https://doi.org/10.31489/2025M1/46-52 *Research article*

On the solvability of the Goursat problem for one class of loaded second-order hyperbolic equations

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In this paper the solvability of the Goursat problems for two locally loaded second-order hyperbolic equations with a wave operator in the main part was explored. The loaded terms for both equations have the same trace, namely a part of one of the characteristics for the given hyperbolic operator, but the trace-forming maps are different. Moreover, in the first case, any point of the domain under consideration and the corresponding point of the load trace always lie on a straight line, which is a characteristic. In the second case, this does not work. It turns out that in the first case the Goursat problem is Voltairean, and in the second case it is Fredholmian.

Keywords: loaded equation, Goursat problem, characteristics of a hyperbolic equation, Volterra equation of the second kind, Fredholm equation of the second kind with a spectral parameter, eigenvalue, eigenfunction.

2020 Mathematics Subject Classification: 35L10.

Introduction

Mathematical modeling of many very important processes in various fields of natural science leads to boundary value problems for loaded partial differential equations. The main questions arising in the theory of boundary value problems for ordinary partial differential equations remain the same for boundary value problems for loaded partial differential equations. However, loads in equations makes their own adjustments to the formulation of the study and the correctness of a particular boundary value problem. For the first time, mentioning of a load eliminating the inequality of characteristics in the second Darboux problem for one second order degenerate hyperbolic equation belongs to A.M. Nakhushev in [1; 87]. Fundamental results on the effects of load, including spectral ones, for significantly loaded parabolic equations of arbitrary order were obtained by M.T. Dzhenaliev and M.I. Ramazanov. The main results in this direction are presented in their joint monograph [2]. Examples of the application of loaded differential equations, the main results of research in this field, as well as a numerous references on this subject are given in monographs [1–3].

We note works [4–9] devoted to the Goursat problem for loaded hyperbolic equations. The works [10–20] are devoted to boundary value problems for equations of the mixed type, boundary value problems with periodic boundary conditions, initial value problems and boundary control problems for loaded partial differential equations with both characteristic and non-characteristic load.

In this paper, research objects are two loaded second-order hyperbolic equations with a wave operator in the main part

$$Lu = \lambda u\left(\frac{\alpha(x+ky)+l}{2}, \frac{\alpha(x+ky)-l}{2}\right), \tag{1}$$

where

$$Lu = u_{xx} - u_{yy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u, \quad k = 1, -1, \quad \lambda = const.$$

It should be noted that the traces of the equations for $y = 0$ coincide.

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1 Main part

Assume $\Omega = \{(x,y) : 0 < x - y < l, 0 < x + y < l\}$.

Goursat problem. In the domain Ω, find a solution to the equation (1) by class $C(\bar{\Omega}) \cap C^2(\Omega)$, satisfying the boundary conditions

$$u\left(\frac{x}{2}, -\frac{x}{2}\right) = \varphi(x), \quad 0 \leq x \leq l, \tag{2}$$

$$u\left(\frac{x}{2}, \frac{x}{2}\right) = \psi(x), \quad 0 \leq x \leq l, \tag{3}$$

where $\bar{\Omega}$ is the boundary for Ω.

It is assumed that

$$\alpha(x) \leq x,\ \alpha(0) = 0,\ \alpha(l) = l,\ 0 \leq x \leq l, \tag{4}$$

$$a, b \in C^1(\Omega),\ \ c \in C(\bar{\Omega}),\ \ \varphi, \psi, \alpha \in C(\bar{J}) \cap C^2(J), \tag{5}$$

where $\bar{J}$ is the boundary for the interval $J = (0, l)$.

Goursat problem (2), (3) for the equation (1) is studied separately for $k = 1$ and $k = -1$.

2 Case $k = 1$

Theorem 1. Goursat problem (2), (3) for equation (1) at $k = 1$ is solvable in a unique way.

First, we present the proof of the theorem for $a \equiv b \equiv c \equiv 0$, $\alpha(x+y) = x + y$, that is, for the model equation

$$u_{xx} - u_{yy} = \lambda u\left(\frac{x+y+l}{2}, \frac{x+y-l}{2}\right) \tag{6}$$

with the characteristic variables $\xi = x - y$, $\eta = x + y$, equation (6) and boundary conditions (2), (3) take the form

$$v_{\xi\eta} = \frac{\lambda}{4} v(l, \eta), \tag{7}$$

$$v(\xi, 0) = \varphi(\xi),\ 0 \leq \xi \leq l, \tag{8}$$

$$v(0, \eta) = \psi(\eta),\ 0 \leq \eta \leq l, \tag{9}$$

where $v(\xi, \eta) = u\left(\frac{\eta+\xi}{2}, \frac{\eta-\xi}{2}\right)$.

The domain Ω goes into the rectangular domain $\Omega_0 = \{(\xi, \eta) : 0 < \xi < l, 0 < \eta < l\}$.

Suppose there is a solution to Goursat problem (8), (9) for equation (7), then it is easy to see that $v(\xi, \eta)$ is a solution to the following loaded integral equation

$$v(\xi, \eta) - \frac{\lambda}{4} \xi \int_0^\eta v(l, t) dt = \varphi(\xi) + \psi(\eta) - \varphi(0). \tag{10}$$

Assuming $\xi = l$ by (10) we get

$$\int_0^\eta v(l, t) dt = \int_0^\eta e^{\frac{\lambda}{4}(\eta - t)} [\varphi(l) - \varphi(0) + \psi(t)] dt.$$

Substituting the resulting value for the integral into (10) and returning to the original variables, we have

$$u(x,y) = \varphi(x-y) + \psi(x+y) - \varphi(0) + \frac{\lambda}{4}(x-y)\int_0^{x+y} e^{\frac{\lambda}{4}(x+y-t)}[\varphi(l) - \varphi(0) + \psi(t)]dt.$$

Let's go back to equation (1) with $k = 1$. In the characteristic variables $\xi = x - y$, $\eta = x + y$ equation (1) is written as follows

$$L_1 v \equiv \frac{\lambda}{4} v(l, \alpha(\eta)), \tag{11}$$

where

$$L_1 v \equiv v_{\xi\eta} + A(\xi,\eta)v_\xi + B(\xi,\eta)v_\eta + C(\xi,\eta)v, \quad a+b = 4A,\ a-b = 4B,\ c = 4C.$$

Condition (5) guarantees the Riemann function $R(\xi_0, \eta_0; \xi, \eta)$ for equation (11). Assume that the right side of the equation is known. Applying the well-known formula [21] for solving the Goursat problem, to find $v(\xi,\eta)$ we obtain the following integral equation

$$v(\xi,\eta) = R(\xi,0;\xi,\eta)\varphi(\xi) + R(0,\eta;\xi,\eta)\psi(\eta) - R(0,0;\xi,\eta)\varphi(0)+$$

$$+\int_0^\xi \left[B(t,0)R(t,0;\xi,\eta) - \frac{\partial}{\partial t}R(t,0;\xi,\eta)\right]\varphi(t)dt+$$

$$+\int_0^\eta \left[A(0,\tau)R(0,\tau;\xi,\eta) - \frac{\partial}{\partial \tau}R(0,\tau;\xi,\eta)\right]\psi(\tau)d\tau+$$

$$+\frac{\lambda}{4}\int_0^\xi dt \int_0^\eta R(t,\tau;\xi,\eta)v(l,\alpha(\tau))d\tau. \tag{12}$$

Changing the order of integration in the double integral and passing to the limit at $\xi \to l$ in (12), to find $v(l,t)$, we obtain the following integral equation

$$v(l,\eta) = \frac{\lambda}{4}\int_0^\eta K(\eta,\tau)v(l,\alpha(\tau))d\tau = F(\eta), \tag{13}$$

$$K(\eta,\tau) = \int_0^l R(t,\tau;l,\eta)dt,$$

$$F(\eta) = \int_0^l \left[B(t,0)R(t,0;l,\eta) - \frac{\partial}{\partial \tau}R(t,0;l,\eta)\right]\varphi(t)dt+$$

$$+\int_0^l \left[A(0,\tau)R(t,\tau;l,\eta) - \frac{\partial}{\partial \tau}R(0,\tau;l,\eta)\right]\psi(\tau)d\tau+$$

$$+R(l,0;l,\eta)\varphi(l) + R(0,\eta;l,\eta)\psi(\eta) - R(0,0;l,\eta)\varphi(0).$$

Substituting $\alpha(\eta)$ into (13) instead of η, we obtain

$$v(l,\alpha(\eta)) - \frac{\lambda}{4}\int\limits_0^{\alpha(\eta)} K(\alpha(\eta),\tau)v(l,\alpha(\tau))d\tau = F(\alpha(\eta)).$$

Condition (4) guarantees the existence of a unique solution $v(l,\alpha(\eta)) \in C(\bar{J})$. After finding $v(l,\alpha(\eta))$, a unique solution to Goursat problem (2), (3) for equation (1) at $k=1$ in the domain Ω due to (12) is given by the formula

$$u(x,y) = u_0(x,y) + \frac{\lambda}{4}\int\limits_0^{\xi}\int\limits_0^{\eta} R(t,\tau;\xi,\eta)v(l,\alpha(\tau))d\tau dt,$$

where $u_0(x,y)$ is a solution to the same problem for equation (1) at $\lambda = 0$.

3 The case with $k=-1$

Theorem 2. Assume $|\lambda| < \frac{4}{Ml}$, where $M = \max_{\bar{\Omega}_0}\left|\int\limits_0^{\alpha(s)} R(t,\tau;l,\alpha(s))d\tau\right|$, $R(\xi_0,\eta_0;\xi,\eta)$ is the Riemann operator for L_1, then Goursat problem (2), (3) for equation (1) at $k=-1$ is solvable in a unique way.

Proof of Theorem 2. In the characteristic variables $\xi = x-y$, $\eta = x+y$ equation (1) for $k=-1$ takes the form

$$L_1 v \equiv \frac{\lambda}{4} v(l,\alpha(\xi)).$$

Applying the same reasoning as for $k=1$, to find $v(l,\alpha(\xi))$, we obtain the following Fredholm integral equation with the spectral parameter

$$v(l,\alpha(\xi)) - \lambda\int\limits_0^{l} K(\alpha(\xi),t)v(l,\alpha(t))dt = F(\alpha(\xi)), \tag{14}$$

where

$$K(\alpha(\xi),t) = \frac{1}{4}\int\limits_0^{\alpha(\xi)} R(t,\tau;l,\alpha(\xi))d\tau.$$

By virtue of continuity $K(\alpha(\xi),t)$ and $F(\alpha(\xi))$ respectively in $\bar{\Omega}_0$ and $\bar{J}$, in $\bar{\Omega}_0$ for $|K(\alpha(\xi),t)|$, we get some maximum value M, $|F(\alpha(\xi))|$ has some maximum value M_1.

Under these conditions, solution (14) can be obtained, for example, by the method of successive substitutions in the form of an absolutely and uniformly convergent series.

Let us now consider the case when $a \equiv b \equiv c \equiv 0$, $\alpha(x-y) = x-y$, that is, the Goursat problem (2), (3) for the equation

$$u_{xx} - u_{yy} = \lambda u\left(\frac{x-y+l}{2}, \frac{x-y-l}{2}\right). \tag{15}$$

Theorem 3. Goursat problem (2), (3) for equation (15) is solvable in a unique way for all $\lambda \neq \frac{8}{l^2}$.

In the case when $\lambda = \frac{8}{l^2}$:

1) The homogeneous problem corresponding to problem (2), (3) for equation (13) has an infinite number of solutions

$$u(x,y) = C(x+y)(x-y)^2,$$

where C is the arbitrary constant.

2) An inhomogeneous problem is solvable if and only if $\int\limits_0^l [\varphi(l)-\varphi(0)+\psi(t)]dt = 0$. If this condition is satisfied, it also has infinitely many solutions.

Proof of Theorem 3. With the characteristic variables $\xi = x - y$, $\eta = x + y$ equation (15) takes the form

$$v_{\xi\eta} = \frac{\lambda}{4} v(l,\xi). \tag{16}$$

Integrating (16) over ξ ranging from 0 to ξ, and then over η ranging from 0 to η, we verify that $v(\xi,\eta)$ is the solution to the following loaded integral equation

$$v(\xi,\eta) - \frac{\lambda}{4}\eta \int\limits_0^\xi v(l,t)dt = \varphi(\xi) + \psi(\eta) - \varphi(0).$$

Replacing ξ by l and η by ξ in the last relation, we obtain

$$v(l,\xi) - \frac{\lambda}{4}\xi \int\limits_0^l v(l,t)dt = \varphi(l) + \psi(\xi) - \varphi(0). \tag{17}$$

Equation (17) is the simplest integral equation with a spectral parameter and a degenerate kernel.

Since the degenerate kernel consists of one term, the corresponding system becomes a single equation

$$q_1 = \frac{\lambda}{4}\left(\int\limits_0^l tdt\right) q_1 + \int\limits_0^l \left[\varphi(l) + \psi(t) - \varphi(0)\right] dt,$$

where $q_1 = \int\limits_0^l v(l,t)dt$, then the result of the Theorem 3 follows directly.

Conclusion

If we consider the Goursat problem

$$u\left(\frac{x}{2},\frac{x}{2}\right) = \varphi(x),\ u\left(\frac{l+x}{2},\frac{l-x}{2}\right) = \psi(x),\ 0 \le x \le l \tag{18}$$

for equation (15), then it is obvious that equation (17) takes the form

$$v(l,\xi) - \frac{\lambda}{4}\int\limits_0^l (\xi - l)v(l,t)dt = \varphi(l) + \psi(\xi) - \varphi(0),$$

whence it follows that it is uniquely solvable for all $\lambda \neq -\frac{l^2}{8}$. We find that for $\lambda = \frac{l^2}{8}$ Goursat problem (2), (3) for equation (15) is not correct, and Goursat problem (18) for of the same equation is correct, but for $\lambda = -\frac{l^2}{8}$, vice versa.

As a consequence, equation (15) can be called an example of an equation for which the effect of inequality of characteristics as carriers of Goursat data takes place.

Conflict of Interest

The author declares no conflict of interest.

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