https://doi.org/10.31489/2025M1/133-142 *Research article*

Inverse boundary value problem for a linearized equations of longitudinal waves in rods

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In this article, a question regarding the solvability of an inverse boundary value problem for the linearized equation of longitudinal waves in rods with an integral condition of the first kind was considered. For the considered inverse boundary value problem, the definition of a classical solution was introduced. Using the Fourier method, the problem was reduced to solving a system of integral equations. The method of contraction mappings is applied to prove the existence and uniqueness of a solution to the system of integral equations. The problem is to deduce the existence and uniqueness of the classical solution for the original problem.

Keywords: Inverse boundary value problem, longitudinal wave equations, Fourier method, classical solution.

2020 Mathematics Subject Classification: 49k10, 49k20.

Introduction

At present, the theory of nonlocal problems is being intensively developed and represents an important branch of the theory of partial differential equations. Problems with nonlocal integral conditions are of great interest in this area. Conditions of this kind may appear in the mathematical modeling of phenomena related to plasma physics [1], heat propagation [2], moisture transfer in capillary-porous media [3], demography, and mathematical biology.

Inverse problems with an integral overdetermination condition for partial differential equations have been studied in many papers. Let us note the articles [4–6] and the references therein.

The study of various aspects of inverse problems of recovering the coefficients of partial differential equations, as well as the study of inverse problems by reducing them to variational formulations, is considered in the works of Kozhanov A.I. [6], Denisov A.M. [7], Ivanchov M.I. [8], and others.

Works are devoted to the study of nonlinear inverse problems for the linearized equation of longitudinal waves in rods. The questions of solvability of problems with nonlocal integral conditions for partial differential equations were studied in [5, 9–18].

1 Statement of the problem and its reduction to an equivalent problem

Let D_T be a Domain, $D_T = \{(x,t) : 0 < x < 1,\ 0 < t \le T\}$. Consider for linearized equation an inverse boundary value problem [6]

$$u_{tt}(x,t) + u_{ttxx}(x,t) - u_{xx}(x,t) = a(t)u(x,t) + f(x,t), \quad (x,t) \in D_T \tag{1}$$

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Received: 6 May 2024; *Accepted:* 17 December 2024.

with the initial condition (IC)

$$u(x,0) = \varphi(x),\ u_t(x,T) = \psi(x) \quad (0 \le x \le 1). \tag{2}$$

Neumann boundary condition (BC)

$$u_x(0,t) = 0 \quad (0 \le t \le T), \tag{3}$$

integral condition (IgC)

$$\int_0^1 u(x,t)dx = 0 \quad (0 \le t \le T) \tag{4}$$

and additional conditions (AD)

$$u(0,t) = h(t) \quad (0 \le t \le T), \tag{5}$$

$f(x,t)$, $\varphi(x)$, $\psi(x)$, and $h(t)$ are given functions, $u(x,t)$ and $a(t)$ are the required functions.

Definition. By the classical solution of the inverse boundary value problem (1)–(5) we mean the pair $\{u(x,t),\, a(t)\}$ of the functions $u(x,t)$, $a(t)$, where $u(x,t) \in \tilde{C}^{2,2}(\bar{D}_T)$, $a(t) \in C[0,T]$, and they satisfy the equations (1)–(5) in the ordinary sense, where

$$\tilde{C}^{2,2}(D_T) = \left\{ u(x,t) \,:\, u(x,t) \in C^2(D_T),\ u_{ttxx}(x,t) \in C(D_T) \right\}.$$

To study problem (1)–(5), we first consider the following:

$$y''(t) = a(t)y(t) \quad (0 \le t \le T), \tag{6}$$

$$y(0) = 0\,,\ \ y'(T) = 0, \tag{7}$$

where $a(t) \in C[0,T]$ are given functions, and $y = y(t)$ is the required function. Furthermore, by solving the problem (6), (7) we mean a function $y(t)$, that, together with all its derivatives in equation (6), is continuous on $[0,T]$ and satisfies conditions (6), (7) in the usual sense. The following lemma holds:

Lemma 1. [7] Let $a(t) \in C[0,T]$ be such that

$$\|a(t)\|_{C[0,T]} \le R = const.$$

Supplementarily, $\frac{1}{2}RT^2 < 1$. Then problem (6), (7) has only a trivial solution.

With the addition of the inverse boundary value problem (1)–(5), consider the following auxiliary inverse boundary value problem. We must define a pair $\{u(x,t), a(t)\}$ of functions $u(x,t) \in \tilde{C}^{2,2}(\bar{D}_T)$ and $a(t) \in C[0,T]$, from (2)-(3),

$$u_x(1,t) = 0 \quad (0 \le t \le T), \tag{8}$$

$$h''(t) + u_{ttxx}(0,t) - u_{xx}(0,t) = a(t)h(t) + f(0,t) \quad (0 \le t \le T). \tag{9}$$

Theorem 1. Let $\varphi(x), \psi(x) \in C^1[0,1]$, $\varphi'(1) = 0$, $\psi'(1) = 0$, $h(t) \in C^2[0,T]$, $f(x,t) \in C(\bar{D}_T)$, $\int_0^1 f(x,t)dx = 0$ $(0 \le t \le T)$, $h(t) \ne 0$ $(0 \le t \le T)$ and

$$\int_0^1 \varphi(x)dx = 0, \quad \int_0^1 \psi(x)dx = 0, \tag{10}$$

$$\varphi(0) = h(0), \quad \psi(0) = h'(T). \tag{11}$$

Then the following statements are true:

1. Each classical solution $\{u(x,t), a(t)\}$ of problem (1)–(5) is also a solution of problem (1)–(3), (8), (9).

2. Each solution $\{u(x,t),a(t)\}$ of problem (1)–(3), (8), (9) is a classical solution of the problem (1)–(5) if

$$\frac{1}{2}\left\|a(t)\right\|_{C[0,T]}T^2<1\,, \tag{12}$$

it is a classical solution (1)–(5).

Proof. Let $\{u(x,t),a(t)\}$ be a classical solution to problem (2)–(5). Integrating equation (2), we get:

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx+u_{ttx}(1,t)-u_{ttx}(0,t)-(u_x(1,t)-u_x(0,t))=$$
$$=a(t)\int_0^1 u(x,t)dx+\int_0^1 f(x,t)dx \quad (0\le t\le T). \tag{13}$$

Suppose it is the case that $\int\limits_0^1 f(x,t)dx=0$ $(0\le t\le T)$, taking into account (3), (4), we identify

$$u_{ttx}(1,t)-u_x(1,t)=0 \quad (0\le t\le T). \tag{14}$$

Due to (2), $\varphi'(1)=0,\ \psi'(1)=0$, therefore

$$u_x(1,0)=\varphi'(1)=0,\quad u_{tx}(1,T)=\psi'(1)=0. \tag{15}$$

Obviously, problem (14), (15) has only a trivial solution, $u_x(1,t)=0$ $(0\le t\le T)$, i.e. conditions (8) are satisfied.

Considering $h(t)\in C^2[0,T]$ and differentiating (5) twice, we obtain:

$$u_{tt}(0,t)=h(t) \quad (0\le t\le T). \tag{16}$$

Further, from (2) we get:

$$\frac{d^2}{dt^2}u(0,t)+u_{ttxx}(0,t)-u_{xx}(0,t)=a(t)h(t)+f(0,t) \quad (0\le t\le T). \tag{17}$$

From (17), regarding to (5) and (16), we obtained (9).

Now, assume that $\{u(x,t),a(t)\}$ is a solution to problem (1)–(3), (8), (9), and (12) is satisfied. Then from (13), taking into account (3) and (8), we deduce:

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx-a(t)\int_0^1 u(x,t)dx=0 \quad (0\le t\le T). \tag{18}$$

As a corollary to (2) and (10), it is evident that

$$\int_0^1 u(x,0)dx=\int_0^1\varphi(x)dx=0,\quad \int_0^1 u_t(x,T)dx=\int_0^1\psi(x)dx=0. \tag{19}$$

Since, by virtue of Lemma 1, problem (18), (19) has only a trivial solution, then $\int\limits_0^1 u(x,t)dx=0$ $(0\le t\le T)$, i.e. condition (4) is satisfied.

Further, from (9) and (17):

$$\frac{d^2}{dt^2}(u(0,t)-h(t))=a(t)(u(0,t)-h(t)) \ (0\le t\le T). \tag{20}$$

From (2) and (11), we get:

$$u(0,0)-h(0)=\varphi(0)-h(0)=0. \tag{21}$$

From (20) and (21), based on Lemma 1, we conclude that condition (5) is fulfilled. The theorem is proven.

2 The Existence and uniqueness of the classical solution of the inverse boundary value problem

Component $u(x,t)$ of solution $\{u(x,t),a(t)\}$ of problem (1)–(3), (8), (9) is studied in the form:

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)\cos\lambda_k x \quad (\lambda_k=\pi k), \tag{22}$$

$u_k(t)=m_k\int_0^1 u(x,t)\cos\lambda_k xdx$ $(k=0,1,2,...)$, and

$$m_k=\begin{cases}1, & k=0,\\ 2, & k=1,2,\dots\end{cases}$$

By applying the scheme of the Fourier method from (2), we get:

$$(1-\lambda_k^2)u_k''(t)+\lambda_k^2u_k(t)=F_k(t;u,a) \quad (k=0,1,2,\dots;\ 0\le t\le T), \tag{23}$$

$$u_k(0)=\varphi_k,\ \ u_k'(T)=\psi_k \quad (k=0,1,2,\dots), \tag{24}$$

$$F_k(t;u,a)=f_k(t)+a(t)u_k(t),\ \ f_k(t)=m_k\int_0^1 f(x,t)\cos\lambda_k xdx,$$

$$\varphi_k=m_k\int_0^1\varphi(x)\cos\lambda_k xdx,\psi_k=m_k\int_0^1\psi(x)\cos\lambda_k xdx \quad (k=0,1,2,\dots).$$

Solving problem (1)–(6), we obtain:

$$u_0(t)=\varphi_0+\int_0^T m(t)u_0(t)dt+\psi_0t+\int_0^T G_0(t,\tau)F_0(\tau;u,a)d\tau, \tag{25}$$

$$u_k(t)\ =\ \frac{ch(\beta_k(T-t))}{ch(\beta_kT)}\varphi_k+$$

$$+\ \frac{sh(\beta_kt)}{\beta_kch(\beta_kT)}\psi_k-\frac{1}{(\lambda_k^2-1)\beta_k}\int_0^T G_k(t,\tau)F_k(\tau;u,a)d\tau \quad (k=1,2,\dots), \tag{26}$$

where

$$G_0(t,\tau)=\begin{cases}-t, & t\in[0,\tau],\\ -\tau, & t\in[\tau,T],\end{cases}$$

$$\beta_k^2=\frac{\lambda_k^2}{\lambda_k^2-1}>0,$$

$$G_k(t,\tau)=\begin{cases}-\frac{[sh(\beta_k(T+t-\tau))-sh(\beta_k(T-(t+\tau)))]}{2ch(\beta_kT)}, & t\in[0,\tau],\\[2mm] -\frac{sh(\beta_k(T-(t+\tau)))-sh(\beta_k(T-(t-\tau)))}{2ch(\beta_kT)}, & t\in[\tau,T].\end{cases}$$

After substituting the expression from (25), (26) into (22), to define a component $u(x,t)$ of the solution of problem (1)–(4), (8), (9), we obtain:

$$u(x,t)=\varphi_0+\psi_0t+\int_0^T G_0(t,\tau)F_0(\tau;u,a)d\tau\ +\sum_{k=1}^{\infty}\left\{\frac{ch(\beta_k(T-t))}{ch(\beta_kT)}\varphi_k+\right.$$

$$\left.+\ \frac{sh(\beta_kt)}{\beta_kch(\beta_kT)}\psi_k-\frac{1}{(\lambda_k^2-1)\beta_k}\int_0^T G_k(t,\tau)F_k(\tau;u,a)d\tau\right\}\cos\lambda_kx. \tag{27}$$

Now, from (8) including (22) and from (23) including (26), we get:

$$a(t) = [h(t)]^{-1}\left\{ h''(t) - f(0,t) - \sum_{k=1}^{\infty}\lambda_k^2(u_k''(t) - u_k(t))\right\}. \quad (28)$$

$$-\lambda_k^2 u_k''(t) + \lambda_k^2 u_k(t) = F_k(t;u,a) - u_k''(t) = \frac{\lambda_k^2}{\lambda_k^2-1}F_k(t;u,a) - \frac{\lambda_k^2}{\lambda_k^2-1}u_k(t) =$$

$$= \frac{\lambda_k^2}{\lambda_k^2-1}F_k(t;u,a) - \frac{\lambda_k^2}{\lambda_k^2-1}\left[\frac{ch(\beta_k(T-t))}{ch(\beta_k T)}\varphi_k + \right.$$

$$\left. + \frac{sh(\beta_k t)}{\beta_k ch(\beta_k T)}\psi_k - \frac{1}{(\lambda_k^2-1)\beta_k}\int_0^T G_k(t,\tau)F_k(\tau;u,a)d\tau\right] \quad (k=1,2,\dots). \quad (29)$$

Aimed at defining an equation for the second component $a(t)$ of the solution $\{u(x,t), a(t)\}$ of problem (1)–(3), (8), (9), we substitute expression (29) into (28):

$$a(t) = [h(t)]^{-1}\left\{ h''(t) - f(0,t) + \sum_{k=1}^{\infty}\left[\frac{\lambda_k^2}{\lambda_k^2-1}F_k(t;u,a) + \right.\right.$$

$$- \frac{\lambda_k^2}{\lambda_k^2-1}\left[\frac{ch(\beta_k(T-t))}{ch(\beta_k T)}\varphi_k + \frac{sh(\beta_k t)}{\beta_k ch(\beta_k T)}\psi_k - \right.$$

$$\left.\left.\left. - \frac{1}{(\lambda_k^2-1)\beta_k}\int_0^T G_k(t,\tau)F_k(\tau;u,a)d\tau\right]\right]\right\} \; (0 \le t \le T). \quad (30)$$

Thus, the solution of problem (1)–(3), (8), (9) is reduced to the solution of system (27), (30) with respect to unknown functions $u(x,t)$ and $a(t)$.

The subsequent lemma plays an essential role in the disquisition of the uniqueness question of the solution to problem (2)-(3), (8), (9).

Lemma 2. If $\{u(x,t), a(t)\}$ is any solution of problem (1)–(3), (8), (9), then the functions

$$u_k(t) = m_k\int_0^1 u(x,t)\cos\lambda_k x dx \quad (k=0,1,2,\dots)$$

satisfy the system consisting of equations (25), (26) on $[0,T]$.

Proof. Let $\{u(x,t), a(t)\}$ be any solution of problem (1)–(3), (8), (9). Then multiplying both sides of equation (2) by the function $m_k\cos\lambda_k x$ $(k=0,1,2,\dots)$, integrating the resulting equality over x from 0 to 1 and using $m_k\int_0^1 u_{tt}(x,t)\cos\lambda_k x dx = \frac{d^2}{dt^2}\left(m_k\int_0^1 u(x,t)\cos\lambda_k x dx\right) = u_k''(t)$ $(k=0,1,2,...)$,

$$m_k\int_0^1 u_{xx}(x,t)\cos\lambda_k x dx = -\lambda_k^2\left(m_k\int_0^1 u(x,t)\cos\lambda_k x dx\right) = -\lambda_k^2 u_k(t) \quad (k=0,1,2,\dots),$$

$$m_k\int_0^1 u_{ttxx}(x,t)\cos\lambda_k x dx = -\lambda_k^2 m_k\int_0^1 u_{tt}(x,t)\cos\lambda_k x dx =$$

$$= -\lambda_k^2\frac{d^2}{dt^2}\left(m_k\int_0^1 u(x,t)\cos\lambda_k x dx\right) = -\lambda_k^2 u_k''(t) \quad (k=0,1,2,\dots),$$

we obtain that Eq. (23) is satisfied.

Similarly, from (2) we obtain that condition (24) is fulfilled.

Thus, $u_k(t)$ $(k=0,1,2,\dots)$ is a solution to problem (23), (24). Moreover, it directly follows that the functions $u_k(t)$ $(k=0,1,2,\dots)$ satisfy the system (25), (26) on $[0,T]$. The lemma is proven.

Therefore, if $u_k(t) = 2\int\limits_0^1 u(x,t)\cos\lambda_k x dx\ \ (k=0,1,2,\dots)$ is a solution to system (25), (26), then the pair $\{u(x,t),a(t)\}$ of functions $u(x,t) = \sum\limits_{k=0}^{\infty} u_k(t)\cos\lambda_k x, a(t)$ and $a(t)$ is a solution to system (27), (30).

Corollary 1. Let system (27), (30) have a unique solution. Then problem (1)–(3), (8), (9) can not have more than one solution, i.e. if problem (1)–(3), (8), (9) has a solution, then it is unique.

To study the problem (1)–(3), (8), (9), we introduce two spaces.

By $B_{2,T}^{\alpha}$ [8], we denote the set of all functions of the form

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t)\cos\lambda_k x\ \ (\lambda_k = k\pi),$$

considered in D_T, where each function $u_k(t)\ \ (k=0,1,\dots)$ is continuous on $[0,T]$ and

$$J(u) = \|u_0(t)\|_{C[0,T]} + \left\{\sum_{k=1}^{\infty}\left(\lambda_k^{\alpha}\|u_k(t)\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}} < +\infty,$$

where $\alpha \geq 0$. We define the norm in this set as follows:

$$\|u(x,t)\|_{B_{2,T}^{\alpha}} = J(u).$$

E_T^{α} denotes the space $B_{2,T}^{\alpha}\times C[0,T]$ with vector functions $z(x,t) = \{u(x,t),a(t)\}$ and norm

$$\|z(x,t)\|_{E_T^{\alpha}} = \|u(x,t)\|_{B_{2,T}^{\alpha}} + \|a(t)\|_{C[0,T]}.$$

$B_{2,T}^{\alpha}$ and E_T^{α} are Banach spaces.

In the space E_T^3, we define an operator:

$$\Phi(u,a,b) = \{\Phi_1(u,a),\Phi_2(u,a)\},$$

with

$$\Phi_1(u,a) = \tilde{u}(x,t) \equiv \sum_{k=0}^{\infty}\tilde{u}_k(t)\cos\lambda_k x,\quad \Phi_2(u,a,b) = \tilde{a}(t),$$

$\tilde{u}_0(t),\tilde{u}_k(t)\ \ (k=1,2,\dots)$ and $\tilde{a}(t)$ are equal to the right-hand sides of (25), (26) and (30).

It is easy to see that

$$\frac{ch(\beta_k(T-t))}{ch(\beta_k T)} < 1,\quad \frac{sh(\beta_k t)}{ch(\beta_k T)} < 1,\quad \frac{sh(\beta_k(T+t-\tau))}{ch(\beta_k T)} < 1\ \ (t\in[0,\tau]),$$

$$\frac{sh(\beta_k(T-(t+\tau)))}{ch(\beta_k T)} < 1,\quad \frac{sh(\beta_k(T-(t-\tau)))}{ch(\beta_k T)} < 1\ \ (t\in[\tau,T]),$$

$$\lambda_k^2 - 1 > \frac{1}{2}\lambda_k^2,\quad 1 < \beta_k = \frac{\lambda_k}{\sqrt{\lambda_k^2-1}} < \sqrt{2},\quad \frac{1}{\sqrt{2}} < \frac{1}{\beta_k} < 1.$$

Then, we have:

$$\|\tilde{u}_0(t)\|_{C[0,T]} \le |\varphi_0| + T\,|\psi_0| + 2T\sqrt{T}\left(\int_0^t |f_0(\tau)|^2\,d\tau\right)^{\frac{1}{2}} + \; 2T^2\,\|a(t)\|_{\;C[0,T]}\,\|u_0(t)\|_{C[0,T]}\,, \tag{31}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|\tilde{u}_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le 2\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\varphi_k|)^2\right)^{\frac{1}{2}} + 2\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\psi_k|)^2\right)^{\frac{1}{2}} +$$

$$+\,4\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k\,|f_k(\tau)|)^2\right)^{\frac{1}{2}} + 4T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}, \tag{32}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \le \left\|[h(t)]^{-1}\right\|_{C[0,T]}\Big\{\left\|h''(t) - f(0,t)\right\|_{C[0,T]} +$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left(\sum_{k=1}^{\infty}(\lambda_k\,\|f_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} + \;\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} +\right.$$

$$+\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\varphi_k|)^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\psi_k|)^2\right)^{\frac{1}{2}} + 2\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k\,|f_k(\tau)|)^2\right)^{\frac{1}{2}} +$$

$$\left.+\; 2T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\Bigg\}. \tag{33}$$

Let us assume that the data of problem (1)–(3), (8), (9), satisfy the following conditions:
1) $\varphi(x) \in C^2[0,1],\; \varphi'''(x) \in L_2(0,1),\; \varphi'(0) = \varphi'(1) = 0;$
2) $\psi(x) \in C^2[0,1],\; \psi'''(x) \in L_2(0,1),\; \psi'(0) = \psi'(1) = 0;$
3) $f(x,t) \in C(D_T),\; f_x(x,t) \in L_2(D_T),\; f_x(0,t) = f_x(1,t) = 0\;\; (0 \le t \le T);$
4) $h(t) \in C^2[0,T],\; h(t) \ne 0\;\; (0 \le t \le T).$

Then from (31)–(33), we have:

$$\|\tilde{u}(x,t)\|_{B_{2,T}^3} \le A_1(T) + B_1(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B_{2,T}^3}\,, \tag{34}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \le\; A_2(T) +\; B_2(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B_{2,T}^3} +$$

$$+\,B_2(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B_{2,T}^3} + C_2(T)\,\|u(x,t)\|_{B_{2,T}^3 + D_2(T)\|a(t)\|_{C[0,T]}}\,, \tag{35}$$

where

$$A_1(T) = \|\varphi(x)\|_{L_2(0,1)} + T\,\|\psi(x)\|_{L_2(0,1)} + 2T\sqrt{T}\,\|f(x,t)\|_{L_2(D_T)} +$$

$$+2\left\|\varphi'''(x)\right\|_{L_2(0,1)} + 2\left\|\psi'''(x)\right\|_{L_2(0,1)} + 4\sqrt{T}\,\|f_x(x,t)\|_{L_2(D_T)}\,,$$

$$B_1(T) = 6T,\quad A_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\Big\{\left\|h''(t) - f(0,t)\right\|_{C[0,T]} +$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left\|\|f_x(x,t)\|_{C[0,T]}\right\|_{L_2(0.1)} + \left\|\varphi'''(x)\right\|_{L_2(0,1)} + \left\|\psi'''(x)\right\|_{L_2(0,1)} +\right.$$

$$\left.+\; 2\sqrt{T}\,\|f_x(x,t)\|_{L_2(D_T)}\right]\Big\},$$

$$B_2(T) = 2\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}(2T+1).$$

From inequalities (34), (35) we conclude:

$$\|\tilde{u}(x,t)\|_{B_{2,T}^3} + \|\tilde{a}(t)\|_{C[0,T]} \le A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B_{2,T}^3}\,, \tag{36}$$

where

$$A(T) = A_1(T) + A_2(T), \quad B(T) = B_1(T) + B_2(T).$$

Theorem 2. Let conditions 1–3 be satisfied and

$$B(T)(A(T)+2)^2 < 1. \tag{37}$$

Then, in $K = K_R$ ($\|\, z\,\|_{E_T^3} \le R = A(T)+2$), in the space E_T^3, problem (1)–(3), (8), (9) has only one solution.

Proof. In the space E_T^3 consider the equation

$$z = \Phi z, \tag{38}$$

where $z = \{u, a\}$ and components $\Phi_i(u,a)$ $(i=1,2)$ of operators $\Phi(u,a)$ are defined by the right-hand sides of equations (27) and (30).

Now, consider the operator $\Phi(u,a)$ in the ball $K = K_R$ of the space E_T^3. Analogously to (36), we obtain that for any $z, z_1, z_2 \in K_R$, the following estimates hold:

$$\|\Phi z\|_{E_T^3} \le A\,(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B_{2,T}^3}\,, \tag{39}$$

$$\|\Phi z_1 - \Phi z_s\|_{E_T^3} \le 2B(T)R\,(\|a_1(t)-a_2(t)\|_{C[0,T]} + \|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^3}). \tag{40}$$

As a result, the operator (38) has a unique solution for $K = K_R$.

Then, from estimates (39) and (40), allowing for (37), it follows that the operator Φ acts in the ball $K = K_R$ and is contractive.

Functions $u(x,t)$, as an element of space $B_{2,T}^3$, are continuous and have continuous derivatives $u_x(x,t)$ and $u_{xx}(x,t)$ in $\bar{D}_T$.

From (23), it is evident that

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u_k''(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le \sqrt{2}\left\{\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} + \right.$$

$$\left. + \left\|\,\|a(t)u_x(x,t) + f_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}\right\}.$$

Hence, it follows that $u_{tt}(x,t)$,$u_{ttx}(x,t)$,$u_{ttxx}(x,t)$ are continuous in $\bar{D}_T$.

It is easy to check that equation (1) and conditions (3), (4), (7), (9) are satisfied in the usual sense. Consequently, $\{u(x,t), a(t)\}$ is a solution to problem (1)–(3), (7)–(9). By the corollary of Lemma 1, it is unique in the ball $K = K_R$. The theorem has been proven.

In the proof of Theorem 1 and Theorem 2, the next Theorem plays, an essential role of unique and solvability of the problem (1)–(5).

Theorem 3. Let all the conditions of Theorem 2 and (10), (11) be satisfied, $\int_0^1 f(x,t)dx = 0$ $(0 \le t \le T)$, and

$$\frac{1}{2}(A(T)+2)\,T^2 < 1\,.$$

Then, problem (1)–(5) has the only classical solution in the ball $K = K_R$ ($\|z\|_{E_T^3} \le A(T)+2$) from E_T^3.

Conclusion

An inverse boundary value problem for a linearized equations of longitudinal waves in rods with integral condition of the first kind are studied.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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