https://doi.org/10.31489/2024M1/194-207 *Research article*

A Novel Numerical Scheme for a Class of Singularly Perturbed Differential-Difference Equations with a Fixed Large Delay

E. Srinivas, K. Phaneendra*

University College of Science, Osmania University, Hyderabad, India
(E-mail: srinivas.erla@gmail.com, kollojuphaneendra@yahoo.co.in)

A trigonometric spline based computational technique is suggested for the numerical solution of layer behavior differential-difference equations with a fixed large delay. The continuity of the first order derivative of the trigonometric spline at the interior mesh point is used to develop the system of difference equations. With the help of singular perturbation theory, a fitting parameter is inserted into the difference scheme to minimize the error in the solution. The method is examined for convergence. We have also discussed the impact of shift or delay on the boundary layer. The maximum absolute errors in comparison to other approaches in the literature are tallied, and layer behavior is displayed in graphs, to demonstrate the feasibility of the suggested numerical method.

Keywords: singularly perturbed differential-difference equation, delay, trigonometric spline, fitting parameter.

2020 Mathematics Subject Classification: 65L11, 65L12.

Introduction

Delay differential equations (DDEs) are frequently encountered in a wide range of application disciplines and are also explained in technological components like control circuits. DDEs are widely occurred in various branches of physiological control systems [1], models of red blood cell system [2], pupil light reflex behaviour [3], hybrid optically bistable devices with delayed feedback [4] and the navigational control of ships and aircraft and in more general control problems [5]. Time delays are virtually always present in systems with feedback controls. These happen because detecting information and responding to it both take time. If the argument for the delay does not appear in the highest order term, the DDE is of the retarded type. Delay differential equations of the retarded type are obtained by restricting the class in which the highest order derivative term is multiplied by a small parameter. Bender and Orszag [6], O'Malley [7], Doolan and Miller [8], Miller et al. [9], Roos et al. [10] have written books detailing several approaches to addressing singularly perturbed problems (SPPs). Driver [11], Bellman and Cooke [12], provided books that explained differential-difference equations. In [13], the researchers elucidated analysis of a class of singularly perturbed differential-difference equations [SPDDEs]. In [14], problems with solutions having a layer structure at one or both of the boundaries are addressed. The layer can alter its nature and possibly be destroyed when the shifts rise but stay small, as demonstrated by the study of the layer equations using Laplace transforms. The same researchers handle two situations in [15]. The first is concerned with the magnitude of the shifts that affect the solution, while the second is concerned with the SPDE's oscillatory solutions. Kadalbajoo and Sharma [16], provided a numerical procedure for solving SPDE with larger or smaller delay argument. To handle the delay term, a mesh is generated so that the delay term falls on nodal points. Kadalbajoo et al. [17], utilize Shishkin mesh to derive the fitted mesh approach to solve singularly perturbed general DDEs. Gabil and Erkan [18], devised a fitted difference scheme for convection-diffusion problems by employing exponential basis functions, integral identities and interpolating quadrature procedures. The authors

*Corresponding author. *E-mail: kollojuphaneendra@yahoo.co.in*
Received: 07 June 2023; *Accepted:* 07 December 2023.

in [19], suggested adaptive grid methods for the solutions to problems with boundary or interior layers. A grid with equidistributing arc-length monitor function is constructed to solve the problem. In [20], a first-order uniform convergence fitted difference approach is built in the discrete maximum norm. Ravi Kanth and Murali [21], devised a numerical scheme for solving nonlinear SPDE. The Quasilinearization technique is implemented on the nonlinear SPDE to get a sequence of linear SPDEs. A fitted spline method is implemented for the solution of these problems. In [22], it is established that the family of periodic boundary value issues for the system of ordinary differential equations with delayed argument and the periodic boundary value problem for the system of hyperbolic equations with delayed argument are related. The construction and convergence of algorithms for solving the comparable problem are demonstrated. The author of [23] investigated a boundary value problem with the Sturm-Liouville type conditions using Green's function method for a linear ordinary differential equation of fractional order with delay. In [24], authors proposed a scheme for the solution of a differential equation with delay and advanced parameters having an interior layer behaviour using a non-standard finite difference method.

1 Statement of problem

Consider the following SPDDE with a fixed delay

$$\varepsilon z''(\vartheta)+p\left(\vartheta\right)z'\left(\vartheta\right)+q\left(\vartheta\right)z\left(\vartheta-1\right)=f\left(\vartheta\right),\ \vartheta\in[0,2] \tag{1}$$

subject to the boundary constraints

$$z\left(\vartheta\right)=\varphi\left(\vartheta\right),\ \vartheta\in\left[-1,0\right];$$

$$z\left(2\right)=\beta, \tag{2}$$

where, $0<\varepsilon\ll 1$ and $p\left(\vartheta\right)\geqslant\alpha>0,\ \theta\leqslant q\left(\vartheta\right)<0$ and $f\left(\vartheta\right)$ are smooth functions on $[0,2]$, $\varphi\left(\vartheta\right)$ is smooth functions on $[-1,0]$ and β is given constant. The solution of Eq. (1) with Eq. (2) reveals a boundary layer at $\vartheta=2$ with the small values of ε.

2 Numerical method using a trigonometric spline

The domain of the integration $[0\,,\,2]$ is partitioned into L equal sub intervals with mesh length $h=\frac{2}{L}$, so that $\vartheta_i=ih,\ i=0,1,2,...,L$ are the nodes with $0=\vartheta_0,\ 2=\vartheta_L$. Let $z\left(\vartheta\right)$ be the exact solution and ϑ_i be an approximation to $z(\vartheta_i)$ by the trigonometric spline $S_i(\vartheta)$ passing through the points $(\vartheta_i,\ z_i)$ and $(\vartheta_{i+1},\ z_{i+1})$. Here $S_i(\vartheta)$ satisfies the conditions of interpolation at ϑ_i and ϑ_{i+1} and also the first order derivative continuity at the common nodes $(\vartheta_i,\ z_i)$ is satisfied. For each i^{th} subinterval, the trigonometric spline function $S_i(\vartheta)$ has the form

$$S_i\left(\vartheta\right)=\ a_i\ +b_i\ \left(\vartheta\ -\ \vartheta_i\ \right)+c_i\sin\tau\left(\vartheta\ -\ \vartheta_i\ \right)+d_i\,cos\tau\ \left(\vartheta\ -\ \vartheta_i\ \right)\ ,\ i\ =\ 0,\,1,\ldots,\,L\ -1. \tag{3}$$

Here a_i, b_i, c_i and d_i are constants and τ is a free parameter.

The trigonometric spline $S_i(\vartheta)$ of class $C^2[0\,,\,2]$ interpolating $z\left(\vartheta\right)$ at the points $\vartheta_i,\ i=0,1,...,L$ depends on τ and deduces to cubic spline in $[0,\,2]$ as $\tau\to 0$. The following are defined to obtain an expression for the coefficients of Eq. (3) in terms of z_i, z_{i+1}, ψ_i and ψ_{i+1}

$$S_i\left(\vartheta_i\right)=z_i,\quad S_i\left(\vartheta_{i+1}\right)=z_{i+1},\quad S_i^{''}\left(\vartheta_i\right)=\psi_i,\quad S_i^{''}\left(\vartheta_{i+1}\right)=\psi_{i+1}.$$

Using these conditions, the following expressions are obtained:

$$a_i=\ z_i+\frac{\psi_i}{\tau^2}\ ,\ b_i=\ \frac{z_i-z_{i+1}}{h}+\frac{\psi_{i+1}-\psi_i}{\tau\theta},$$

$$c_i = \frac{\psi_i cos\theta \ - \psi_{i+1}}{\tau^2 sin\theta} \ \text{ and } d_i = -\frac{\psi_i}{\tau^2},$$

where $\theta = \tau h$, for $i = 0, 1, \ldots, L-1$. Using the continuity of the first order derivative at $(\vartheta_i,\, z_i)$, that is $S^{'}_{i+1}(\vartheta_i) = S^{'}_i(\vartheta_i)$, we get the following relation for $i = 1, 2, \ldots, L-1$.

$$\alpha\psi_{i+1} + 2\beta\psi_i + \alpha\psi_{i-1} = \frac{z_{i-1} - 2z_i + z_{i+1}}{h^2}, \tag{4}$$

where

$$\alpha = \frac{-1}{\theta^2} + \frac{1}{\theta sin\theta} \ \text{ and } \ \beta = \frac{1}{\theta^2} - \frac{cos\theta}{\theta sin\theta},$$
$$\psi_j = z^{''}(\vartheta_j)\,,\; j = \; i \pm 1,\; i.$$

At the mesh point ϑ_j, the suggested approach can be discretized by the convection-diffusion equation (1) as

$$\psi_j = \frac{1}{\varepsilon}\Big(f(\vartheta_j) - p(\vartheta_j)\, z^{'}(\vartheta_j) - q(\vartheta_j)\, z(\vartheta_j - 1)\Big) \text{ for } j = \; i \pm 1,\; i \tag{5}$$

using Eq. (5), Eq. (4) can be represented as

$$\frac{\alpha}{\varepsilon}\Big(f(\vartheta_{i+1}) - \mathrm{p}(\vartheta_{i+1})\, z^{'}(\vartheta_{i+1}) - \mathrm{q}(\vartheta_{i+1})\, z(\vartheta_{i+1}-1)\Big) +$$
$$+\frac{2\beta}{\varepsilon}\Big(f(\vartheta_i) - \mathrm{p}(\vartheta_i)\, z^{'}(\vartheta_i) - \mathrm{q}(\vartheta_i)\, z(\vartheta_i - 1)\Big) +$$
$$+\frac{\alpha}{\varepsilon}\Big(f(\vartheta_{i-1}) - \mathrm{p}(\vartheta_{-1})\, \mathrm{z}^{'}(\vartheta_{j-1}) - \mathrm{q}(\vartheta_{i-1})\, z(\vartheta_{i-1}-1)\Big) = \left(\frac{z_{i-1} - 2z_i + z_{i+1}}{h^2}\right),$$
$$\alpha\Big(f(\vartheta_{i+1}) - \mathrm{p}(\vartheta_{i+1})\, z^{'}(t_{i+1}) - \mathrm{q}(\vartheta_{i+1})\, z(\vartheta_{i+1}-1)\Big) +$$
$$+2\beta\Big(f(\vartheta_i) - \mathrm{p}(\vartheta_i)\, z^{'}(\vartheta_i) - \mathrm{q}(\vartheta_i)\, z(\vartheta_i - 1)\Big) +$$
$$+\alpha\Big(f(\vartheta_{i-1}) - \mathrm{p}(\vartheta_{i-1})\, z^{'}(\vartheta_{i-1}) - \mathrm{q}(\vartheta_{i-1})\, z(\vartheta_{i-1}-1)\Big) = \frac{\varepsilon}{h^2}(z_{i-1} - 2z_i + z_{i+1})\,. \tag{6}$$

Using the finite differences

$$z^{'}(\vartheta_{i+1}) = \left(\frac{z_{\mathrm{i}-1} - 4z_{\mathrm{i}} - 3z_{\mathrm{i}+1}}{2\mathrm{h}}\right),\; z^{'}(\vartheta_i) = \left(\frac{z_{\mathrm{i}+1} - z_{\mathrm{i}-1}}{2\mathrm{h}}\right),$$
$$z^{'}(\vartheta_{i-1}) = \left(\frac{z_{\mathrm{i}+1} - 4z_{\mathrm{i}} - 3z_{\mathrm{i}-1}}{2\mathrm{h}}\right).$$

Eq. (6) is reduced to

$$\alpha\left(\mathrm{f}(\vartheta_{\mathrm{i}+1}) - \mathrm{p}(\vartheta_{\mathrm{i}+1})\left(\frac{z_{\mathrm{i}-1} - 4z_{\mathrm{i}} - 3z_{\mathrm{i}+1}}{2\mathrm{h}}\right) - \mathrm{q}(\vartheta_{\mathrm{i}+1})\, z(\vartheta_{i+1}-1)\right) +$$
$$+2\beta\left(\mathrm{f}(\vartheta_{\mathrm{i}}) - \mathrm{p}(\vartheta_{\mathrm{i}})\left(\frac{z_{\mathrm{i}+1} - z_{\mathrm{i}-1}}{2\mathrm{h}}\right) - \mathrm{q}(\vartheta_{\mathrm{i}})\; z(\vartheta_i - 1)\right) +$$
$$+\alpha\left(\mathrm{f}(\vartheta_{\mathrm{i}-1}) - \mathrm{p}(\vartheta_{\mathrm{i}-1})\left(\frac{z_{\mathrm{i}+1} - 4z_{\mathrm{i}} - 3z_{\mathrm{i}-1}}{2\mathrm{h}}\right) - \mathrm{q}(\vartheta_{\mathrm{i}-1})\; z(\vartheta_{i-1}-1)\right) =$$
$$= \frac{\varepsilon}{h^2}(z_{\mathrm{i}-1} - 2z_{\mathrm{i}} + z_{\mathrm{i}+1})\,,$$

$$\left(\frac{\varepsilon}{h^2}-\frac{\beta a_i}{h}+\left(\frac{\alpha}{2h}\left(p_{i+1}-3p_{i-1}\right)\right)\right)z_{i-1}+\left(\frac{-2\varepsilon}{h^2}-\frac{2\alpha}{h}\left(p_{i+1}-p_{i-1}\right)\right)z_i+$$
$$+\left(\frac{\varepsilon}{h^2}+\frac{\beta a_i}{h}+\frac{\alpha}{2h}\left(3p_{i+1}-p_{i-1}\right)\right)z_{i+1}=\alpha f(\vartheta_{i-1})-\alpha \mathrm{q}(\vartheta_{i-1})z(\vartheta_{i-1}-1)+$$
$$+2\beta f(\vartheta_i)-2\beta \mathrm{q}(\vartheta_i)z(\vartheta_i-1)+\alpha f\left(\vartheta_{i+1}\right)-\alpha \mathrm{q}\left(\vartheta_{i+1}\right)z\left(\vartheta_{i+1}-1\right). \tag{7}$$

To reduce the error value in the solution over the domain $\Omega_1=(0,1)$, we insert a fitting parameter $\sigma(\rho)$ in the above numerical scheme Eq. (7) for the equation

$$\varepsilon\sigma(\rho)\, z''(\vartheta)+p(\vartheta)\, z'(\vartheta)+q(\vartheta)\, z(\vartheta-1)=f(\vartheta).$$

The value of the fitting parameter is $\sigma(\rho)=\rho(\alpha+\beta)\coth\left(\frac{\rho p_i}{2}\right)$, where $\rho=\frac{h}{\varepsilon}$.

The scheme Eq. (7) with a fitting factor can be written as

$$E_i z_{i-1}+F_i z_i+G_i z_{i-1}-H_i=0 \quad \text{for } i=1,2,...,L-1, \tag{8}$$

where

$$E_i=\left(\frac{\sigma\varepsilon}{h^2}-\frac{\beta a_i}{h}+\left(\frac{\alpha}{2h}\left(p_{i+1}-3p_{i-1}\right)\right)\right),$$
$$F_i=\left(\frac{-2\sigma\varepsilon}{h^2}-\frac{2\alpha}{h}\left(p_{i+1}-p_{i-1}\right)\right),$$
$$G_i=\left(\frac{\varepsilon\sigma}{h^2}+\frac{\beta a_i}{h}+\frac{\alpha}{2h}\left(3p_{i+1}-p_{i-1}\right)\right)$$

and

$$H_i=\alpha\left(f\left(\vartheta_{i-1}\right)-q\left(\vartheta_{i-1}\right)\varphi_{i-1}\right)+2\beta\left(f\left(\vartheta_i\right)-q\left(\vartheta_i\right)\varphi_i\right)+\alpha\left(f\left(\vartheta_{i+1}\right)-q\left(\vartheta_{i+1}\right)\varphi_{i+1}\right),$$

here

$$\varphi_{i+1}=z\left(\vartheta_{i+1}-1\right),\quad \varphi_i=z\left(\vartheta_i-1\right),\quad \varphi_{i-1}=z\left(\vartheta_{i-1}-1\right) \text{ in } [0,1].$$

Now, to find the solution in $\Omega_2=(1,2)$, we consider the finite difference scheme Eq. (7) with fitting factor $\sigma(\rho)$ in the equation

$$\varepsilon\sigma(\rho)\, z''(\vartheta)+p(\vartheta)\, z'(\vartheta)+q(\vartheta)\, z(\vartheta-1)=f(\vartheta).$$

Here the value of $\sigma(\rho)$ is $\sigma(\rho)=\rho(\alpha+\beta)\coth\left(\frac{\rho p_i}{2}\right)$, where $\rho=\frac{h}{\varepsilon}$.

Then, the scheme Eq. (7) with a fitting factor can be written as

$$E_i z_{i-1}+F_i z_i+G_i z_{i+1}-H_i=0 \quad \text{for } i=L,L+1,...,2L-1, \tag{9}$$

where

$$E_i=\left(\frac{\sigma\varepsilon}{h^2}-\frac{\beta a_i}{h}+\left(\frac{\alpha}{2h}\left(p_{i+1}-3p_{i-1}\right)\right)\right),$$
$$F_i=\left(\frac{-2\sigma\varepsilon}{h^2}-\frac{2\alpha}{h}\left(p_{i+1}-p_{i-1}\right)\right),$$
$$G_i=\left(\frac{\varepsilon\sigma}{h^2}+\frac{\beta a_i}{h}+\frac{\alpha}{2h}\left(3p_{i+1}-p_{i-1}\right)\right)$$

and

$$H_i=\alpha\left(f\left(\vartheta_{i-1}\right)-\mathrm{q}\left(\vartheta_{i-1}\right)z\left(\vartheta_{i-1}-L\right)\right)+\alpha\left(f\left(\vartheta_{i+1}\right)-\mathrm{q}\left(\vartheta_{i+1}\right)z\left(\vartheta_{i+1}-L\right)\right)+$$
$$+2\beta\left(f\left(\vartheta_i\right)-\mathrm{q}\left(\vartheta_i\right)z\left(\vartheta_i-L\right)\right).$$

To solve the system of equations Eq. (8) and Eq. (9), the condition $z(L)$ is required. To get the value of $z(L)$, we utilize the reduced problem of Eq. (1) by setting $\varepsilon=0$ and Runge-Kutta 4^{th} order method is used to solve the reduced differential equation.

3 Local error estimate

The local error estimate for the numerical scheme of Eq. (8) is

$$\mathcal{T}_i\,(h) = [2\alpha + 2\beta - 1]\,\varepsilon h^2 z_i^{''} + \left(\left(\alpha - \frac{1}{12}\right)\varepsilon z_i^{iv} - \left(\frac{-2\alpha}{3} + \frac{\beta}{3}\right) p_i z_i^{'''}\right) h^4 + O(h^6). \tag{10}$$

Hence, with $\alpha = \frac{1}{12}$ and $\alpha + \beta = \frac{1}{2}$, the truncation error is fourth order.

4 Convergence analysis

Considering the matrix version of Eq. (8) with the boundary conditions, we have

$$(A + P)\,Z + Q + \mathcal{T}(h) = 0, \tag{11}$$

where

$$A = \begin{bmatrix} -2\varepsilon\sigma & \varepsilon\sigma & 0 & 0 & ... & 0 \\ \varepsilon\sigma & -2\varepsilon\sigma & \varepsilon\sigma & .. & ... & 0 \\ 0 & .. & .. & .. & ... & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & .. & \varepsilon\sigma & -2\varepsilon\sigma \end{bmatrix}$$

and

$$P = [l_i, m_i, k_i] = \begin{bmatrix} m_1 & k_1 & 0 & 0 & .... & 0 \\ l_2 & m_2 & k_2 & 0 & .... & 0 \\ 0 & l_3 & m_3 & k_3 & .... & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & 0 & l_{L-1} & m_{L-1} \end{bmatrix},$$

where

$$l_i = \frac{h}{2}\left(-3\alpha p_{i-1} - 2\beta p_i + \alpha p_{i+1}\right) + h^2 \alpha q_{i-1}\varphi_{i-1},$$

$$m_i = \frac{h}{2}\left(4\alpha p_{i-1} - \alpha p_{i+1}\right) + 2h^2 \beta q_i \varphi_i,$$

$$k_i = \frac{h}{2}\left(-\alpha p_{i-1} + 2\beta p_i + 3\alpha p_{i+1}\right) + h^2 \alpha q_{i+1}\varphi_{i+1}, \text{ for } 1 \leqslant i \leqslant L-1$$

and

$$Q = \left[r_1 + (\varepsilon\sigma + k_1)\,\varphi_0, r_2, r_3, ..., r_{L-2}, r_{L-1} + (\varepsilon\sigma + k_{L-1})\gamma\right]^T,$$

where

$$q_i = h^2\left[\alpha k_{i+1} + 2\beta k_i + \alpha k_{i-1}\right], \quad 1 \leqslant i \leqslant L-1,$$

$\mathcal{T}\,(h) = O\left(h^4\right)$ and $Z = [Z_1, Z_2, \ldots, Z_{L-1}]^T$, $\mathcal{T}\,(h) = [\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_{L-1}]^T$, $O = [0, 0, \ldots, 0]^T$ are associated vectors of Eq. (11).

Let $n = [n_1, n_2, ..., n_{L-1}]^T \cong Z$ satisfy the equation

$$(A + P)\,n + Q = 0. \tag{12}$$

Let $e_i = n_i - Z_i$, $i = 1, 2..., L-1$ be the discretized error $E = [e_1, e_2, ..., e_{L-1}]^T = n - Z$.
Using Eq. (12) and Eq. (11), we get the error equation as

$$(A + P)\, E = \mathcal{T}(h). \tag{13}$$

Let $|p(s)| \le D_1$ and $|q\,(s)| \le D_2$, where $D_1,\ D_2$ are positive constants. Let $(i,j)^{th}$ element of the matrix $(A + P)$ be $\zeta_{i,j}$ then

$$|\zeta_{i,i+1}| \leqslant (\varepsilon) + h(\alpha+\beta)D_1 + h^2(\alpha q_{i+1}\varphi_{i+1} + 2\beta q_i\varphi_i),\ 1 \leqslant i \leqslant L-2,$$

$$|\zeta_{i,i-1}| \leqslant (\varepsilon) + h(\alpha+\beta)D_1 + h^2(\alpha q_{i-1}\varphi_{i-1} + 2\beta q_i\varphi_i),\ 1 \leqslant i \leqslant L-1.$$

Hence, for small values of h, we have

$$|\zeta_{i,i+1}| < \varepsilon\sigma, \quad 1 \leqslant i \leqslant L-2$$

and

$$|\zeta_{i,i-1}| < \varepsilon\sigma, \quad 2 \leqslant i \leqslant L-1.$$

Hence $(A + P)$ is irreducible [25].

Let $\mathrm{S_i}$ be the i^{th} row elements sum, of the matrix $(A + P)$, then we have

$$S_i = -\varepsilon + h(\alpha+\beta)p_i + h^2(\alpha q_{i+1}\varphi_{i+1} + 2\beta q_i\varphi_i) \quad \text{for} \quad i = 1,$$

$$S_i = h^2\left(\alpha q_{i-1}\varphi_{i-1} + 2\beta q_i\varphi_i + \alpha q_{i+1}\varphi_{i+1}\right) \quad \text{for} \quad i = 2, 3, ..., L-2,$$

$$S_i = -\varepsilon - h(\alpha+\beta)p_i + h^2(\alpha q_{i-1}\varphi_{i-1} + 2\beta q_i\varphi_i) \quad \text{for} \quad i = L-1.$$

Let $D_{1^*} = |p(s)|$ and $D_1^* = |p(s)|$, $\ D_{2^*} = |q(s)|$ and $D_2^* = |q(s)|$. Since $0 < \varepsilon \ll 1$, and $\varepsilon \propto O\,(h)$ it is verified that for sufficiently small $h.\,(A+P)$ is monotone [25, 26]. Hence $(A+P)^{-1}$ exists and $(A+P)^{-1} \geqslant 0$. Thus using Eq. (13), we have

$$\|E\| \leqslant \left\|(A+P)^{-1}\right\| \|\mathcal{T}\|\,. \tag{14}$$

Let $(A+P)^{-1}_{i,k}$ be the $(i,k)^{th}$ element of $(A+P)^{-1}$ and define

$$\|\,(A+P)^{-1}\,\| = \max_{1 \le i \le L-1} \sum_{k=1}^{L-1} (A+P)^{-1}_{i,k} \quad \text{and} \quad \|\mathcal{T}(h)\| = \max_{1 \le i \le L-1} |T(h)|.$$

Since

$$(A+P)^{-1}_{i,k} \geqslant 0 \ \text{ and } \sum_{k=1}^{L-1} (A+P)^{-1}_{i,k}\,, \quad S_k = 1 \ \text{ for } 1 \leqslant i \leqslant L-1, \tag{15}$$

we have

$$(A+P)^{-1}_{i,k} \leqslant \frac{1}{\max\limits_{1 \le i \le L-1} S_i} < \frac{1}{h^2 D_2}, \quad i = 1, \tag{16}$$

$$(A+P)^{-1}_{i,k} \leqslant \frac{1}{S_i} < \frac{1}{h^2 D_2}, \quad i = L-1. \tag{17}$$

Further

$$\sum_{k=2}^{L-2} (A+P)^{-1}_{i,k} \leqslant \frac{1}{\max\limits_{2 \le i \le L-2} S_i} < \frac{1}{h^2 D_2},\ \text{for } 2 \leqslant i \leqslant L-2. \tag{18}$$

From Eq. (10), Eq. (14) and using of Eqs. (15)–(18) we get

$$\|E\| \leqslant O\left(h^2\right).$$

Second-order convergence of the proposed scheme is thus observed in the first half of the interval. Similarly, we can demonstrate that the scheme exhibits second-order convergence in the second half of the interval by using Eq. (9).

5 Numerical examples

Three examples are used to demonstrate the proposed scheme. The maximum absolute errors (MAEs) in the solution are computed using the double mesh principle [4].

Utilizing the following formula

$$R^L = \frac{log\left|\frac{E_\varepsilon^L}{E_\varepsilon^{2L}}\right|}{log2}$$

the numerical convergence for each case has been determined.

Example 1. $\varepsilon z^{''}(\vartheta) - 3z^{'}(\vartheta) + z(\vartheta - 1) = 0$, with $z(\vartheta) = 1;\ -1 \leqslant \vartheta \leqslant 0,\ z(2) = 2$.

Example 2. $\varepsilon z^{''}(\vartheta) - 2z^{'}(\vartheta) + 5z(\vartheta - 1) = 0$, with $z(\vartheta) = 1;\ -1 \leqslant \vartheta \leqslant 0,\ z(2) = 2$.

Example 3. $\varepsilon z^{''}(\vartheta) - 5z^{'}(\vartheta) + \frac{1}{2}z(\vartheta - 1) = \begin{cases} -1,\ 0 \leqslant \vartheta \leqslant 1 \\ 1,\ 1 \leqslant \vartheta \leqslant 2 \end{cases}$, with

$$z(\vartheta) = 1; -1 \leqslant \vartheta \leqslant 0, z(2) = 2.$$

6 Discussions and conclusion

To solve a SPDE with a fixed large delay, a trigonometric spline-based numerical technique is proposed. The strategy is designed by utilizing the continuity of the first order derivative of the spline. The convergence of the method is investigated, and it reached second order convergence. Three examples of the scheme with the right end boundary layer are provided. The maximum absolute errors (MAEs) in the solutions are tabulated in Tables 1, 2 and 3 in comparison to the method given in [27]. The rate of convergence in the solutions is also computed. The layer structure is depicted in Figures 1, 2 and 3. In the illustration, it can be seen that the width of the right end layer similarly reduces as the perturbation value does.

7 *Tables and Figures*

Table 1

MAEs in Example 1

$\varepsilon \downarrow L \rightarrow$	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
suggested method						
2^{-5}	9.5547e-06	2.3939e-06	5.9883e-07	1.4975e-07	3.7573e-08	9.7531e-09
	1.9968	1.9991	1.9996	1.9948	1.9458	
2^{-6}	1.9455e-05	4.9061e-06	1.2292e-06	3.0749e-07	7.6942e-08	1.9565e-08
	1.9458	1.9875	1.9969	1.9987	1.9755	
2^{-7}	3.8180e-05	9.8729e-06	2.4898e-06	6.2383e-07	1.5608e-07	3.9153e-08
	1.9513	1.9874	1.9968	1.9989	1.9951	
2^{-8}	6.8209e-05	1.9242e-05	4.9774e-06	1.2552e-06	3.1452e-07	7.8749e-08
	1.8257	1.9508	1.9875	1.9967	1.9978	
2^{-9}	9.7588e-05	3.4253e-05	9.6682e-06	2.5001e-06	6.3049e-07	1.5801e-07
	1.5105	1.8249	1.9513	1.9874	1.9965	
2^{-10}	1.0755e-04	4.8915e-05	1.7177e-05	4.8460e-06	1.2532e-06	3.1606e-07
	1.1367	1.5098	1.8256	1.9512	1.9873	
2^{-11}	1.0808e-04	5.3879e-05	2.4506e-05	8.6011e-06	2.4264e-06	6.2744e-07
	1.0043	1.1366	1.5105	1.8257	1.9513	
2^{-12}	1.0808e-04	5.4147e-05	2.6966e-05	1.2265e-05	4.3037e-06	1.2141e-06
	0.9971	1.0057	1.1366	1.5109	1.8257	
2^{-13}	1.0808e-04	5.4147e-05	2.7100e-05	1.3490e-05	6.1354e-06	2.1526e-06
	0.9971	0.9986	1.0064	1.1367	1.5111	
Results in [27]						
2^{-5}	3.8774(-5)	9.6108(-6)	2.3975(-6)	5.9904(-7)	1.4972(-7)	3.7294(-8)
2^{-6}	8.2126(-5)	1.9910(-5)	4.9349(-6)	1.2310(-6)	3.0757(-7)	7.6825(-8)
2^{-7}	1.8429(-4)	4.1727(-5)	1.0104(-5)	2.5044(-6)	6.2472(-7)	1.5606(-7)
2^{-8}	4.5000(-4)	9.2878(-5)	2.1029(-5)	5.0938(-6)	1.2626(-6)	3.1493(-7)
2^{-9}	1.0778(-3)	2.2589(-4)	4.6641(-5)	1.0566(-5)	2.5585(-6)	6.3417(-7)
2^{-10}	2.3688(-3)	5.4102(-4)	1.1321(-4)	2.3389(-5)	5.2961(-6)	1.2825(-6)
2^{-11}	4.9529(-3)	1.1891(-3)	2.7104(-4)	5.6715(-5)	1.1712(-5)	2.6518(-6)
2^{-12}	1.0121(-2)	2.4862(-3)	5.9570(-4)	1.3565(-4)	2.8385(-5)	5.8601(-6)
2^{-13}	2.0458(-2)	5.0805(-3)	1.2455(-3)	2.9814(-4)	6.7859(-5)	1.4200(-5)

Table 2

MAEs in Example 2

$\varepsilon \downarrow L \rightarrow$	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
suggested method						
2^{-5}	3.5024e-04	8.7651e-05	2.1918e-05	5.4801e-06	1.3706e-06	3.4554e-07
	1.9985	1.9997	1.9998	1.9994	1.9880	
2^{-6}	7 .2423e-04	1.8177e-04	4.5489e-05	1.1375e-05	2.8445e-06	7.1278e-07
	1.9943	1.9944	1.9986	1.9996	1.9996	
2^{-7}	1.4561e-03	3.6979e-04	9.2808e-05	2.3225e-05	5.8078e-06	1.4533e-06
	1.9773	1.9944	1.9986	1.9996	1.9987	
2^{-8}	2.7805e-03	7.3682e-04	1.8705e-04	4.6948e-05	1.1749e-05	2.9382e-06
	1.9160	1.9778	1.9943	1.9985	1.9995	
2^{-9}	4.6105e-03	1.3988e-03	3.7082e-04	9.4139e-05	2.3626e-05	5.9127e-06
	1.7207	1.9154	1.9779	1.9944	1.9985	
2^{-10}	5.8590e-03	2.3143e-03	7.0214e-04	1.8607e-04	4.7237e-05	1.1855e-05
	1.3401	1.7207	1.9159	1.9779	1.9944	
2^{-11}	6.0756e-03	2.9352e-03	1.1594e-03	3.5176e-04	9.3215e-05	2.36604e-05
	1.0496	1.3401	1.7207	1.9160	1.9779	
2^{-12}	6.0797e-03	3.0438e-03	1.4691e-03	5.8029e-04	1.7606e-04	4.6657e-05
	0.9981	1.0509	1.3401	1.7207	1.9159	
2^{-13}	6.05797e-03	3.0458e-03	1.5234e-03	7.3489e-04	2.9031e-04	8.8084e-05
	0.9920	0.9953	1.0517	1.3399	1.7206	
Results in [27]						
2^{-5}	1.4101(-3)	3.5121(-4)	8.7724(-5)	2.1926(-5)	5.4809(-6)	1.3713(-6)
2^{-6}	2.9715(-3)	7.3183(-4)	1.8226(-4)	4.5523(-5)	1.1378(-5)	2.8438(-6)
2^{-7}	6.3962(-3)	1.5166(-3)	3.7366(-4)	9.3055(-5)	2.3241(-5)	5.8086(-6)
2^{-8}	1.4877(-2)	3.2366(-3)	7.6743(-4)	1.8901(-4)	4.7072(-5)	1.1757(-5)
2^{-9}	3.6799(-2)	7.4974(-3)	1.6281(-3)	3.8622(-4)	9.5120(-5)	2.3688(-5)
2^{-10}	8.4871(-2)	1.8472(-2)	3.7635(-3)	8.1727(-4)	1.9379(-4)	4.7729(-5)
2^{-11}	1.8184(-1)	4.2603(-2)	9.2539(-3)	1.8854(-3)	4.0944(-4)	9.7084(-5)
2^{-12}	3.7579(-1)	9.1277(-2)	2.1343(-2)	4.6315(-3)	9.4363(-4)	2.0493(-4)
2^{-13}	7.6369(-1)	1.8863(-1)	4.5728(-2)	1.0682(-2)	2.3169(-3)	4.7207(-4)

Table 3

MAEs in Example 3

$\varepsilon \downarrow L \rightarrow$	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
suggested method						
2^{-5}	4.7341e-05	1.3845e-06	3.7561e-06	9.7905e-07	2.5003e-07	6.3369e-08
	1.7737	1.8821	1.9398	1.9693	1.9803	
2^{-6}	7.1268e-05	2.3726e-05	6.9366e-06	1.8816e-06	4.9045e-07	1.2535e-07
	1.5868	1.7742	1.8823	1.9398	1.9681	
2^{-7}	8.9077e-05	3.5691e-05	1.1878e-05	3.4721e-06	9.4179e-07	2.4552e-07
	1.3195	1.5873	1.7744	1.8823	1.9396	
2^{-8}	9.4225e-05	4.4593e-05	2.2312e-05	5.9429e-06	1.7371e-06	4.7118e-07
	1.0793	0.9990	1.9086	1.7745	1.8823	
2^{-9}	9.4324e-05	4.7161e-05	2.3593e-05	8.9346e-06	2.9725e-06	8.6884e-07
	1.0000	0.9992	1.4009	1.5877	1.7745	
2^{-10}	9.4287e-05	4.7190e-05	2.3593e-05	1.1160e-05	4.4684e-06	1.4866e-06
	0.9986	0.9996	1.0800	1.3205	1.5877	
2^{-11}	9.4287e-05	4.7172e-05	2.3602e-05	1.1799e-05	5.5810e-06	2.2345e-06
	0.9991	0.9996	1.0002	1.0801	1.3206	
2^{-12}	9.4287e-05	4.7172e-05	2.3593e-05	1.1803e-05	5.9004e-06	2.7908e-06
	0.9991	0.9996	0.9992	1.0003	1.0801	
2^{-13}	9.4287e-05	4.7172e-05	2.3593e-05	1.1798e-05	5.9019e-06	2.9504e-06
	0.9991	0.9996	0.9998	0.9993	1.0003	
Results in [27]						
2^{-5}	1.9126(-4)	6.2523(-6)	1.7063(-5)	4.3683(-6)	1.0987(-6)	2.7510(-7)
2^{-6}	2.2400(-4)	9.7185(-5)	3.1769(-5)	8.6781(-6)	2.2223(-6)	5.5895(-7)
2^{-7}	2.2703(-4)	1.1381(-4)	4.8981(-5)	1.6049(-5)	4.3824(-6)	1.1219(-6)
2^{-8}	2.2705(-4)	1.1535(-4)	5.7355(-5)	2.4657(-5)	8.0719(-6)	2.2029(-6)
2^{-9}	2.2705(-4)	1.1536(-4)	5.8131(-5)	2.8790(-5)	1.2377(-5)	4.0479(-6)
2^{-10}	2.2705(-4)	1.1536(-4)	5.8136(-5)	2.9180(-5)	1.4423(-5)	6.2006(-6)
2^{-11}	2.2705(-4)	1.1536(-4)	5.8136(-5)	2.9182(-5)	1.4618(-5)	7.2188(-6)
2^{-12}	2.2705(-4)	1.1536(-4)	5.8136(-5)	2.9182(-5)	1.4620(-5)	7.3164(-6)
2^{-13}	2.2705(-4)	1.1536(-4)	5.8136(-5)	2.9182(-5)	1.4620(-5)	7.3171(-6)

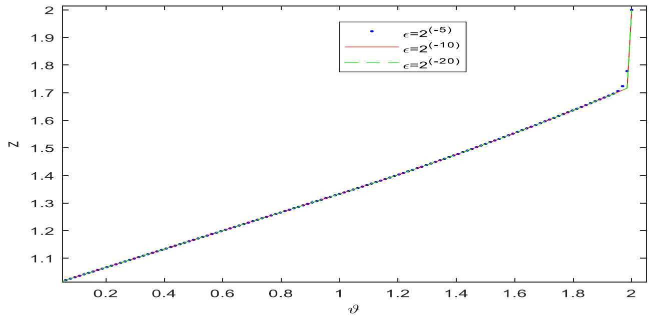

Figure 1. Layer profile in the solution Example 1 with $\varepsilon = 2^{-5}, 2^{-10}, 2^{-20}$.

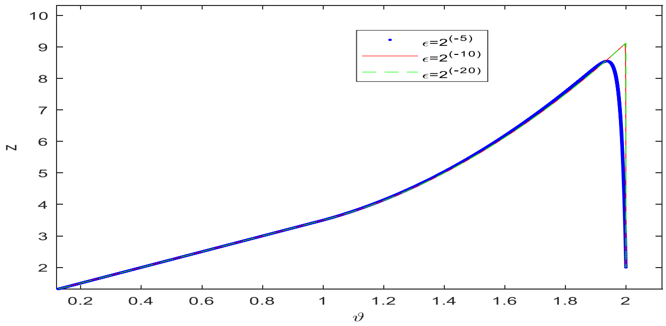

Figure 2. Layer profile in the solution Example 2 with $\varepsilon = 2^{-5}, 2^{-10}, 2^{-20}$.

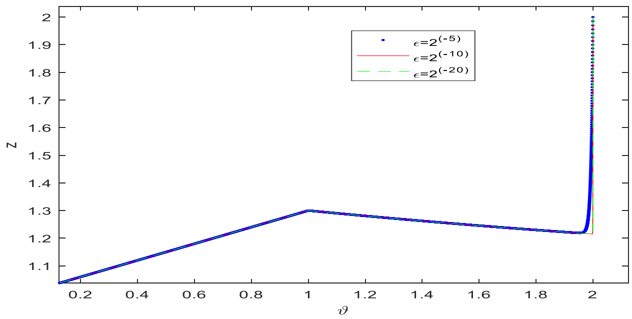

Figure 3. Layer profile in the solution Example 3 with $\varepsilon = 2^{-5}, 2^{-10}, 2^{-20}$.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Mackey, M.C., & Glass, L. (1977). Oscillations and chaos in physiological control systems. *Science 197*, 287–289.

2 Wazewska-Czyzewska, M., & Lasota, A. (1976). Mathematical models of the red cell system. *Mat Sots., 6*, 25–40.

3 Longtin, A., & Milton, J. (1998). Complex oscillations in the human pupil light reflex with mixed and delayed feedback. *Math Biosci., 90*, 183–199.

4 Derstine, M.W., Gibbs, H.M., & Kaplan, D.L. (1982). Bifurcation gap in a hybrid optical system. *Phys. Rev. A., 26*, 3720–3722. https://doi.org/10.1103/PhysRevA.26.3720

5 Mallet-Paret, J., & Nussbaum, R.D. (1996). *A differential-delay equations arising in optics and physiology*. Singapore: World scientific. https://doi.org/10.1137/0520019

6 Bender, C.M., & Orszag , S.A. (1978). *Advanced mathematical methods for scientists and engineers*. New York: Mc Graw-Hill.

7 O'Malley, R.E. (1974). *Introduction to singular perturbations*. New York: Academic press.

8 Doolan, E.P., Miller, J.J.H., & Schilders, W.H.A. (1980). *Uniform numerical methods for problems with initial and boundary layers*. Dublin: Boole Press.

9 Miller, J.J.H., O'Riordan E., & Shishkin, G.I. (1996). *Fitted numerical methods for singular perturbation problems*. Singapore: World Scientific.

10 Roos, H.G., Stynes, M., & Tobiska, L. (1996). *Numerical methods for singularly perturbed differential equations*. Berlin: Springer-Verlag.

11 Driver, R.D. (1977). *Ordinary and delay differential equations*. Heidelberg, Springer, New York, Berlin.

12 Bellman, R., & Cooke, K.L. (1963). *Differential-difference equations*. New York: Academic press.

13 Lange, C.G., & Miura, R.M. (1982). Singular perturbation analysis of boundary value problems for differential-difference equations. *SIAM J Appl Math., 42*(3), 502–530. https://doi.org/10.1137/0142036

14 Lange, C.G., & Miura, R.M. (1994). Singular perturbation analysis of boundary-value problems for differential difference equations. V. Small shifts with layer behaviour. *SIAM J. Appl. Math., 54*(1), 249–272. https://doi.org/10.1137/S0036139992228120

15 Lange, C.G., & Miura, R.M. (1994). Singular perturbation analysis of boundary-value problems for differential difference equations. VI. Small shifts with rapid oscillations. *SIAM J. Appl. Math., 54*, 273–283.

16 Kadalbajoo, M.K., Patidar, K.C., & Sharma, K.K. (2006).ε-uniformly convergent fitted methods for the numerical solution of the problems arising from singularly perturbed general DDEs. *Appl Math Comput., 182*, 119–139. https://doi.org/10.1016/j.amc.2006.03.043

17 Kadalbajoo, M.K., & Sharma, K.K. (2008). A numerical method based on finite difference for boundary value problems for singularly perturbed delay differential equations. *Appl Math Comput., 197*, 692–707. https://doi.org/10.1016/j.amc.2007.08.089

18 Amiraliyev, G.M., & Cimen, E. (2010). Numerical method for a singularly perturbed convection-diffusion problem with delay. *Appl Math Comput., 216*, 2351–2359. https://doi.org/10.1016/j.amc.2010.03.080

19 Mohapatra, J., & Natesan, S. (2010). Uniform convergence analysis of finite difference scheme for singularly perturbed delay differential equation on an adaptively generated grid. Numer. *Math. Theory Methods Appl., 3*, 1–22. https://doi.org/10.4208/nmtma.2009.m8015

20 Erdogan, F., & Amiraliyev, G.M. (2012). Fitted finite difference method for singularly perturbed delay differential equations. *Numer. Algorithms., 59*, 131–145. https://doi.org/10.1007/s11075-011-9480-7

21 Kanth, A.R., & P. Murali, M.K. (2018). A numerical technique for solving nonlinear singularly perturbed delay differential equations. *Mathematical Modelling and Analysis, 23*(1), 64–78. https://doi.org/10.3846/mma.2018.005

22 Assanova, A.T., Iskakova, N.B., & Orumbayeva, N.T. (2018). Well-posedness of a periodic boundary value problem for the system of hyperbolic equations with delayed argument. *Bulletin of the Karaganda university. Mathematics series,* 1(89), 8–14. https://doi.org/10.31489/2018m1/8-14

23 Mazhgikhova, M.G. (2020). Green function method for a fractional–order delay differential equation. *Bulletin of the Karaganda university. Mathematics series,* 1(97), 87–96. https://doi.org/10.31489/2020m1/87-96

24 Omkar, R., Lalu, M., & Phaneendra, K. (2023). Numerical solution of differential – difference equations having an interior layer using nonstandard finite differences. *Bulletin of the Karaganda university. Mathematics series,* 2(110), 104–115. https://doi.org/10.31489/2023m2/104-115

25 Varga, R.S. (1962). *Matrix iterative analysis.* Englewood Cliffs, NJ Prentice-Hall.

26 Young, D.M. (1971). *Iterative solution of large linear systems.* New York: Acadamic press.

27 Kumar, N.S., & Rao, R.N. (2020). A Second Order Stabilized Central Difference Method for Singularly Perturbed Differential Equations with a Large Negative Shift. *Differential Equations and Dynamical Systems, 31*, 787–804. https://doi.org/10.1007/s12591-020-00532-w

Бекітілген көп кешігуі бар сингулярлы ауытқыған дифференциалдық-айырымдық теңдеулер класының жаңа сандық схемасы

Э. Шринивас, К. Фанеендра

Университеттің ғылым колледжі, Османия университеті, Хайдарабад, Үндістан

Тригонометриялық сплайнға негізделген есептеу әдісі бекітілген көп кешігуі бар қабаттың әрекеті үшін дифференциалдық-айырымдық теңдеулерді сандық шешу үшін ұсынылған. Айырымдық теңдеулер жүйесін құру үшін тордың ішкі нүктесіндегі тригонометриялық сплайнның бірінші ретті туындысының үзіліссіздігі қолданылады. Сингулярлы ауытқыған теориясын қолдана отырып, шешімдегі қатені азайту үшін айырымдық схемасына сәйкестендіретін параметр енгізіледі. Әдіс жинақтылыққа тексерілген. Сонымен қатар шекаралық қабатқа ығысу немесе кешігу әсері қарастырылды. Әдебиеттерде келтірілген басқа тәсілдермен салыстырғанда максималды абсолютті қателер есептеледі және ұсынылған сандық әдістің орындылығын көрсету үшін қабаттардың өзгеруі графиктерде көрсетілді.

Кілт сөздер: сингулярлы ауытқыған дифференциалдық-айырымдық теңдеу, кешігу, тригонометриялық сплайн, сәйкестендіретін параметр.

Новая численная схема для класса сингулярно возмущенных дифференциально-разностных уравнений с фиксированным большим запаздыванием

Э. Шринивас, К. Фанеендра

Университетский колледж науки; Университет Османии, Хайдарабад, Индия

Предложен вычислительный метод на основе тригонометрического сплайна для численного решения дифференциально-разностных уравнений поведения слоя с фиксированной большой задержкой. Для построения системы разностных уравнений используется непрерывность производной первого порядка тригонометрического сплайна во внутренней точке сетки. С помощью теории сингулярных возмущений в разностную схему вводится подгоночный параметр, позволяющий минимизировать ошибку в решении. Метод проверен на сходимость. Мы также рассмотрели влияние сдвига или задержки на пограничный слой. Подсчитаны максимальные абсолютные погрешности по сравнению с другими подходами, описанными в литературе, а поведение слоев отображено на графиках, чтобы продемонстрировать осуществимость предложенного численного метода.

Ключевые слова: сингулярно возмущенное дифференциально-разностное уравнение, запаздывание, тригонометрический сплайн, подгоночный параметр.

Author Information[*]

Erla Srinivas — Research Scholar, Department of Mathematics, University College of Science, Osmania University, Hyderabad, 500007, India; e-mail: *Srinivas.erla@gmail.com*; https://orcid.org/0009-0006-9380-5255

Kolloju Phaneendra (*corresponding author*) — Doctor of mathematical sciences, Associate Professor, Department of Mathematics, University College of Science, Osmania University, Hyderabad, 500007, India; e-mail: *kollojuphaneendra@yahoo.co.in*; https://orcid.org/0000-0002-2852-8494

[*]The author's name is presented in the order: First, Middle and Last Names.