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On solvability of the inverse problem for a fourth-order parabolic equation with a complex-valued coefficient

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In this paper, the inverse problem for a fourth-order parabolic equation with a variable complex-valued coefficient is studied by the method of separation of variables. The properties of the eigenvalues of the Dirichlet and Neumann boundary value problems for a non-self-conjugate fourth-order ordinary differential equation with a complex-valued coefficient are established. Known results on the Riesz basis property of eigenfunctions of boundary value problems for ordinary differential equations with strongly regular boundary conditions in the space $L_2(-1,1)$ are used. On the basis of the Riesz basis property of eigenfunctions, formal solutions of the problems under study are constructed and theorems on the existence and uniqueness of solutions are proved. When proving theorems on the existence and uniqueness of solutions, the Bessel inequality for the Fourier coefficients of expansions of functions from space $L_2(-1,1)$ into a Fourier series in the Riesz basis is widely used. The representations of solutions in the form of Fourier series in terms of eigenfunctions of boundary value problems for a fourth-order equation with involution are derived. The convergence of the obtained solutions is discussed.

Keywords: parabolic equation, inverse problem, classical solution, Fourier method, strongly regular boundary conditions, Riesz basis.

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Introduction

With the further development of the theory of solvability of differential equations, with the advent of new mathematical models in various fields of natural sciences, it becomes more and more important to formulate new mathematical problems and to study more general cases of classical differential equations. These are direct and inverse problems for the fourth-order partial differential equations. A lot of papers are devoted to the study of boundary value problems for the fourth-order partial differential equations (see, for example, [1, 2], and references therein).

It should be noted that boundary value problems with complex-valued coefficients are of particular interest. The existence and uniqueness of the solution of mixed problems for the heat equation with a complex-valued coefficient was established in [3]. The solvability of mixed problems for a perturbed wave equation with involution and with a variable complex-valued coefficient was studied in [4, 5]. The solvability of inverse problems for the perturbed heat equation with involution and with a variable complex-valued coefficient was considered in [6–8].

The results on the existence of a unique solution to inverse problems for a fourth-order partial differential equation with real coefficients depending on x and t can be found in [9, 10].

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This article presents the results of studies of inverse problems for a fourth-order parabolic equation with a variable complex-valued coefficient. The existence and uniqueness of the solution of mixed inverse problems for a one-dimensional fourth-order equation is established

$$u_t(x,t) + \frac{\partial^4}{\partial x^4} u(x,t) + q(x)\, u(x,t) = f(x), \tag{1}$$

where $q(x) = q_1(x) + iq_2(x)$. We will use $\Omega = \{-1 < x < 1,\ 0 < t < T\}$ to denote an open domain, and $\bar{\Omega} = \{-1 \le x \le 1,\ 0 \le t \le T\}$ to denote a closed domain.

The space $C^{k,l}_{x,t}(\Omega)$ consists of all functions $u(x,t)$ having continuous derivatives with respect to t and x of the order l, k respectively, in the domain Ω.

1 Problem Statement

Let us introduce a non-self-conjugate fourth-order differential operator $L_q : D(L_q) \subset L_2(-1,1) \to L_2(-1,1)$ by the formula

$$L_q y = y^{IV}(x) + q(x)\, y(x), \ \ -1 \le x \le 1,$$

with the domain of definition

$$D(L_q) = \left\{ y(x) \in W_2^4[-1,1] :\ U_i(y) = 0,\ \ i = 1,2,3,4, \right\}, \tag{2}$$

where the linear forms $U_i(y)$ are written as $U_i(y)$

$$U_i(y) = a_{i1}y'''(-1) + a_{i2}y'''(1) + a_{i3}y''(-1) + a_{i4}y''(1) + a_{i5}y'(-1) + a_{i6}y'(1) + a_{i7}y(-1) + a_{i8}y(1),$$

with given complex coefficients a_{ij}, $W_2^4[-1,1] = \left\{ y(x) \in C^3[-1,1] : y^{IV}(x) \in L_2(-1,1) \right\}$ is the Sobolev space. Assume that the linear forms $U_1(y)$, $U_2(y)$, $U_3(y)$, $U_4(y)$ are linearly independent. The order of the highest derivative of the form will be called the order of the form. Then the maximum number of forms of order 3 will be not more than two. Boundary conditions (2) can easily be reduced to the form

$$\begin{gathered}
a_{11}y'''(-1) + a_{12}y'''(1) + a_{13}y''(-1) + a_{14}y''(1) + a_{15}y'(-1) + a_{16}y'(1) + a_{17}y(-1) + a_{18}y(1) = 0, \\
a_{21}y'''(-1) + a_{22}y'''(1) + a_{23}y''(-1) + a_{24}y''(1) + a_{25}y'(-1) + a_{26}y'(1) + a_{27}y(-1) + a_{28}y(1) = 0, \\
a_{33}y''(-1) + a_{34}y''(1) + a_{35}y'(-1) + a_{36}y'(1) + a_{37}y(-1) + a_{38}y(1) = 0, \\
a_{43}y''(-1) + a_{44}y''(1) + a_{45}y'(-1) + a_{46}y'(1) + a_{47}y(-1) + a_{48}y(1) = 0,
\end{gathered} \tag{3}$$

called the normalized boundary conditions [11; 66]. For the sake of simplicity, we have not changed the notation of the coefficients. We proceed similarly if the order of the highest derivative of the forms is less than 3.

Let us rewrite equation (1) in the form

$$u_t(x,t) + L_q u(x,t) = f(x), \ \ (x,t) \in \Omega, \tag{4}$$

and then consider a differential operator L_q with domain generated by one of the following two boundary conditions:

D: Dirichlet boundary conditions

$$\begin{gathered}
U_1(u) = u(-1,t) = 0,\ U_2(u) = u(1,t) = 0,\ U_3(u) = u_{xx}(-1,t) = 0, \\
U_4(u) = u_{xx}(1,t) = 0,\ t \in (0,T).
\end{gathered} \tag{5}$$

N: Neumann boundary conditions

$$U_1(u) = u_x(-1,t) = 0,\ U_2(u) = u_x(1,t) = 0,\ U_3(u) = u_{xxx}(-1,t) = 0,$$

$$U_4(u) = u_{xxx}(1,t) = 0,\ \ t \in (0,T). \tag{6}$$

We have to find a pair of functions $u(x,t)$ and $f(x)$ satisfying equation (4) in the domain Ω and conditions

$$u(x,0) = \varphi(x),\ u(x,T) = \psi(x),\ x \in (-1,1), \tag{7}$$

where $\varphi(x)$ and $\psi(x)$ are given sufficiently smooth functions.

Definition 1. A pair of functions $u(x,t)$ and $f(x)$ is called a solution to inverse problem (4), (5), and (7) if the following three conditions are satisfied:

1) the function $u(x,t) \in C\left(\bar{\Omega}\right) \bigcap C^{2,0}_{x,t}\left(\bar{\Omega}\right)$;

2) there are derivatives $u_t(x,t)$, $u_{xxx}(x,t)$ and $u_{xxxx}(x,t)$ continuous in the open domain Ω, $f(x) \in C[-1,1]$;

3) functions $u(x,t)$ and $f(x)$ satisfy equation (4), and the function$u(x,t)$ satisfies conditions (5), (7) in the usual sense.

The notion of a solution to inverse problem (4), (6) with boundary conditions (7) is defined similarly.

To prove the existence and uniqueness of a solution to the inverse problem posed, we use the Fourier method. The advantage of this method is that we will have a representation of the solution to the inverse problem in the form of Fourier series. A disadvantage of the Fourier method may be increased requirements for initial data. However, the aim of this work is not to reduce the smoothness of the initial data.

In this regard, it is necessary to solve the inverse problem of convergence of expansions of functions from a certain class in terms of eigenfunctions of the following spectral problem:

$$L_q X(x) = \lambda X(x),\ \ -1 \le x \le 1. \tag{8}$$

2 Properties of eigenfunctions of spectral problems

It is easier to prove the convergence of expansions of operator L_q in eigenfunctions if the system of eigenfunctions $\{X_k(x)\}$ forms a Riesz basis in the class $L_2(-1,1)$. Therefore, in this section, we study the basis property of the eigenfunctions of a differential operator L_q. The differential operator L_q is not a self-conjugate operator. The conjugate spectral problem is written as

$$L_q^* Z(x) = \overline{\lambda} Z(x), \tag{9}$$

where $L_q^* Z(x) = Z^{IV}(x) + \overline{q}(x) Z(x)$ is the operator conjugate to the operator L_q. The domain of definition of the conjugate operator L_q^* is given by one of the boundary conditions (D) or (N) so that $D(L_q) = D\left(L_q^*\right)$. Suppose that all eigenvalues of the operators L_q are simple and zero is not an eigenvalue. The systems of eigenfunctions $\{X_k(x)\}$ and $\{Z_k(x)\}$ satisfy the biorthogonality condition [11; 30]

$$(X_k, Z_n) = \int_{-1}^{1} X_k(x)\, \bar{Z}_n(x)\, dx = \delta_{kn},$$

where δ_{kn} is the Kronecker symbol. In the case of positive self-conjugate operators, the eigenvalues are real and positive. In the case of nonself-conjugate operators, the eigenvalues can be complex numbers. Therefore, it is necessary to study the condition of non-negativity of their real parts.

Lemma 1. Let $q(x) \in C[-1,1]$. Then the inequality $|\mathrm{Im}\ \lambda_k| \le \max|q_2(x)|$ holds for all eigenvalues λ_k of the operator L_q. Under an additional condition $\mathrm{Re}\ q(x) = q_1(x) \ge 0$ in the interval $-1 \le x \le 1$, all eigenvalues λ_k of the operator L_q satisfy the inequality $\mathrm{Re}\ \lambda_k > 0$.

Proof. Consider equation (8) with boundary conditions (5) or (6). We multiply both parts of equation (8) by the complex conjugate function $\bar{X}_k(x)$ and integrate the resulting equality twice by parts over the interval $(-1,1)$. After this, the non-integral terms that arise disappear, and we obtain the equality

$$\int_{-1}^{1} \left|X''_k(x)\right|^2 dx + \int_{-1}^{1} q(x)\,|X_k(x)|^2 dx = \lambda_k \int_{-1}^{1} |X_k(x)|^2 dx.$$

Writing out the real and imaginary parts of the last equality separately, we get the following two relations:

$$\int_{-1}^{1} q_2(x)\,|X_k(x)|^2 dx = \mathrm{Im}\,\lambda_k \int_{-1}^{1} |X_k(x)|^2 dx,$$

$$\int_{-1}^{1} \left|X''_k(x)\right|^2 dx + \int_{-1}^{1} q_1(x)\,|X_k(x)|^2 dx = \mathrm{Re}\,\lambda_k \int_{-1}^{1} |X_k(x)|^2 dx.$$

From the first equality we obtain the first assertion of the lemma

$$\max_{x\in[-1,1]} |q_2(x)| \ge |\mathrm{Im}\,\lambda_k|,\ k \in N.$$

To prove the second assertion of the lemma, we assume the contrary. Let there be a subsequence $\{\lambda_{n_k}\}$ satisfying the condition $\mathrm{Re}\,\lambda_{n_k} < 0$. Then the second relation implies the inequality

$$\int_{-1}^{1} \left|X''_{n_k}(x)\right|^2 dx + \int_{-1}^{1} q_1(x)\,|X_{n_k}(x)|^2 dx = \mathrm{Re}\,\lambda_{n_k} \int_{-1}^{1} |X_{n_k}(x)|^2 dx < 0,$$

whence, by virtue of $q_1(x) \ge 0$, we get a contradiction, which proves the lemma.

Note that this lemma is valid for continuous $q(x) \in C[-1,1]$. In this case $\mathrm{Re}\,\lambda_k > 0$, starting from some number k_0, as $\mathrm{Re}\,\lambda_k \ge |\min q_1(x)|$ for $k \ge k_0$, if $\min q_1(x) < 0$.

For further presentation, let us dwell on some well-known facts. Let $\lambda = \rho^4$. In the complex ρ-plane, consider a fixed region $S_\nu,\ \nu = 0,1,2,...,7,$ defined by the inequality $\frac{\nu\pi}{4} \le \arg\rho \le \frac{(\nu+1)\pi}{4}$. We enumerate $\omega_1, \omega_2, \omega_3, \omega_4$ different roots of the number $\sqrt[4]{-1}$ so that for $\rho \in S_\nu$, $\mathrm{Re}\,(\rho\omega_1) \le \mathrm{Re}\,(\rho\omega_2) \le \mathrm{Re}\,(\rho\omega_3) \le \mathrm{Re}\,(\rho\omega_4)$.

It is well known that the normalized boundary conditions (3) are called regular (see, for example, [11; 67] if the numbers θ_{-1}, θ_1 defined by the equality

$$\frac{\theta_{-1}}{s} + \theta_0 + \theta_1 s = \begin{vmatrix} a_{11}\omega_1^3 & (a_{11}+sa_{12})\,\omega_2^3 & \left(a_{11}+\frac{1}{s}a_{12}\right)\omega_3^3 & a_{12}\omega_4^3 \\ a_{21}\omega_1^3 & (a_{21}+sa_{22})\,\omega_2^3 & \left(a_{21}+\frac{1}{s}a_{22}\right)\omega_3^3 & a_{22}\omega_4^3 \\ a_{33}\omega_1^2 & (a_{33}+sa_{34})\,\omega_2^2 & \left(a_{33}+\frac{1}{s}a_{34}\right)\omega_3^2 & a_{34}\omega_4^2 \\ a_{43}\omega_1^2 & (a_{43}+sa_{44})\,\omega_2^2 & \left(a_{43}+\frac{1}{s}a_{44}\right)\omega_3^2 & a_{44}\omega_4^2 \end{vmatrix}$$

are different from zero. Here the power of the number ω_j is equal to the order of the highest derivative of the corresponding boundary condition. We proceed similarly if the order of the highest derivative of the forms is less than 3.

If the additional condition $\theta_0^2 - 4\theta_{-1}\theta_1 \ne 0$ is satisfied, then the boundary conditions (3) are called strongly regular.

Note that the differential operator L_q generated by strongly regular boundary conditions can have only a finite number of multiple eigenvalues.

The papers [12], [13] imply the following important theorem.

Theorem 1. [12], [13]. If the operator L_q is generated by strongly regular boundary conditions, then the eigenfunctions and associated functions of this operator form a Riesz basis in the space $L_2(-1,1)$.

It is easy to check that the boundary conditions (5) (and (6)) are strongly regular, so the system of eigenfunctions $\{X_k(x)\}$ of the operator L_q forms a Riesz basis in the space $L_2(-1,1)$. This is also valid for the system of eigenfunctions $\{Z_k(x)\}$ of the operator L_q^*.

Everywhere below we will assume that all eigenvalues of the operator L_q are single.

Lemma 2. For any function $\varphi \in D(L_q)$ each of the Fourier series

$$\varphi(x) = \sum_{k=1}^{\infty} (\varphi, Z_k) X_k(x), \quad \varphi(x) = \sum_{k=1}^{\infty} (\varphi, X_k) Z_k(x), \tag{10}$$

by eigenfunctions $\{X_k(x)\}$, $\{Z_k(x)\}$ converges uniformly for $-1 \le x \le 1$.

Proof. Let us rewrite equation (8) in the form (the number $\lambda = 0$ is not an eigenvalue)

$$X_k(x) = \frac{X_k^{IV}(x) + q(x) X_k(x)}{\lambda_k}.$$

Then

$$(\varphi, X_k) = \int_{-1}^{1} \varphi(x) \bar{X}_k(x)\, dx = \int_{-1}^{1} \varphi(x) \frac{\bar{X}_k^{IV}(x) + \overline{q}(x) \bar{X}_k(x)}{\overline{\lambda}_k} dx =$$

$$= \frac{1}{\overline{\lambda}_k} \int_{-1}^{1} \left[\varphi^{IV}(x) + q(x)\varphi(x)\right] \bar{X}_k(x)\, dx = \frac{1}{\overline{\lambda}_k} \left(L_q\varphi, \bar{X}_k\right).$$

Using this relation, the second series in (10) can be written as

$$\varphi(x) = \sum_{k=1}^{\infty} \frac{A_k}{\overline{\lambda}_k} Z_k(x), \tag{11}$$

where

$$A_k = \int_{-1}^{1} \left[\varphi^{IV}(x) + q(x)\varphi(x)\right] \bar{X}_k(x)\, dx.$$

On the other hand, it is well known that the conjugate spectral problem is equivalent to the integral equation

$$Z_k(x) = \overline{\lambda_k} \int_{-1}^{1} G^*(x,t) \bar{Z}_k(t)\, dt,$$

where $G^*(x,t)$ is the Green's function of the conjugate boundary value problem for $\lambda = 0$. By definition [11; 45], the Green's function $G^*(x,t)$ is continuous for $x \in [-1,1]$ and $t \in [-1,1]$ and therefore it is bounded. Let's denote $C_k(x) = \int_{-1}^{1} G^*(x,t) \bar{Z}_k(t)\, dt$. Then equality (11) takes the form

$$\sum_{k=1}^{\infty} \frac{A_k}{\overline{\lambda}_k} Z_k(x) = \sum_{k=1}^{\infty} A_k C_k(x).$$

Further, using the inequality $ab \leq \frac{1}{2}\left(a^2+b^2\right)$, we obtain the following estimate

$$\sum_{k=1}^{\infty}\left|\frac{A_k}{\overline{\lambda_k}}Z_k(x)\right| = \sum_{k=1}^{\infty}|A_k C_k(x)| \leq \sum_{k=1}^{\infty}|A_k|^2 + \sum_{k=1}^{\infty}|C_k(x)|^2. \tag{12}$$

As the quantities A_k are the Fourier coefficients of the expansion in the Riesz basis $Z_k(x)$, $k = 1,2,3,...$, and $C_k(x)$ are the Fourier coefficients of the expansion of the Green's function $G(x,t)$ in the Riesz basis $\{X_k(x)\}$, due to the Bessel inequality for the Riesz bases, both series on the right side of inequality (12) converge and

$$\sum_{k=1}^{\infty}|C_k(x)|^2 \leq \int_{-1}^{1}|G^*(x,t)|^2 dt \leq M_0,\ \forall x \in [-1,1].$$

This implies absolute and uniform convergence of the second series (10) . The absolute and uniform convergence of the first series (10) is proved similarly. The lemma is proved.

3 Formal solution to the inverse problem

In this section, we construct a formal solution to the inverse problem for equation (4) with Dirichlet boundary conditions (5) and conditions (8). Recall that if the domain $D(L_q)$ of the operator L_q is generated by one of the boundary conditions (D), (N), then each of the systems $\{X_k(x)\}$ and $\{Z_k(x)\}$, consisting of the eigenfunctions of the operators L_q and L_q^*, respectively, forms a Riesz basis in the space $L_2(-1,1)$. The functions $u(x,t)$ and $f(x)$ can be represented as Fourier series

$$u(x,t) = \sum_{k=0}^{\infty} C_k(t) X_k(x), \tag{13}$$

$$f(x) = \sum_{k=0}^{\infty} f_k X_k(x), \tag{14}$$

$$C_k(t) = \int_{-1}^{1} u(x,t)\,\bar{Z}_k(x)\,dx,\ \ f_k = \int_{-1}^{1} f(x)\,\bar{Z}_k(x)\,dx, \tag{15}$$

where $C_k(t)$ are unknown functions and f_k are unknown constants. Substituting (13) and (14) into equation (4), we obtain the equation

$$C'_k(t) + \lambda_k C_k(t) = f_k,$$

whose solution will be written in the form

$$C_k(t) = D_k \cdot e^{-\lambda_k t} + \frac{f_k}{\lambda_k}. \tag{16}$$

As, according to condition (7) and formula (15),

$$C_k(0) = \int_{-1}^{1} u(x,0)\,\bar{Z}_k(x)\,dx = \int_{-1}^{1} \varphi(x)\,\bar{Z}_k(x)\,dx = \varphi_k,$$

$$C_k(T) = \int_{-1}^{1} u(x,T)\,\bar{Z}_k(x)\,dx = \int_{-1}^{1} \psi(x)\,\bar{Z}_k(x)\,dx = \psi_k,$$

from equality (16) we get

$$\begin{cases} D_k + \frac{f_k}{\lambda_k} = \varphi_k, \\ D_k e^{-\lambda_k T} + \frac{f_k}{\lambda_k} = \psi_k. \end{cases}$$

Solving this system of equations, we find the unknown quantities

$$D_k = \frac{\varphi_k - \psi_k}{1 - e^{-\lambda_k T}},\ f_k = (\varphi_k - D_k)\,\lambda_k,$$

using which from relation (16) we find

$$C_k(t) = \varphi_k - \frac{1 - e^{-\lambda_k t}}{1 - e^{-\lambda_k T}}\left[\varphi_k - \psi_k\right].$$

Substituting the found values of the unknowns $C_k(t)$ and f_k into (13) and (14), we find the formal solution to the inverse problem in the form of the following series

$$u(x,t) = \varphi(x) + \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - e^{-\lambda_k T}} \left(e^{-\lambda_k t} - 1\right) X_k(x), \tag{17}$$

and

$$f(x) = L_q\varphi(x) - \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - e^{-\lambda_k T}} \lambda_k X_k(x). \tag{18}$$

Now we have to prove that the functions (17) and (18) will be the classical solution to the studied inverse problems.

4 Main results

In [9], the authors of this work proposed a new approach to prove the uniform convergence of formally differentiated series, which represent a formal solution to the inverse problem for the equation of a fourth-order hyperbolic equation with complex-valued coefficients. This approach has two advantages: 1) the first advantage is the use of estimates of the norms of eigenfunctions derivatives through the norm of eigenfunctions [14]; the second advantage is the use of the properties of uniform boundedness of Riesz bases consisting of eigenfunctions of the differential operator [15]. In this section, this approach is developed for the case of inverse problems for a fourth-order parabolic equation with complex-valued coefficients. It is clear that the formal solutions to hyperbolic and parabolic equations have completely different structures. The conditions for the existence of solutions are also different.

Let us formulate the main result of the present work. The solvability of the inverse problem (4), (7) with the Dirichlet boundary conditions (5) is formulated as the following theorem.

Theorem 2. Let $q(x) \in C^4[-1,1]$, and functions $\varphi,\ \psi$ are such that $\varphi,\ \psi,\ L_q\varphi,\ L_q\psi \in D(L_q)$. Then inverse problem (4), (5), (7) has a unique solution, which can be represented as Fourier series (17), (18).

Proof. We have to show that the resulting formal solution in the form of series (17), (18) satisfies equation (4) and conditions (5), (7). Let us first show that series (17), (18), as well as the formal derivative with respect to the variable t and formal derivatives up to the fourth order with respect to

the variable x of series (17), converge uniformly in the open domain Ω, i.e. let us prove the uniform convergence of the series (17), (18) and the uniform convergence of the formally differentiated series

$$u_t(x,t) = -\sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - e^{-\lambda_k T}} \lambda_k e^{-\lambda_k t} X_k(x), \tag{19}$$

$$u_x(x,t) = \varphi'(x) + \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - e^{-\lambda_k T}} \left(e^{-\lambda_k t} - 1\right) X'_k(x), \tag{20}$$

$$u_{xx}(x,t) = \varphi''(x) + \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - e^{-\lambda_k T}} \left(e^{-\lambda_k t} - 1\right) X''_k(x), \tag{21}$$

$$u_{xxx}(x,t) = \varphi'''(x) + \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - e^{-\lambda_k T}} \left(e^{-\lambda_k t} - 1\right) X'''_k(x), \tag{22}$$

$$u_{xxxx}(x,t) = \varphi^{IV}(x) + \sum_{k=0}^{\infty} \frac{\varphi_k - \psi_k}{1 - e^{-\lambda_k T}} \left(e^{-\lambda_k t} - 1\right) X_k^{IV}(x). \tag{23}$$

The uniform convergence of series (17) follows from the obvious inequality

$$|u(x,t)| \le |\varphi(x)| + \left|\sum_{k=0}^{\infty} (\varphi, Z_k) X_k(x)\right| + \left|\sum_{k=0}^{\infty} (\psi, Z_k) X_k(x)\right|,$$

and Lemma 2, taking into account Lemma 1 ($\operatorname{Re}\lambda_k > 0$).

To prove series (18) in the expressions $\varphi_k = (\varphi, Z_k)$, $\psi_k = (\psi, Z_k)$, the function $Z_k(x)$ is replaced by the conjugate equation (9). Then

$$\lambda_k \varphi_k = \lambda_k (\varphi, Z_k) = \left(\varphi, L_q^* Z_k\right) = (L_q \varphi, Z_k), \ \lambda_k \psi_k = (L_q \psi, Z_k). \tag{24}$$

Substituting them into (18), we obtain

$$f(x) = L_q \varphi(x) - \sum_{k=0}^{\infty} \frac{(L_q \varphi, Z_k) - (L_q \psi, Z_k)}{1 - e^{-\lambda_k T}} \cdot X_k(x).$$

Hence we get the inequality

$$|f(x)| \le |L_q \varphi(x)| + \left|\sum_{k=0}^{\infty} (L_q \varphi, Z_k) X_k(x)\right| + \left|\sum_{k=0}^{\infty} (L_q \psi, Z_k) X_k(x)\right|.$$

As, by the condition of the theorem $L_q\varphi$, $L_q\psi \in D(L_q)$, then, by virtue of Lemma 2, both series on the right-hand side of the last inequality converge uniformly. The uniform convergence of the series (17), (18) is proved. The uniform convergence of the series (19) is proved as well as the convergence of the series (18), taking into account the boundedness of the quantities $\lambda_k e^{-\lambda_k \tau} \to 0$, $k \to \infty$.

Let us prove the uniform convergence of series (20)–(23). Applying (24) to the series (20) we obtain the relation

$$|u_x(x,t)| \le \left|\varphi'(x)\right| + \left|\sum_{k=0}^{\infty} \frac{(L_q \varphi, Z_k) - (L_q \psi, Z_k)}{\lambda_k \left(1 - e^{-\lambda_k T}\right)} \left(e^{-\lambda_k t} - 1\right) X'_k(x)\right|.$$

In [14] the validity of the estimates

$$\max\left|X_k^{(s)}(x)\right| \le c_1\left(\sqrt[4]{|\lambda_k|}\right)^s \max|X_k(x)|, \ s=1,2,3, \tag{25}$$

for the eigenfunctions of the fourth-order differential operator is shown. Using estimates (25), from the last inequality we obtain the estimate

$$|u_x(x,t)| \le \left|\varphi'(x)\right| + c_1\sum_{k=0}^{\infty}\frac{|(L_q\varphi,Z_k)-(L_q\psi,Z_k)|}{\left(\sqrt[4]{|\lambda_k|}\right)^3}\max|X_k(x)|.$$

It follows from [15] that only uniformly bounded systems of eigenfunctions of ordinary differential operators can be Riesz bases. Therefore, due to the conditions of the theorem $L_q\varphi,\ L_q\psi \in D(L_q)$, the Bessel inequality for the Riesz bases, and the asymptotics of the eigenvalues [11; 99], the series on the right-hand side of the following inequality

$$|u_x(x,t)| \le \left|\varphi'(x)\right| + c_1\sum_{k=0}^{\infty}\left[(L_q\varphi,Z_k)^2+(L_q\psi,Z_k)^2+\frac{2}{\left(\sqrt{\lambda_k}\right)^3}\right]$$

converges. The uniform convergence of series (20) is proved.

Using the estimates (25), the convergence of series (21), (22) in the open domain Ω is similarly proved. Consider the uniform convergence of the series

$$u_{xxxx}(x,t) = \varphi^{IV}(x) + \sum_{k=0}^{\infty}\frac{(L_q\varphi,Z_k)-(L_q\psi,Z_k)}{\lambda_k\left(1-e^{-\lambda_k T}\right)}\left(e^{-\lambda_k t}-1\right)X_k^{IV}(x).$$

Replacing the fourth derivative with the help of equation (8), we obtain the estimate

$$|u_{xxxx}(x,t)| \le \left|\varphi^{IV}(x)\right| + \left|\sum_{k=0}^{\infty}\frac{q(x)}{\lambda_k}\left[(L_q\varphi,Z_k)X_k(x)-(L_q\psi,Z_k)X_k(x)\right]\right|$$

$$+\left|\sum_{k=0}^{\infty}\left[(L_q\varphi,Z_k)X_k(x)-(L_q\psi,Z_k)X_k(x)\right]\right|. \tag{26}$$

The second series on the right-hand side of (26) converges by virtue of the conditions of the theorem $L_q\varphi,\ L_q\psi \in D(L_q)$ and Lemma 2. The convergence of the first series in (26) follows from the uniform boundedness of the system $\{X_k(x)\}$ [15], the Bessel inequality for the Riesz bases, the asymptotics of the eigenvalues [11; 99], and the boundedness of the function $q(x)$. This proves the uniform convergence of the series $u_{xxxx}(x,t)$ in the open domain Ω. Thus, we have shown that series (17), (18) satisfy equation (4).

Obviously, the formal solution (17) satisfies conditions (7):

$$\lim_{t\to 0+0}u(x,t) = \lim_{t\to 0+0}\left[\varphi(x)+\sum_{k=0}^{\infty}\frac{\varphi_k-\psi_k}{1-e^{-\lambda_k T}}\left(e^{-\lambda_k t}-1\right)X_k(x)\right] = \varphi(x),$$

$$\lim_{t\to T-0}u(x,t) = \lim_{t\to T-0}\left[\varphi(x)+\sum_{k=0}^{\infty}\frac{\varphi_k-\psi_k}{1-e^{-\lambda_k T}}\left(e^{-\lambda_k t}-1\right)X_k(x)\right] = \psi(x).$$

The boundary conditions are satisfied as each term of the series (17) satisfies them. The existence of a classical solution to problem (4), (5), (7) has been completely proved. To prove the uniqueness of the

solution, we assume the contrary. Suppose that there are two sets of solutions $\{u_1(x,t),\ f_1(x)\}$ and $\{u_2(x,t),\ f_2(x)\}$ to the inverse problem (4), (5), (7). Denote

$$u(x,t) = u_1(x,t) - u_2(x,t)$$

and

$$f(x) = f_1(x) - f_2(x).$$

Then the functions $u(x,t)$ and $f(x)$ satisfy equation (4), boundary conditions (5), and homogeneous conditions

$$u(x,0) = 0,\ u(x,T) = 0,\ x \in [-1,1]. \tag{27}$$

Consider the Fourier coefficients:

$$u_k(t) = \int_{-1}^{1} u(x,t)\,\bar{X}_k(x)\,dx,\ k \in N, \tag{28}$$

$$f_k = \int_{-1}^{1} f(x)\,\bar{X}_k(x)\,dx,\ k \in N, \tag{29}$$

and note that the homogeneous conditions (27) lead to equalities

$$u_k(x,0) = u_k(x,T) = 0.$$

Differentiating equality (28) with respect to the variable t, we obtain

$$u'_k(t) = \int_{-1}^{1} u'_t(x,t)\,\bar{X}_k(x)\,dx,$$

where the derivative $u_t(x,t)$ will be replaced using equation (4)

$$u'_k(t) = \int_{-1}^{1} \left[-u_{xxxx}(x,t) - q(x)\,u(x,t)\right]\bar{X}_k(x)\,dx + \int_{-1}^{1} f(x)\,\bar{X}_k(x)\,dx,$$

or

$$u'_k(t) = \int_{-1}^{1} \left[-u_{xxxx}(x,t) - q(x)\,u(x,t)\right]\bar{X}_k(x)\,dx + f_k.$$

After integrating by parts four times, we get

$$u'_k(t) = \int_{-1}^{1} \left[-\bar{X}_k^{IV}(x) - \overline{q}(x)\,\bar{X}_k(x)\right]u(x,t)\,dx + f_k,$$

or

$$u'_k(t) = \int_{-1}^{1} -\overline{\lambda_k}\bar{X}_k(x)\,u(x,t)\,dx + f_k.$$

The last equality can be rewritten as

$$u'_k(t) + \overline{\lambda}_k u_k(t) = f_k.$$

As (29) is satisfied, i.e., $u_k(0) = u_k(T) = 0$, the last relation implies

$$f_k = 0,\ u_k(t) \equiv 0.$$

The basis property of the system $\{X_k(x)\}$ implies the equality

$$f(x) \equiv 0,\ u(x,t) \equiv 0,\ (x,t) \in \Omega.$$

The uniqueness of the solution is proved. The theorem is completely proved. The assertion of the theorem is fully applicable to the case of inverse problem (4), (6), (7).

Conclusion

The inverse problem of determining the right side for a fourth-order parabolic equation with a complex-valued variable coefficient is studied. The existence of a unique solution to the inverse problem with Dirichlet and Neumann boundary conditions is established

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Janno, J., & Seletski, A. (2015). Reconstruction of coefficients of higher order nonlinear wave equations by measuring solitary waves. *Wave Motion, 52,* 15–25. https://doi.org/10.1016/j.wavemoti.2014.08.005

2 Kabanikhin, S.I. (2011). Inverse and Ill-posed Problems: Theory and Applications. Berlin, Germany: De Gruyter. https://doi.org/10.1515/9783110224016

3 Ionkin, N.I., & Moiseev, E.I. (1979). A problem for a heat equation with two-point boundary conditions. *Differ. Uravn., 15*(7), 1284–1295. https://www.mathnet.ru/eng/de3763

4 Mussirepova, E., Sarsenbi, A.A., & Sarsenbi, A.M. (2022). Solvability of mixed problems for the wave equation with reflection of the argument. *Math Meth Appl Sci., 45*(17), 11262–11271. https://doi.org/10.1002/mma.8448

5 Ashyralyev, A., Ashyralyyev, A., & Abdalmohammed, B. (2023). On the hyperbolic type differential equation with time involution. *Bulletin of the Karaganda University. Mathematics series,* 1(109), 38–47. https://doi.org/10.31489/2023m1/38-47

6 Mussirepova, E., Sarsenbi, A., & Sarsenbi, A. (2022). The inverse problem for the heat equation with reflection of the argument and with a complex coefficient. *Bound Value Probl.,* 99. https://doi.org/10.1186/s13661-022-01675-1

7 Ashyralyev, A., Erdogan, A.S., & Sarsenbi, A. (2020). A note on the parabolic identification problem with involution and Dirichlet condition. *Bulletin of the Karaganda University. Mathematics series,* 3(99), 130–139. https://doi.org/10.31489/2020m3/130-139

8 Sarsenbi, A.A. (2018). A solvability conditions of mixed problems for equations of parabolic type with involution. *Bulletin of the Karaganda University. Mathematics series,* 4(92), 87–93. https://doi.org/10.31489/2018M4/87-93

9 Imanbetova, A., Sarsenbi, A., & Seilbekov, B. (2023). Inverse problem for a fourth-order hyperbolic equation with a complex-valued coefficient. *MDPI Mathematics, 11*(15), 3432. https://doi.org/10.3390/math11153432

10 Mehraliyev, Y., Allahverdiyeva, S., & Ramazanova, A. (2023). On one coefficient inverse boundary value problem for a linear pseudoparabolic equation of the fourth order. *AIMS Mathematics, 8*(2), 2622–2633. https://doi.org/10.3934/math.2023136

11 Naimark, M.A. (1968). *Linear Differential Operators.* New York: Ungar.

12 Kesel'man, G.M. (1964). On the unconditional convergence of eigenfunction expansions of certain differential operators. *Izv. Vyssh. Uchebn. Zaved. Mat.,* (2), 82–93. https://www.mathnet.ru/eng/ivm2384

13 Mikhailov, V.P. (1962). On Riesz bases in $L_2(0,1)$. *Dokl. Akad. Nauk SSSR, 144*(5), 981–984. https://www.mathnet.ru/eng/dan26607

14 Kerimov, N.B. (1996). On the basis property and uniform minimality of systems of root functions of differential operators. I. *Differential Equations, 32*(3), 321–327.

15 Sarsenbi, A.M. (2008). Necessary and sufficient conditions for the Riesz basis property of the eigen- and associated functions of high-order differential operators on an interval. *Doklady Mathematics, 77*(2), 290–292. https://doi.org/10.1134/S1064562408020336

Комплекс мәнді коэффициенті бар төртінші ретті параболалық теңдеу үшін кері есептің шешімділігі туралы

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Мақалада айнымалыларды ажырату әдісімен айнымалы комплексті коэффициенті бар төртінші ретті параболалық теңдеу үшін кері есеп зерттелген. Комплекс мәнді коэффициенті бар өзіне-өзі түйіндес емес төртінші ретті біртекті дифференциалдық теңдеу үшін Дирихле және Нейман шекаралық есептерінің меншікті мәндерінің қасиеттері белгіленген. Күшті регулярлы шекаралық шарттары бар біртекті дифференциалдық теңдеулер үшін шекаралық есептердің меншікті функцияларының $L_2(-1,1)$ кеңістігіндегі Рис базистік қасиеті бойынша белгілі нәтижелер пайдаланылады. Меншікті функциялардың Рис базистік қасиетінің негізінде зерттелетін есептердің формальды шешімдері құрастырылып, шешімдердің бар болуы мен жалғыздығы туралы теоремалар дәлелденген. Шешімдердің бар

екендігі мен жалғыздығы туралы теореманы дәлелдеу кезінде Бессель теңсіздігі Фурье коэффициенттері $L_2(-1,1)$ кеңістігінен Рис базисі бойынша Фурье қатарына функциялардың жіктелінуі үшін кеңінен қолданылады. Инволюциялы төртінші ретті теңдеу үшін шеттік есептердің меншікті функциялары бойынша Фурье қатарлары түрінде шешімдердің түр сипаты жазылды. Алынған шешімдердің жинақтылығы да талқыланған.

Кілт сөздер: параболалық теңдеу, кері есеп, классикалық шешім, Фурье әдісі, күшті регулярлы шекаралық шарттар, Рис базисі.

О разрешимости обратной задачи для параболического уравнения четвертого порядка с комплекснозначным коэффициентом

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В настоящей работе методом разделения переменных изучена обратная задача для параболического уравнения четвертого порядка с переменным комплекснозначным коэффициентом. Установлены свойства собственных значений краевых задач Дирихле и Неймана для несамосопряженного обыкновенного дифференциального уравнения четвертого порядка с комплекснозначным коэффициентом. Использованы известные результаты о базисности Рисса в пространстве $L_2(-1,1)$ собственных функций краевых задач для обыкновенных дифференциальных уравнений с усиленно регулярными краевыми условиями. На основании базисности Рисса собственных функций построены формальные решения изучаемых задач и доказаны теоремы о существовании и единственности решения. При доказательстве теорем о существовании и единственности решений применено неравенство Бесселя для коэффициентов Фурье разложений функций из пространства $L_2(-1,1)$ в ряд Фурье по базису Рисса. Выписаны представления решений в виде рядов Фурье по собственным функциям краевых задач для уравнения четвертого порядка с инволюцией. Также обсуждена сходимость полученных решений.

Ключевые слова: параболическое уравнение, обратная задача, классическое решение, метод Фурье, усиленно регулярные краевые условия, базис Рисса.

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