalization of the aforementioned convex functions.

Definition 9. [26] Let the functions $h : (0;1) \to (0;1)$, $g : \mathbb{R}^n \to \mathbb{R}^n$ and $\varepsilon : \mathfrak{I} \subset \mathbb{R} \to [0,+\infty)$. If for all $\eta, \mu \in \mathbb{R}^n :\ g(\eta), g(\mu) \in K_g$ and $\varpi \in [0,1]$

$$\varepsilon(\varpi g(\eta) + (1-\varpi) g(\mu)) \le h(\varpi)\, \varepsilon(g(\eta)) + h(1-\varpi)\varepsilon(g(\mu))$$

holds, then ε is a g_h-convex function.

Among the subclasses of Definition 9 we mention the classes of g_{MT}- and g_β-convex functions as follows:

Definition 10. Let $\varepsilon : K_g \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a nonnegative and $g : \mathbb{R} \to \mathbb{R}$, if for all $\eta, \mu \in \mathbb{R}$: $g(\eta), g(\mu) \in K_g$, and $\varpi \in (0,1)$

$$\varepsilon(\varpi g(\eta) + (1-\varpi) g(\mu)) \le \tfrac{\sqrt{\varpi}}{2\sqrt{1-\varpi}} \varepsilon(g(\eta)) + \tfrac{\sqrt{1-\varpi}}{2\sqrt{\varpi}} \varepsilon(g(\mu))$$

holds, then ε is a g_{MT}-convex function.

Definition 11. Let $\varepsilon : K_g \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a nonnegative and $g : \mathbb{R} \to \mathbb{R}$, if for all $\eta, \mu \in \mathbb{R}$: $g(\eta), g(\mu) \in K_g$, $p,\ q > -1$ and $\varpi \in (0,1)$

$$\varepsilon(\varpi g(\eta) + (1-\varpi) g(\mu)) \le \varpi^p (1-\varpi)^q \varepsilon(g(\eta)) + \varpi^q (1-\varpi)^p \varepsilon(g(\mu))$$

holds, then ε is a g_β-convex function.

Definition 12. [27–29] The Riemann-Liouville integrals $I^{\alpha}_{\eta^+}\varepsilon$ and $I^{\alpha}_{\mu^-}\varepsilon$ of order $\alpha > 0$ with $\eta \geq 0$ where $\varepsilon \in L^1[\eta, \mu]$ are defined by

$$I^{\alpha}_{\eta^+}\varepsilon(x) = \frac{1}{\Gamma(\alpha)} \int_{\eta}^{x} (x-\varpi)^{\alpha-1}\varepsilon(\varpi)d\varpi, \quad x > \eta, \tag{3}$$

and

$$I^{\alpha}_{\mu^-}\varepsilon(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\mu} (\varpi - x)^{\alpha-1}\varepsilon(\varpi)d\varpi, \quad x < \mu, \tag{4}$$

respectively, where $\Gamma(\alpha) = \int_0^{\infty} e^{-\varpi}\varpi^{\alpha-1}d\varpi$, $\alpha > 0$ is the gamma function. Here $I^{0}_{\eta^+}\varepsilon(x) = I^{0}_{\mu^-}\varepsilon(x) = \varepsilon(x)$. For $\alpha = 1$, (3) and (4) recapture the classical integral.

Definition 13. [30] The left- and right-sided generalized Riemann-Liouville fractional integrals of order $\alpha > 0$, where $\varepsilon \in L^1[g(\eta), g(\mu)]$, with $g(\eta) < g(\mu)$, are given by

$$I^{\alpha}_{g(\eta)^+}\varepsilon(x) = \frac{1}{\Gamma(\alpha)} \int_{g(\eta)}^{x} (x-\varpi)^{\alpha-1}\varepsilon(\varpi)d\varpi, \quad 0 \leq g(\eta) < x < g(\mu), \tag{5}$$

and

$$I^{\alpha}_{g(\mu)^-}\varepsilon(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{g(\mu)} (\varpi - x)^{\alpha-1}\varepsilon(\varpi)d\varpi, \quad 0 \leq g(\eta) < x < g(\mu). \tag{6}$$

It is clear from (5) and (6) that $I^{\alpha}_{g(\eta)^+}\varepsilon(g(\eta)) = 0$ and $I^{\alpha}_{g(\mu)^-}\varepsilon(g(\mu)) = 0$.

2 *Main results*

Throughout this paper $K_g = [g(r), g(k)]$, $g(r) < g(k)$.

2.1 *Ostrowski type fractional integral inequalities for g_{MT}-convex functions*

Lemma 1. Let $\varepsilon : [g(r), g(k)] \to \mathbb{R}$ be a differentiable function on $(g(r), g(k))$ with $g(r) < g(k)$, where $g : \mathbb{R} \to \mathbb{R}$ is a mapping satisfying for all $x \in (r, k) : g(r) < g(x) < g(k)$. If $\varepsilon' \in L^1[g(r), g(k)]$, then for all $x \in [r, k]$ such that $g(x) \in [g(r), g(k)]$, and $\alpha > 0$, then

$$\begin{aligned}
&\tfrac{(g(x)-g(r))^{\alpha}+(g(k)-g(x))^{\alpha}}{k-r}\varepsilon(g(x)) - \tfrac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{g(x)^-}\varepsilon(g(r)) + I^{\alpha}_{g(x)^+}\varepsilon(g(k))\right] \\
&= \tfrac{(g(x)-g(r))^{\alpha+1}}{k-r} \int_0^1 \varpi^{\alpha}\varepsilon'(\varpi g(x) + (1-\varpi)g(r))\, d\varpi \\
&\quad - \tfrac{(g(k)-g(x))^{\alpha+1}}{k-r} \int_0^1 \varpi^{\alpha}\varepsilon'(\varpi g(x) + (1-\varpi)g(k))d\varpi.
\end{aligned} \tag{7}$$

Proof. Integrating by parts, it yields

$$\begin{aligned} I_1 &= \int_0^1 \varpi^\alpha \varepsilon' (\varpi g(x) + (1-\varpi)g(r)) d\varpi \\ &= \tfrac{\varepsilon(g(x))}{g(x)-g(r)} - \tfrac{\alpha}{g(x)-g(r)} \int_0^1 \varpi^{\alpha-1} \varepsilon (\varpi g(x) + (1-\varpi)g(r)) d\varpi, \end{aligned}$$

with the change of variable $u = \varpi g(x) + (1-\varpi)g(r)$, it follows that

$$\begin{aligned} I_1 &= \tfrac{\varepsilon(g(x))}{g(x)-g(r)} - \tfrac{\Gamma(\alpha+1)}{(g(x)-g(r))^{\alpha+1}} \tfrac{1}{\Gamma(\alpha)} \int_{g(r)}^{g(x)} (u-g(r))^{\alpha-1} \varepsilon(u) du \\ &= \tfrac{\varepsilon(g(x))}{g(x)-g(r)} - \tfrac{\Gamma(\alpha+1)}{(g(x)-g(r))^{\alpha+1}} I^\alpha_{g(x)^-} \varepsilon(g(r)). \end{aligned}$$

Similarly, we obtain

$$I_2 = \int_0^1 \varpi^\alpha \varepsilon' (\varpi g(x) + (1-\varpi)g(k)) d\varpi = \tfrac{\varepsilon(g(x))}{g(x)-g(k)} + \tfrac{\Gamma(\alpha+1)}{(g(k)-g(x))^{\alpha+1}} I^\alpha_{g(x)^+} \varepsilon(g(k)).$$

Multiplying I_1 by $\frac{(g(x)-g(r))^{\alpha+1}}{k-r}$, and I_2 by $\frac{(g(k)-g(x))^{\alpha+1}}{k-r}$, we have

$$\begin{aligned} &\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r} \int_0^1 \varpi^\alpha \varepsilon' (\varpi g(x) + (1-\varpi)g(r)) \, d\varpi \\ &= \tfrac{(g(x)-g(r))^{\alpha}}{k-r} \varepsilon(g(x)) - \tfrac{\Gamma(\alpha+1)}{k-r} I^\alpha_{g(x)^-} \varepsilon(g(r)) \end{aligned} \tag{8}$$

and

$$\begin{aligned} &\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r} \int_0^1 \varpi^\alpha \varepsilon' (\varpi g(x) + (1-\varpi)g(k)) \, dt \\ &= -\tfrac{(g(k)-g(x))^{\alpha}}{k-r} \varepsilon(g(x)) + \tfrac{\Gamma(\alpha+1)}{k-r} I^\alpha_{g(x)^+} \varepsilon(g(k)). \end{aligned} \tag{9}$$

Subtracting (9) from (8), we get (7).

Remark 3. Lemma 1 gives Lemma 1 from [4], for $g(x) = x$.

Theorem 5. Let $\varepsilon : K_g \subset \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on K_g° such that $\varepsilon' \in L^1 [g(r), g(k)]$. If $|\varepsilon'|$ is g_{MT}-convex function with respect to g where $g : \mathbb{R} \to \mathbb{R}$ is a mapping satisfying $g(r) < g(x) < g(k)$ for all $x \in (r, k)$ and $|\varepsilon'(z)| \le M$, $z \in K_g$, then

$$\begin{aligned} &\left| \tfrac{(g(x)-g(r))^\alpha + (g(k)-g(x))^\alpha}{k-r} \varepsilon(g(x)) - \tfrac{\Gamma(\alpha+1)}{k-r} \left[I^\alpha_{g(x)^-} \varepsilon(g(r)) + I^\alpha_{g(x)^+} \varepsilon(g(k)) \right] \right| \\ &\le \tfrac{\Gamma\left(\alpha+\frac{1}{2}\right)}{2\Gamma(\alpha+1)} \tfrac{\left((g(x)-g(r))^{\alpha+1} + (g(k)-g(x))^{\alpha+1} \right) \sqrt{\pi}}{k-r} M \end{aligned}$$

holds for all $x \in [r, k]$ with $g(x) \in K_g$ and $\alpha > 0$ and Γ is the gamma function.

Proof. By Lemma 1 and modulus, we have

$$\begin{aligned}
&\left|\tfrac{(g(x)-g(r))^{\alpha}+(g(k)-g(x))^{\alpha}}{k-r}\varepsilon(g(x))-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{g(x)^{-}}\varepsilon(g(r))+I^{\alpha}_{g(x)^{+}}\varepsilon(g(k))\right]\right| \\
\leq&\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\int\limits_{0}^{1}\varpi^{\alpha}\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|d\varpi \\
&+\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\int\limits_{0}^{1}\varpi^{\alpha}\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|d\varpi.
\end{aligned}$$

Since $|\varepsilon'|$ is g_{MT}-convex with respect to the function g, and taking into account that $|\varepsilon'(x)|\leq M$, $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, we obtain

$$\begin{aligned}
&\left|\tfrac{(g(x)-g(r))^{\alpha}+(g(k)-g(x))^{\alpha}}{k-r}\varepsilon(g(x))-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{g(x)^{-}}\varepsilon(g(r))+I^{\alpha}_{g(x)^{+}}\varepsilon(g(k))\right]\right| \\
\leq&\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\int\limits_{0}^{1}\varpi^{\alpha}\left(\tfrac{\sqrt{\varpi}}{2\sqrt{1-\varpi}}\left|\varepsilon'(g(x))\right|+\tfrac{\sqrt{1-\varpi}}{2\sqrt{\varpi}}\left|\varepsilon'(g(r))\right|\right)d\varpi \\
&+\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\int\limits_{0}^{1}\varpi^{\alpha}\left(\tfrac{\sqrt{\varpi}}{2\sqrt{1-\varpi}}\left|\varepsilon'(g(x))\right|+\tfrac{\sqrt{1-\varpi}}{2\sqrt{\varpi}}\left|\varepsilon'(g(k))\right|\right)d\varpi \\
\leq&\tfrac{M(g(x)-g(r))^{\alpha+1}}{2(k-r)}\int\limits_{0}^{1}\left(\varpi^{\alpha+\frac{1}{2}}(1-\varpi)^{\frac{-1}{2}}+\varpi^{\alpha-\frac{1}{2}}(1-\varpi)^{\frac{1}{2}}\right)d\varpi \\
&+\tfrac{M(g(k)-g(x))^{\alpha+1}}{2(k-r)}\int\limits_{0}^{1}\left(\varpi^{\alpha+\frac{1}{2}}(1-\varpi)^{\frac{-1}{2}}+\varpi^{\alpha-\frac{1}{2}}(1-\varpi)^{\frac{1}{2}}\right)d\varpi \\
=&\tfrac{\left((g(x)-g(r))^{\alpha+1}+(g(k)-g(x))^{\alpha+1}\right)}{2(k-r)}\left(\beta(\alpha+\tfrac{3}{2},\tfrac{1}{2})+\beta(\alpha+\tfrac{1}{2},\tfrac{3}{2})\right)M \\
=&\tfrac{\left((g(x)-g(r))^{\alpha+1}+(g(k)-g(x))^{\alpha+1}\right)\sqrt{\pi}}{2(k-r)}\tfrac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha+1)}M,
\end{aligned}$$

where β is the beta function, defined by: $\beta(x,y)=\int\limits_{0}^{1}\varpi^{x-1}(1-\varpi)^{y-1}d\varpi=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\ x>0,\ y>0.$

Remark 4. For $g(x)=x$, Theorem 5 becomes Theorem 2.

Theorem 6. Let the differentiable mapping $\varepsilon:K_g\subset\mathbb{R}_{+}\to\mathbb{R}$ with $\varepsilon'\in L^{1}\left[g(r),g(k)\right]$. If $|\varepsilon'|^{q}$ is g_{MT}-convex function with respect to g, where $g:\mathbb{R}\to\mathbb{R}$ is a mapping satisfying $g(r)<g(x)<g(k)$ and $|\varepsilon'(z)|\leq M$, $z\in K_g$, then

$$\begin{aligned}
&\left|\tfrac{(g(x)-g(r))^{\alpha}+(g(k)-g(x))^{\alpha}}{k-r}\varepsilon(g(x))-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{g(x)^{-}}\varepsilon(g(r))+I^{\alpha}_{g(x)^{+}}\varepsilon(g(k))\right]\right| \\
\leq&\left(\tfrac{1}{1+\alpha p}\right)^{\frac{1}{p}}\left(\tfrac{\pi}{2}\right)^{\frac{1}{q}}\tfrac{(g(x)-g(r))^{\alpha+1}+(g(k)-g(x))^{\alpha+1}}{k-r}M
\end{aligned}$$

holds for all $x\in[r,k]$, p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $\alpha>0$.

Proof. By Lemma 1 and Hölder's inequality, we obtain

$$
\begin{aligned}
&\left|\tfrac{(g(x)-g(r))^\alpha+(g(k)-g(x))^\alpha}{k-r}\varepsilon(g(x))-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I^\alpha_{g(x)^-}\varepsilon(g(r))+I^\alpha_{g(x)^+}\varepsilon(g(k))\right]\right|\\
\le&\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\int_0^1\varpi^\alpha\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|d\varpi\\
&+\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\int_0^1\varpi^\alpha\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|d\varpi\\
\le&\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\left(\int_0^1\varpi^{\alpha p}d\varpi\right)^{\frac1p}\left(\int_0^1\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^q d\varpi\right)^{\frac1q}\\
&+\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\left(\int_0^1\varpi^{\alpha p}d\varpi\right)^{\frac1p}\left(\int_0^1\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|^q d\varpi\right)^{\frac1q}\\
=&\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\left(\tfrac{1}{1+\alpha p}\right)^{\frac1p}\left(\int_0^1\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^q d\varpi\right)^{\frac1q}\\
&+\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\left(\tfrac{1}{1+\alpha p}\right)^{\frac1p}\left(\int_0^1\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|^q d\varpi\right)^{\frac1q}.
\end{aligned}
\tag{10}
$$

Since $|\varepsilon'|^q$ is g_{MT}-convex with respect to g and $|\varepsilon'(x)|\le M$, we have

$$
\begin{aligned}
\int_0^1\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^q d\varpi&\le\int_0^1\left(\tfrac{\sqrt{\varpi}}{2\sqrt{1-\varpi}}\left|\varepsilon'(g(x))\right|^q+\tfrac{\sqrt{1-\varpi}}{2\sqrt{\varpi}}\left|\varepsilon'(g(r))\right|^q\right)d\varpi\\
&\le M^q\int_0^1\left(\tfrac{\sqrt{\varpi}}{2\sqrt{1-\varpi}}+\tfrac{\sqrt{1-\varpi}}{2\sqrt{\varpi}}\right)d\varpi=\tfrac{\pi}{2}M^q.
\end{aligned}
\tag{11}
$$

From (10) and (11), we get the result.

Remark 5. For $g(x)=x$, Theorem 6 will be reduced to Theorem 3.

Theorem 7. Let the differentiable mapping $\varepsilon : K_g\subset\mathbb{R}_+\to\mathbb{R}$ with $\varepsilon'\in L^1\left[g(r),g(k)\right]$. If $|\varepsilon'|^q$ is g_{MT}-convex function with respect to g, where $g:\mathbb{R}\to\mathbb{R}$ is a mapping satisfying $g(r)<g(x)<g(k)$ for all $x\in(r,k)$, $q\ge1$, and $|\varepsilon'(z)|\le M$, $z\in K_g$, then

$$
\begin{aligned}
&\left|\tfrac{(g(x)-g(r))^\alpha+(g(k)-g(x))^\alpha}{k-r}\varepsilon(g(x))-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I^\alpha_{g(x)^-}\varepsilon(g(r))+I^\alpha_{g(x)^+}\varepsilon(g(k))\right]\right|\\
\le&\left(\tfrac{(1+\alpha)\Gamma(\alpha+\frac12)}{2\Gamma(\alpha+1)}\sqrt{\pi}\right)^{\frac1q}\tfrac{(g(x)-g(r))^{\alpha+1}+(g(k)-g(x))^{\alpha+1}}{(1+\alpha)(k-r)}M
\end{aligned}
$$

holds for all $x\in[r,k]$ with $g(x)\in K_g$ and $\alpha>0$, where Γ is the gamma function.

Proof. By the identity of Lemma 1, modulus and the so-called power mean inequality, it yields

$$\begin{aligned}
&\left|\tfrac{(g(x)-g(r))^{\alpha}+(g(k)-g(x))^{\alpha}}{k-r}\varepsilon(g(x)) - \tfrac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{g(x)^{-}}\varepsilon(g(r)) + I^{\alpha}_{g(x)^{+}}\varepsilon(g(k))\right]\right| \\
\leq& \tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\int\limits_0^1 \varpi^{\alpha}\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right| d\varpi \\
&+ \tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\int\limits_0^1 \varpi^{\alpha}\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right| d\varpi \\
\leq& \tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\left(\int\limits_0^1 \varpi^{\alpha} d\varpi\right)^{1-\frac{1}{q}}\left(\int\limits_0^1 \varpi^{\alpha}\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^{q} dt\right)^{\frac{1}{q}} \\
&+ \tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\left(\int\limits_0^1 \varpi^{\alpha} d\varpi\right)^{1-\frac{1}{q}}\left(\int\limits_0^1 \varpi^{\alpha}\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|^{q} dt\right)^{\frac{1}{.q}} \\
=& \tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\tfrac{1}{(1+\alpha)^{1-\frac{1}{q}}}\left(\int\limits_0^1 \varpi^{\alpha}\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^{q} d\varpi\right)^{\frac{1}{q}} \\
&+ \tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\tfrac{1}{(1+\alpha)^{1-\frac{1}{q}}}\left(\int\limits_0^1 \varpi^{\alpha}\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|^{q} d\varpi\right)^{\frac{1}{.q}}.
\end{aligned} \tag{12}$$

Since $|\varepsilon'|^{q}$ is g_{MT}-convex with respect to a function g, and $|\varepsilon'(x)| \leq M$, we get

$$\begin{aligned}
&\int\limits_0^1 \varpi^{\alpha}\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^{q} d\varpi \\
\leq& \int\limits_0^1 \left(\tfrac{\varpi^{\alpha}\sqrt{\varpi}}{2\sqrt{1-\varpi}}\left|\varepsilon'(g(x))\right|^{q} + \tfrac{\varpi^{\alpha}\sqrt{1-\varpi}}{2\sqrt{\varpi}}\left|\varepsilon'(g(r))\right|^{q}\right) d\varpi \\
\leq& M^{q}\int\limits_0^1 \left(\tfrac{\varpi^{\alpha}\sqrt{\varpi}}{2\sqrt{1-\varpi}} + \tfrac{\varpi^{\alpha}\sqrt{1-\varpi}}{2\sqrt{\varpi}}\right) d\varpi \\
=& \tfrac{1}{2}M^{q}\int\limits_0^1 \left(\varpi^{\alpha+\frac{1}{2}}(1-\varpi)^{-\frac{1}{2}} + \varpi^{\alpha-\frac{1}{2}}(1-\varpi)^{\frac{1}{2}}\right) d\varpi \\
=& M^{q}\left(\tfrac{\Gamma(\alpha+\frac{3}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+2)} + \tfrac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{3}{2})}{2\Gamma(\alpha+2)}\right) \\
=& M^{q}\left(\left(\alpha+\tfrac{1}{2}\right)+\tfrac{3}{2}\right)\tfrac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+2)} = \tfrac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+1)}M^{q}
\end{aligned} \tag{13}$$

and

$$\int\limits_0^1 \varpi^{\alpha}\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)k\right)\right|^{q} d\varpi \leq M^{q}\tfrac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+1)}. \tag{14}$$

From (12)–(14), we get the result.

Remark 6. For $g(x) = x$, Theorem 7 will be reduced to Theorem 4.

2.2 Fractional Ostrowski's inequalities for g_β-convexity

Theorem 8. Let the differentiable mapping $\varepsilon : K_g \subset \mathbb{R}_+ \to \mathbb{R}$ with $\varepsilon^{'} \in L^1[g(r), g(k)]$. If $|\varepsilon'|$ is g_β-convex function with respect to g where $g : \mathbb{R} \to \mathbb{R}$ is a mapping satisfying for all $x \in (r, k)$: $g(r) < g(x) < g(k)$ and $|\varepsilon'(z)| \leq M$, $z \in K_g$, then

$$\left|\tfrac{(g(x)-g(r))^\alpha+(g(k)-g(x))^\alpha}{k-r}\varepsilon(g(x)) - \tfrac{\Gamma(\alpha+1)}{k-r}\left[I^\alpha_{g(x)^-}\varepsilon(g(r)) + I^\alpha_{g(x)^+}\varepsilon(g(k))\right]\right|$$
$$\leq \tfrac{\Gamma(\alpha+p+1)\Gamma(q+1)+\Gamma(\alpha+q+1)\Gamma(p+1)}{\Gamma(\alpha+p+q+2)}\left(\tfrac{(g(x)-g(r))^{\alpha+1}+(g(k)-g(x))^{\alpha+1}}{k-r}\right)M \tag{15}$$

holds for all $x \in [r, k]$ with $g(x) \in K_g$ and $\alpha > 0$, p, $q > -1$ and Γ is the gamma function.

Proof. Using Lemma 1 and modulus, we get

$$\left|\tfrac{(g(x)-g(r))^\alpha+(g(k)-g(x))^\alpha}{k-r}\varepsilon(g(x)) - \tfrac{\Gamma(\alpha+1)}{k-r}\left[I^\alpha_{g(x)^-}\varepsilon(g(r)) + I^\alpha_{g(x)^+}\varepsilon(g(k))\right]\right|$$
$$\leq \tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\int_0^1 \varpi^\alpha \left|\varepsilon'(\varpi g(x) + (1-\varpi)g(r))\right| d\varpi$$
$$+ \tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\int_0^1 \varpi^\alpha \left|\varepsilon'(\varpi g(x) + (1-\varpi)g(k))\right| d\varpi.$$

The fact that $|\varepsilon'|$ is g_β-convex with respect to g and $|\varepsilon'(x)| \leq M$, gives

$$\left|\tfrac{(g(x)-g(r))^\alpha+(g(k)-g(x))^\alpha}{k-r}\varepsilon(g(x)) - \tfrac{\Gamma(\alpha+1)}{k-r}\left[I^\alpha_{g(x)^-}\varepsilon(g(r)) + I^\alpha_{g(x)^+}\varepsilon(g(k))\right]\right|$$
$$\leq \tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\int_0^1 \varpi^\alpha \left[\varpi^p(1-\varpi)^q\left|\varepsilon'(g(x))\right| + \varpi^q(1-\varpi)^p\left|\varepsilon'(g(r))\right|\right] d\varpi$$
$$+ \tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\int_0^1 \varpi^\alpha \left[\varpi^p(1-\varpi)^q\left|\varepsilon'(g(x))\right| + \varpi^q(1-\varpi)^p\left|\varepsilon'(g(k))\right|\right] d\varpi$$
$$\leq \tfrac{M(g(x)-g(r))^{\alpha+1}}{k-r}\int_0^1 \left[\varpi^{\alpha+p}(1-\varpi)^q + \varpi^{\alpha+q}(1-\varpi)^p\right] d\varpi$$
$$+ \tfrac{M(g(k)-g(x))^{\alpha+1}}{k-r}\int_0^1 \left[\varpi^{\alpha+p}(1-\varpi)^q + \varpi^{\alpha+q}(1-\varpi)^p\right] d\varpi$$
$$= (\beta(\alpha+p+1, q+1) + \beta(\alpha+q+1, p+1))\left(\tfrac{(g(x)-g(r))^{\alpha+1}+(g(k)-g(x))^{\alpha+1}}{k-r}\right)M$$
$$= \tfrac{\Gamma(\alpha+p+1)\Gamma(q+1)+\Gamma(\alpha+q+1)\Gamma(p+1)}{\Gamma(\alpha+p+q+2)}\left(\tfrac{(g(x)-g(r))^{\alpha+1}+(g(k)-g(x))^{\alpha+1}}{k-r}\right)M.$$

The proof is completed.

Corollary 1. In (15), if we choose $g(x) = x$, i.e., $|\varepsilon'|$ is β-convex, we get

$$\left|\tfrac{(x-r)^\alpha+(k-x)^\alpha}{k-r}\varepsilon(x) - \tfrac{\Gamma(\alpha+1)}{k-r}\left[I^\alpha_{x^-}\varepsilon(r) + I^\alpha_{x^+}\varepsilon(k)\right]\right|$$
$$\leq \tfrac{\Gamma(\alpha+p+1)\Gamma(q+1)+\Gamma(\alpha+q+1)\Gamma(p+1)}{\Gamma(\alpha+p+q+2)}\left(\tfrac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{k-r}\right)M.$$

We now give some special cases which can be derived from the preceding corollary.

Corollary 2. In Corollary 1, taking $p = q = 0$, we get

$$\begin{aligned}&\left|\tfrac{(x-r)^{\alpha}+(k-x)^{\alpha}}{k-r}\varepsilon(x) - \tfrac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{x^-}\varepsilon(r) + I^{\alpha}_{x^+}\varepsilon(k)\right]\right|\\ &\leq \tfrac{2\left((x-r)^{\alpha+1}+(k-x)^{\alpha+1}\right)}{(\alpha+1)(k-r)}M.\end{aligned}$$

Corollary 3. In Corollary 1, taking $p = s$, $q = 0$, we get

$$\begin{aligned}&\left|\tfrac{(x-r)^{\alpha}+(k-x)^{\alpha}}{k-r}\varepsilon(x) - \tfrac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{x^-}\varepsilon(r) + I^{\alpha}_{x^+}\varepsilon(k)\right]\right|\\ &\leq \tfrac{\Gamma(\alpha+s+1)+\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\left(\tfrac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{k-r}\right)M.\end{aligned}$$

Corollary 4. In Corollary 1, taking $p = q = 1$, we get

$$\begin{aligned}&\left|\tfrac{(x-r)^{\alpha}+(k-x)^{\alpha}}{k-r}\varepsilon(x) - \tfrac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{x^-}\varepsilon(r) + I^{\alpha}_{x^+}\varepsilon(k)\right]\right|\\ &\leq \tfrac{2\left((x-r)^{\alpha+1}+(k-x)^{\alpha+1}\right)}{(\alpha+3)(\alpha+2)(k-r)}M.\end{aligned}$$

Corollary 5. For $x = \frac{r+k}{2}$ Corollary 1 gives the following midpoint inequality:

$$\begin{aligned}&\left|\varepsilon(\tfrac{r+k}{2}) - \tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(k-r)^{\alpha}}\left[I^{\alpha}_{(\frac{r+k}{2})^-}\varepsilon(r) + I^{\alpha}_{(\frac{r+k}{2})^+}\varepsilon(k)\right]\right|\\ &\leq \left(\tfrac{\Gamma(\alpha+p+1)\Gamma(q+1)+\Gamma(\alpha+q+1)\Gamma(p+1)}{\Gamma(\alpha+p+q+2)}\right)\tfrac{(k-r)M}{2}.\end{aligned}$$

Theorem 9. Let the differentiable mapping $\varepsilon : K_g \subset \mathbb{R}_+ \to \mathbb{R}$ with $\varepsilon' \in L^1[g(r), g(k)]$. If $|\varepsilon'|^{\mu}$ is g_{β}-convex function with respect to g, where $g : \mathbb{R} \to \mathbb{R}$ is a mapping satisfying for all $x \in (r, k)$: $g(r) < g(x) < g(k)$ and $|\varepsilon'(z)| \leq M$, $z \in K_g$, then

$$\begin{aligned}&\left|\tfrac{(g(x)-g(r))^{\alpha}+(g(k)-g(x))^{\alpha}}{k-r}\varepsilon(g(x)) - \tfrac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{g(x)^-}\varepsilon(g(r)) + I^{\alpha}_{g(x)^+}\varepsilon(g(k))\right]\right|\\ &\leq \left(\tfrac{1}{\alpha\lambda+1}\right)^{\frac{1}{\lambda}}\left(\tfrac{2\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}\right)^{\frac{1}{\mu}}\left(\tfrac{(g(x)-g(r))^{\alpha+1}+(g(k)-g(x))^{\alpha+1}}{k-r}\right)M \qquad (16)\end{aligned}$$

holds for all $x \in [r, k]$ with $g(x) \in K_g$ and $\alpha > 0$, p, $q > -1$, λ, $\mu > 1$ with $\frac{1}{\lambda} + \frac{1}{\mu} = 1$, where Γ is the gamma function.

Proof. By Lemma 1, modulus and Hölder's inequality, we have

$$\begin{aligned}
&\left|\tfrac{(g(x)-g(r))^\alpha+(g(k)-g(x))^\alpha}{k-r}\varepsilon(g(x))-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I^\alpha_{g(x)^-}\varepsilon(g(r))+I^\alpha_{g(x)^+}\varepsilon(g(k))\right]\right|\\
\leq&\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\int_0^1\varpi^\alpha\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|d\varpi\\
&+\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\int_0^1\varpi^\alpha\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|d\varpi\\
\leq&\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\left(\int_0^1\varpi^{\alpha\lambda}d\varpi\right)^{\frac{1}{\lambda}}\left(\int_0^1\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^\mu d\varpi\right)^{\frac{1}{\mu}}\\
&+\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\left(\int_0^1\varpi^{\alpha\lambda}d\varpi\right)^{\frac{1}{\lambda}}\left(\int_0^1\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|^\mu d\varpi\right)^{\frac{1}{\mu}}\\
=&\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\left(\tfrac{1}{\alpha\lambda+1}\right)^{\frac{1}{\lambda}}\left(\int_0^1\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^\mu d\varpi\right)^{\frac{1}{\mu}}\\
&+\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\left(\tfrac{1}{\alpha\lambda+1}\right)^{\frac{1}{\lambda}}\left(\int_0^1\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|^\mu d\varpi\right)^{\frac{1}{\mu}}. \qquad (17)
\end{aligned}$$

Since $|\varepsilon'|^\mu$ is g_β-convex with respect to a function g, and $|\varepsilon'(x)|\leq M$, we get

$$\begin{aligned}
&\int_0^1\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^\mu d\varpi\\
\leq&\int_0^1\left[\varpi^p(1-\varpi)^q\left|\varepsilon'(g(x))\right|^\mu+\varpi^q(1-\varpi)^p\left|\varepsilon'(g(r))\right|^\mu\right]d\varpi\\
\leq&M^\mu\int_0^1\left[\varpi^p(1-\varpi)^q+\varpi^q(1-\varpi)^p\right]d\varpi\\
=&2M^\mu\beta(p+1,q+1)=2M^\mu\tfrac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}. \qquad (18)
\end{aligned}$$

From (17) and (18), we get the result.

Corollary 6. In (16), if we choose $g(x)=x$, i.e. $|\varepsilon'|^\mu$ is β-convex, we have

$$\begin{aligned}
&\left|\tfrac{(x-r)^\alpha+(k-x)^\alpha}{k-r}\varepsilon(x)-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I^\alpha_{x^-}\varepsilon(r)+I^\alpha_{x^+}\varepsilon(k)\right]\right|\\
\leq&\left(\tfrac{2\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}\right)^{\frac{1}{\mu}}\left(\tfrac{1}{\alpha\lambda+1}\right)^{\frac{1}{\lambda}}\left(\tfrac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{k-r}\right)M.
\end{aligned}$$

Some particular situations that may be derived from the earlier corollary are given below.

Corollary 7. In Corollary 6, taking $p = q = 0$, we get

$$\begin{aligned}&\left|\tfrac{(x-r)^{\alpha}+(k-x)^{\alpha}}{k-r}\varepsilon(x)-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{x^-}\varepsilon(r)+I^{\alpha}_{x^+}\varepsilon(k)\right]\right|\\ &\leq 2^{\frac{1}{\mu}}\left(\tfrac{1}{\alpha\lambda+1}\right)^{\frac{1}{\lambda}}\left(\tfrac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{k-r}\right)M.\end{aligned}$$

Corollary 8. In Corollary 6, taking $p = s$, $q = 0$, we get

$$\begin{aligned}&\left|\tfrac{(x-r)^{\alpha}+(k-x)^{\alpha}}{k-r}\varepsilon(x)-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{x^-}\varepsilon(r)+I^{\alpha}_{x^+}\varepsilon(k)\right]\right|\\ &\leq \left(\tfrac{2}{s+1}\right)^{\frac{1}{\mu}}\left(\tfrac{1}{\alpha\lambda+1}\right)^{\frac{1}{\lambda}}\left(\tfrac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{k-r}\right)M.\end{aligned}$$

Corollary 9. In Corollary 6, taking $p = q = 1$, we get

$$\begin{aligned}&\left|\tfrac{(x-r)^{\alpha}+(k-x)^{\alpha}}{k-r}\varepsilon(x)-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{x^-}\varepsilon(r)+I^{\alpha}_{x^+}\varepsilon(k)\right]\right|\\ &\leq \left(\tfrac{1}{3}\right)^{\frac{1}{\mu}}\left(\tfrac{1}{\alpha\lambda+1}\right)^{\frac{1}{\lambda}}\left(\tfrac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{k-r}\right)M.\end{aligned}$$

Corollary 10. For $x = \frac{r+k}{2}$, Corollary 6 gives the following midpoint inequality:

$$\begin{aligned}&\left|\varepsilon\left(\tfrac{r+k}{2}\right)-\tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(k-r)^{\alpha}}\left[I^{\alpha}_{\left(\frac{r+k}{2}\right)^-}\varepsilon(r)+I^{\alpha}_{\left(\frac{r+k}{2}\right)^+}\varepsilon(k)\right]\right|\\ &\leq \left(\tfrac{2\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}\right)^{\frac{1}{\mu}}\left(\tfrac{1}{\alpha\lambda+1}\right)^{\frac{1}{\lambda}}\tfrac{(k-r)M}{2}.\end{aligned}$$

Theorem 10. Let the differentiable mapping $\varepsilon : K_g \subset \mathbb{R}_+ \to \mathbb{R}$ with $\varepsilon' \in L^1[g(r), g(k)]$. If $|\varepsilon'|^{\mu}$ is g_{β}-convex function with respect to g, where $g : \mathbb{R} \to \mathbb{R}$ is a mapping satisfying for all $x \in (r, k)$: $g(r) < g(x) < g(k)$ and $|\varepsilon'(z)| \leq M$, $z \in K_g$, then

$$\begin{aligned}&\left|\tfrac{(g(x)-g(r))^{\alpha}+(g(k)-g(x))^{\alpha}}{k-r}\varepsilon(g(x))-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{g(x)^-}\varepsilon(g(r))+I^{\alpha}_{g(x)^+}\varepsilon(g(k))\right]\right|\\ &\leq \left(\tfrac{(\alpha+1)\Gamma(\alpha+p+1)\Gamma(q+1)+(\alpha+1)\Gamma(\alpha+q+1)\Gamma(p+1)}{\Gamma(\alpha+p+q+2)}\right)^{\frac{1}{\mu}}\tfrac{(g(x)-g(r))^{\alpha+1}+(g(k)-g(x))^{\alpha+1}}{(\alpha+1)(k-r)}M \qquad (19)\end{aligned}$$

holds for all $x \in [r, k]$ with $g(x) \in K_g$ and $\alpha > 0$ and $p, q > -1$, $\mu > 1$, where Γ is the gamma function.

Proof. By Lemma 1, modulus and power mean inequality, we get

$$
\begin{aligned}
&\left|\tfrac{(g(x)-g(r))^\alpha+(g(k)-g(x))^\alpha}{k-r}\varepsilon(g(x))-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I^\alpha_{g(x)^-}\varepsilon(g(r))+I^\alpha_{g(x)^+}\varepsilon(g(k))\right]\right|\\
&\leq\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\int_0^1\varpi^\alpha\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|d\varpi\\
&\quad+\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\int_0^1\varpi^\alpha\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|d\varpi\\
&\leq\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\left(\int_0^1\varpi^\alpha d\varpi\right)^{1-\frac{1}{\mu}}\left(\int_0^1\varpi^\alpha\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^\mu d\varpi\right)^{\frac{1}{\mu}}\\
&\quad+\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\left(\int_0^1\varpi^\alpha d\varpi\right)^{1-\frac{1}{\mu}}\left(\int_0^1\varpi^\alpha\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|^\mu d\varpi\right)^{\frac{1}{.\mu}}\\
&=\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\tfrac{1}{(\alpha+1)^{1-\frac{1}{\mu}}}\left(\int_0^1\varpi^\alpha\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^\mu d\varpi\right)^{\frac{1}{\mu}}\\
&\quad+\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\tfrac{1}{(\alpha+1)^{1-\frac{1}{\mu}}}\left(\int_0^1\varpi^\alpha\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|^\mu d\varpi\right)^{\frac{1}{.\mu}}.
\end{aligned}
\tag{20}
$$

Since $|\varepsilon'|^\mu$ is g_β-convex with respect to g, and $|\varepsilon'(x)|\leq M$, we get

$$
\begin{aligned}
&\int_0^1\varpi^\alpha\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^\mu d\varpi\\
&\leq\int_0^1\varpi^\alpha\left[\varpi^p(1-\varpi)^q\left|\varepsilon'(g(x))\right|^\mu+\varpi^q(1-\varpi)^p\left|\varepsilon'(g(r))\right|^\mu\right]d\varpi\\
&\leq M^\mu\int_0^1\left[\varpi^{\alpha+p}(1-\varpi)^q+\varpi^{\alpha+q}(1-\varpi)^p\right]d\varpi\\
&=\left(\beta(\alpha+p+1,q+1)+\beta(\alpha+q+1,p+1)\right)M^\mu\\
&=\tfrac{\Gamma(\alpha+p+1)\Gamma(q+1)+\Gamma(\alpha+q+1)\Gamma(p+1)}{\Gamma(\alpha+p+q+2)}M^\mu.
\end{aligned}
\tag{21}
$$

From (20) and (21), we get the result.

Corollary 11. In (19), if we choose $g(x)=x$, i.e. $|\varepsilon'|^\mu$ is β-convex, we have

$$
\begin{aligned}
&\left|\tfrac{(x-r)^\alpha+(k-x)^\alpha}{k-r}\varepsilon(x)-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I^\alpha_{x^-}\varepsilon(r)+I^\alpha_{x^+}\varepsilon(k)\right]\right|\\
&\leq\left(\tfrac{(\alpha+1)\Gamma(\alpha+p+1)\Gamma(q+1)+(\alpha+1)\Gamma(\alpha+q+1)\Gamma(p+1)}{\Gamma(\alpha+p+q+2)}\right)^{\frac{1}{\mu}}\left(\tfrac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{(\alpha+1)(k-r)}\right)M.
\end{aligned}
$$

We will now show some special cases that can be extracted from the previous result.

Corollary 12. In Corollary 11, taking $p = q = 0$, we get

$$\left|\frac{(x-r)^{\alpha}+(k-x)^{\alpha}}{k-r}\varepsilon(x) - \frac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{x^-}\varepsilon(r) + I^{\alpha}_{x^+}\varepsilon(k)\right]\right| \leq 2^{\frac{1}{\mu}}\left(\frac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{(\alpha+1)(k-r)}\right)M.$$

Corollary 13. In Corollary 11, taking $p = s$, $q = 0$, we get

$$\left|\frac{(x-r)^{\alpha}+(k-x)^{\alpha}}{k-r}\varepsilon(x) - \frac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{x^-}\varepsilon(r) + I^{\alpha}_{x^+}\varepsilon(k)\right]\right|$$
$$\leq \left(\frac{(\alpha+1)\Gamma(\alpha+s+1)+\Gamma(\alpha+2)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right)^{\frac{1}{\mu}}\left(\frac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{(\alpha+1)(k-r)}\right)M.$$

Corollary 14. In Corollary 11, taking $p = q = 1$, we get

$$\left|\frac{(x-r)^{\alpha}+(k-x)^{\alpha}}{k-r}\varepsilon(x) - \frac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{x^-}\varepsilon(r) + I^{\alpha}_{x^+}\varepsilon(k)\right]\right|$$
$$\leq \left(\frac{2(\alpha+1)}{(\alpha+3)(\alpha+2)}\right)^{\frac{1}{\mu}}\left(\frac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{(\alpha+1)(k-r)}\right)M.$$

Corollary 15. For $x = \frac{r+k}{2}$ Corollary 11 gives the following midpoint inequality:

$$\left|\varepsilon\left(\frac{r+k}{2}\right) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(k-r)^{\alpha}}\left[I^{\alpha}_{\left(\frac{r+k}{2}\right)^-}\varepsilon(r) + I^{\alpha}_{\left(\frac{r+k}{2}\right)^+}\varepsilon(k)\right]\right|$$
$$\leq \left(\frac{(\alpha+1)\Gamma(\alpha+p+1)\Gamma(q+1)+(\alpha+1)\Gamma(\alpha+q+1)\Gamma(p+1)}{\Gamma(\alpha+p+q+2)}\right)^{\frac{1}{\mu}}\frac{(k-r)M}{2(\alpha+1)}.$$

Conclusion

In this study, we have explored fractional Ostrowski inequalities for functions whose modulus of the first derivatives exhibit g_{MT}-convexity and g_{β}-convexity. Several new results have been established, contributing to the advancement of fractional integral inequalities. Additionally, by considering specific cases, we have successfully recovered some well-known results, demonstrating the broad applicability and generality of our approach. This work extends classical Ostrowski inequalities and provides deeper insights into their behavior under generalized convexity assumptions. Future research could further investigate the potential applications of these inequalities in fields such as numerical analysis, optimization, and approximation theory.

Author Contributions

All authors participated in the conception and preparation of the manuscript and approved the final submission. All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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