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A Second Order Convergence Method for Differential Difference Equation with Mixed Shifts using Mixed Non-Polynomial Spline

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A proposed numerical approximation method is presented for solving a singularly perturbed second-order differential-difference equation with both the delay and advance shifts. The algorithm utilises a non-polynomial spline with a fitting factor finite difference scheme. The application of finite difference approximations for higher order derivatives leads to the derivation of a tri-diagonal system. To efficiently solve this system of equations, an algorithm based on discrete invariant imbedding is employed and the stability of the method is analysed. An assessment of the applicability and efficiency of the proposed scheme is conducted by performing three numerical experiments and comparing the results with other methods. The maximum absolute errors are used as the basis for comparison. The impact of minor shifts on the boundary layer behaviour of the solution is illustrated using plotted graphs featuring different degrees of shifts. The method is theoretically and numerically analysed using uniformly convergent solutions with quadric convergence rate.

Keywords: Differential-Difference equation, Singular Perturbation problem, boundary layer, finite difference approximation, Stability.

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Introduction

In science and engineering, singularly perturbed differential-difference equations (SPDDEs) appear frequently in the mathematical modelling of real-life situations [1, 2]. The presence of small-time parasitic parameters such as moments of resistance, inertia, inductances, and capacitances in the mathematical modelling of a physical system, as in control theory, increases the order and stiffness of these systems. They are termed as singular perturbation systems, then they are called as singularly perturbed delay differential equations. Delay differential equations appear in first-exit time problems in practical bioscience phenomena. A differential-difference equation with the presence of shift terms induces large amplitudes and exhibits oscillations, resonance, turning point behaviour, and boundary and interior layers. As a result, simple and efficient numerical techniques are required to control such behaviour.

The extension methods developed in the papers [3,4] for ordinary differential equations to obtain approximate solution of SPDDEs with mixed shifts are published by the various authors. M. Adilaxmi, D. Bhargavi, and K. Phaneendra [5] devised a method for finding the Numerical Solution of SPDDEs using multiple fitting factors. Habtamu Garoma Debela and Gemechis File Duressa [6] consider SPDDEs with mixed small shift and the resulting singularly perturbed boundary value problem to solve the problem using fitted non-polynomial spline method. A fourth order exponentially fitted numerical scheme on uniform mesh is developed by Habtamu Garoma Debela, Solomon Bati Kejela

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and Ayana Deressa Negassa [7] to solve SPDDEs. For the numerical solution of singularly perturbed differential equations with delay and shift, Arshad Khan and Akmal Raza [8] proposed an efficient Haar wavelet collocation method. The authors [9] proposed a numerical scheme, involving cubic spline for Robin boundary conditions and the classical central difference scheme for solving a singularly perturbed reaction-diffusion problem. A numerical approach is proposed by the authors [10] using a hybrid-difference technique on a propitious layer-adaptive piecewise-uniform Shishkin mesh to examine a higher order convergent approximation for a class of singularly perturbed two-dimensional (2D) convection-diffusion-reaction elliptic problems with discontinuous convection and source terms which leads to an almost second-order estimate. The authors [11] applied the adaptive mesh based numerical approximations for solving the Darcy scale precipitation-dissolution reactive transport 1D and 2D models consist of a convection-diffusion-reaction PDE with reactions being described by an ODE having a nonlinear, discontinuous, possibly multi-valued right-hand side describing precipitate concentration in a porous medium effectively. A hybrid difference scheme involving the trapezoidal and the backward difference schemes is chosen by the authors [12] for integral boundary value problems of nonlinear singularly perturbed parameterized form consists of a priori and a posteriori error analysis. A higher order numerical approximation for analysing a class of multi-term time fractional partial integro-differential equations involving Volterra integral operators is explained by the authors [13] using an adaptive mesh.

With this motivation, an exponential fitting factor is introduced in non-polynomial method for the solution of SPDDEs with delay and advanced parameters. Problem description is explained in Section 1 and Section 2 explains the procedure of mixed non polynomial spline. Section 3 presents a numerical scheme for solving the problem, and Section 4 deals with the proposed scheme's convergence analysis. To demonstrate the efficacy of the proposed method, numerical experiments for several test problems are performed, and the results are presented in Section 5. The conclusion is given for the proposed work in the final section.

1 Problem description

Consider a linear singularly perturbed differential-difference equation of the following form

$$\varepsilon u''(v) + p(v)\, u'(v) + q(v)\, u(v-\delta) + r(v)\, u(v) + s(v)\, u(v+\omega) = f(v) \tag{1}$$

on $(0, 1)$, under the boundary conditions

$$\begin{aligned} u(v) &= \varphi(v), \quad -\delta \le v \le 0, \\ \text{and } u(1) &= \gamma(v), \quad 1 \le v \le 1+\omega. \end{aligned} \tag{2}$$

Here ε is a small parameter such that $0 < \varepsilon < 1$ and $\delta > 0$, $\omega > 0$ are known as the delay (negative shift) and the advance (positive shift) parameters respectively. When $0 < \delta = O(\varepsilon)$ and $0 < \omega = O(\varepsilon)$ then $p(v)$, $q(v)$, $r(v)$, $s(v)$ and $f(v)$ are smooth functions in the given domain and the higher order derivatives of $u(v-\delta)$ and $u(v+\omega)$ will vanish if the powers of δ and ω increase.

Since $0 < \delta < 1$ and $0 < \omega < 1$, by applying Taylor's series expansion for $u(v-\delta)$ and $u(v+\omega)$ then

$$u(v-\ \delta) = u(v) - \delta u'(v) + \mathrm{O}(\delta^2), \tag{3}$$

$$u(v+\ \omega) = u(v) + \omega u'(v) + \mathrm{O}\left(\omega^2\right). \tag{4}$$

Substituting Eqs. (3) and (4) in Eq. (1), then Eq. (1) becomes

$$\varepsilon u''(v) + a(v)\, u'(v) + b(v)\, u(v) = f(v) + \mathrm{O}(\delta^2 + \omega^2), \tag{5}$$

where

$$a(v) = p(v) - \delta q(v) + \omega s(v) \ \text{ and } \ b(v) = q(v) + r(v) + s(v).$$

Eq. (5) is an asymptotically equivalent second order singular perturbation problem of Eq. (1) with boundary conditions as

$$u(0) = \varphi(0) \text{ and } u(1) = \omega(1). \tag{6}$$

Thus, the solution of Eq. (5) provides a good approximation to the solution of Eq. (1). If $a(v) > 0$, the solution of Eq. (1) with Eq. (2) exhibits layer at the left end of the interval and if $a(v) < 0$, the layer exhibits at the right end of the interval.

2 Mixed non-polynomial spline

Let $a = v_0 < v_1 < v_2 < \cdots < v_n = b$, we first divide the interval $[a, b]$ into 'n' equal parts by introducing $v_i = a + ih$, $i = 0, 1, \ldots, n$ and $h = \frac{b-a}{n}$.

Let

$$P_i(v) = a_i \exp[\tau(v - v_i)] + b_i [\cos(\tau(v - v_i)) + \sin(\tau(v - v_i))] + c_i \tag{7}$$

be a mixed non-polynomial quadratic spline defined on [a, b] which interpolates the uniform mesh points v_i, depends on a parameter τ, reduces to an ordinary quadratic spline in [a, b] as $\tau \to 0$. To determine the coefficients a_i, b_i and c_i, the following interpolation conditions are defined as

$$P_i(v_i) = u_i, \quad P_i(v_{i+1}) = u_{i+1}, \quad P_i''(v_i) = \frac{1}{2}(Z_i + Z_{i+1}), \text{ for } i = 0, 1, \ldots, n.$$

By using the above conditions, the coefficients in Eq. (7) are calculated as

$$a_i = \frac{u_{i+1} - u_i}{\sin\theta + \cos\theta + \exp\theta - 2} - \frac{h^2}{2\theta^2}\left(\frac{\exp\theta - 1}{\sin\theta + cos\theta + \exp\theta - 2} - 1\right)(Z_i + Z_{i+1}),$$

$$b_i = \frac{u_{i+1} - u_i}{\sin\theta + \cos\theta + \exp\theta - 2} - \frac{h^2}{2\theta^2}\left(\frac{\exp\theta - 1}{\sin\theta + \cos\theta + \exp\theta - 2}\right)(Z_i + Z_{i+1}),$$

$$c_i = \frac{u_{i+1} + (\sin\theta + \cos\theta + \exp\theta)y_i}{\sin\theta + \cos\theta + \exp\theta - 2} + \frac{h^2}{2\theta^2}\left(\frac{2\exp\theta - 1}{\sin\theta + \cos\theta + \exp\theta - 2} - 1\right)(Z_i + Z_{i+1}),$$

where $\theta = \tau h$.

Using the continuity of first derivative, $P_{i-1}^m(v_i) = P_i^m(v_i)$, $m = 0, 1$, the following consistency relation derived

$$\alpha u_{i-1} + \beta u_i + \gamma u_{i+1} = h^2 (\alpha_1 Z_{i-1} + \beta_1 Z_i + \gamma_1 Z_{i+1}), \quad i = 0, 1, \ldots, n, \tag{8}$$

where

$$\alpha = \frac{\exp\theta + \cos\theta + \sin\theta}{2},$$
$$\beta = \frac{\sin\theta - \cos\theta - \exp\theta + 2}{2},$$
$$\gamma = 1,$$
$$\alpha_1 = \frac{(2\sin\theta - 1)\exp\theta + \cos\theta - \sin\theta}{4\theta^2},$$
$$\beta_1 = \frac{\sin\theta\exp\theta - \sin\theta}{2\theta^2},$$
$$\gamma_1 = \frac{\exp\theta - \sin\theta - \cos\theta}{4\theta^2}.$$

Remark: The proposed method reduces to Al-Said [14] based on quadratic spline when $(\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1) = (1, -2, 1, \frac{1}{4}, \frac{1}{2}, \frac{1}{4})$.

3 Application of the method

At the grid points v_i, Eq. (5) can be written as

$$\varepsilon u_i'' = -a(v_i)\,u_i' - b(v_i)\,u_i + f(v_i),$$

using $Z_i = u_i''$ in the above equation, then it becomes

$$\varepsilon Z_j = -a_j(v_i)\,u_i' - b_j(v_i)\,u_i + f_j(v_i) \quad \text{for} \quad j = i-1,\ i,\ i+1. \tag{9}$$

Using Eq. (9) in Eq. (8) and the following approximations for the first derivative of u as

$$u_i' = \frac{(u_{i+1} - u_{i-1})}{2h}, \quad u_{i+1}' = \frac{(3u_{i+1} - 4u_i + u_{i-1})}{2h} \text{ and } u_{i-1}' = \frac{(-u_{i+1} + 4u_i - 3u_{i-1})}{2h},$$

then

$$\begin{aligned}\frac{\varepsilon}{h^2}(\alpha u_{i-1} + \beta u_i + \gamma u_{i+1}) &= -\alpha_1 a_{i-1}\frac{(-u_{i+1} + 4u_i - 3u_{i-1})}{2h} - \beta_1 a_i \frac{(u_{i+1} - u_{i-1})}{2h} \\ &- \gamma_1 a_{i+1}\frac{(3u_{i+1} - 4u_i + u_{i-1})}{2h} - \alpha_1 b_{i-1} u_{i-1} - \beta_1 b_i u_i - \gamma_1 b_{i+1} u_{i+1} \\ &+ (\alpha_1 f_{i-1} + \beta_1 f_i + \gamma_1 f_{i+1}).\end{aligned} \tag{10}$$

By introducing a constant fitting factor $\sigma\ (\rho)$ in the above scheme (10), we have

$$\begin{aligned}\frac{\varepsilon\sigma(\rho)}{h^2}(\alpha u_{i-1} + \beta u_i + \gamma u_{i+1}) &= -\alpha_1 a_{i-1}\frac{(-u_{i+1} + 4u_i - 3u_{i-1})}{2h} - \beta_1 a_i \frac{(u_{i+1} - u_{i-1})}{2h} \\ &- \gamma_1 a_{i+1}\frac{(3u_{i+1} - 4u_i + u_{i-1})}{2h} - \alpha_1 b_{i-1} u_{i-1} - \beta_1 b_i u_i - \gamma_1 b_{i+1} u_{i+1} \\ &+ \left(\alpha_1 f_{i-1} + \beta_1 f_i + \gamma_1 f_{i+1}\right).\end{aligned} \tag{11}$$

On simplification, the obtained tridiagonal system as

$$E_i u_{i-1} + F_i u_i + G_i u_{i+1} = H_i, \quad i = 1, 2, \ldots, N-1. \tag{12}$$

A brief explanation and simplification about tridiagonal system are given in [15–19], where

$$E_i = \varepsilon\alpha\sigma - \frac{3\alpha_1 h a_{i-1}}{2} - \frac{\beta_1 h a_i}{2} + \frac{\gamma_1 h a_{i+1}}{2} + h^2\alpha_1 b_{i-1},$$

$$F_i = \varepsilon\beta\sigma + 2\alpha_1 h a_{i-1} - 2\gamma_1 h a_{i+1} + h^2\beta_1 b_i,$$

$$G_i = \varepsilon\gamma\sigma - \frac{\alpha_1 h a_{i-1}}{2} + \frac{\beta_1 h a_i}{2} + \frac{3\gamma_1 h a_{i+1}}{2} + h^2\gamma_1 b_{i+1},$$

$H_i = h^2\left(\alpha_1 f_{i-1} + \beta_1 f_i + \gamma_1 f_{i+1}\right)$ with the truncation error is

$$\begin{aligned}t_i &= (\alpha + \beta + \gamma)\,u_i + (-\alpha + \gamma)\,h u_i' + \left[\frac{\alpha + \gamma}{2!} - (\alpha_1 + \beta_1 + \gamma_1)\right] h^2 u_i'' + \left[\frac{-\alpha + \gamma}{3!} - (-\alpha_1 + \gamma_1)\right] h^3 u_i''' \\ &+ \left[\left(\frac{\alpha + \gamma}{4!} - \frac{\alpha_1 + \gamma_1}{2!}\right)\varepsilon u_i^{(iv)} + \frac{1}{6}\left(\beta_1 - 2(\alpha_1 + \gamma_1)\right) a_i y_i'''\right] h^4 + O(h^5) \text{ for } i = 1, \ldots, n-1.\end{aligned}$$

For the choice of parameters $(\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1) = (1, -2, 1, \frac{1}{12}, \frac{10}{12}, \frac{1}{12})$, then the order of the truncation error is fourth order.

To calculate fitting parameter from singular perturbations theory [20, 21], the following is an approximation for the solution of the homogeneous problem of Eq. (1)

$$u(v) = u_0(v) + \frac{a(0)}{a(v)}(\alpha_1 - u_0(0))\, e^{-\int_0^v \left(\frac{a(v)}{\varepsilon} - \frac{b(v)}{a(v)}\right)dv} + O(\varepsilon),$$

where $u_0(x)$ is the solution of $a(v)\, u_0{}'(v) + b(v)\, u_0(v) = f(v)$, $u_0(1) = \psi_1$.

By using the Taylor's series expansion for a (v) and $b(v)$ about the point zero and limiting to their first terms, Eq. (11) becomes

$$u(v) = u_0(v) + (\phi_0 - u_0(0))\, e^{-\left(\frac{a(0)}{\varepsilon} - \frac{b(0)}{a(0)}\right)v} + O(\varepsilon).$$

From Eq. (11), it is clear that

$$\lim_{h\to 0} u(ih) = u_0(0) + (\phi_0 - u_0(0))\, e^{-\left(\frac{a^2(0) - \varepsilon b(0)}{a(0)}\right)i\rho},$$

$$\lim_{h\to 0} u(ih + h) = u_0(0) + (\phi_0 - u_0(0))\, e^{-\left(\frac{a^2(0) - \varepsilon b(0)}{a(0)}\right)(i\rho+\rho)},$$

$$\lim_{h\to\ 0} u(ih\ -\ h) = u_0(0) + (\phi_0 - u_0(0))\, e^{-\left(\frac{u^2(0) - \varepsilon b(0)}{a(0)}\right)(i\rho\ -\ \rho)}.$$

Using these limit values in Eq. (11), the fitting parameter obtains as

$$\sigma(\rho) = (\alpha_1 + 0.5\beta_1)\, a_i\rho Coth\left(\frac{a_i\rho}{2}\right), \text{ where } \rho = \frac{h}{\varepsilon},$$

which is the required value of the constant fitting factor $\sigma(\rho)$ in this case of problems having boundary layer at right end and left end of the given interval.

4 Convergence Analysis

Theorem 1. Under the assumptions that $q(v) \geq M > 0$ and $r(v) < 0$, $\forall v \in [0, 1]$, the solution to the system of difference equations (12) together with the given boundary conditions exists, is unique and satisfies $\|u\| \leq M^{-1}\|f\| + \max[|\varphi(0)| + |\gamma(1)|]$.

Proof. Proof of the above theorem can be found in [22–28].

Incorporating the boundary conditions in Eq. (6), the system of Eq. (12) with the truncation error can be written in the matrix form as:

$$(\mathbb{D}{+}\mathbb{P})\, U + \hat{M} + T(h) = 0, \tag{13}$$

where

$$\mathbb{D} = \begin{bmatrix} -2\varepsilon\sigma & \varepsilon\sigma & 0 & 0 & ... & 0 \\ \varepsilon\sigma & -2\varepsilon\sigma & \varepsilon\sigma & .. & ... & 0 \\ 0 & .. & .. & .. & ... & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & .. & \varepsilon\sigma & -2\varepsilon\sigma \end{bmatrix},$$

$$\mathbb{P}=[p_i,q_i,r_i]=\begin{bmatrix} p_1 & r_1 & 0 & 0 & .... & 0\\ p_2 & q_2 & r_2 & 0 & .... & 0\\ 0 & p_3 & q_3 & r_3 & .... & 0\\ .. & .. & .. & .. & .. & ..\\ .. & .. & .. & .. & .. & ..\\ 0 & .. & .. & 0 & p_{N-1} & q_{N-1}\end{bmatrix},$$

where

$$p_i=-\frac{3\alpha_1 h a_{i-1}}{2}-\frac{\beta_1 h a_i}{2}+\frac{\gamma_1 h a_{i+1}}{2}+h^2\alpha_1 b_{i-1},\ q_i=2\alpha_1 h a_{i-1}-2\gamma_1 h a_{i+1}+h^2\beta_1 b_i,$$

$$r_i=-\frac{\alpha_1 h a_{i-1}}{2}+\frac{\beta_1 h a_i}{2}+\frac{3\gamma_1 h a_{i+1}}{2}+h^2\gamma_1 b_{i+1},\ \text{for } i=1,2,\dots,N-1,$$

$$\hat{M}\ =\ [g_1+(\varepsilon\sigma+p_1)\,\phi\,(0)\,,g_2,g_3\ ,\dots,g_{N-2},g_{N-1}+(\varepsilon\sigma+r_{N-1})\,\gamma_1]^T,$$

where $g_i = h^2\left(\alpha_1 f_{\ i-1} + \beta_1 f_i + \gamma_1 f_{i+1}\right)$, for $i = 1,2,\dots,N-1$, $T(h) = O\left(h^4\right)$ and $u=[u_1,u_2,...,u_{N-1}]^T$, $T(h)=[T_1,T_2,...,T_{N-1}]^T$, $O=[0,0,...,0]^T$ are corresponding vectors of Eq. (12).

Let $U=[U_1,U_2,...,U_{N-1}]^T\cong U$, which satisfies the equation

$$(\mathbb{D}+\mathbb{P})\,U+\hat{M}\ =\ 0. \tag{14}$$

Let $e_i=U-u_i,\ i=1,\ 2.\ \dots,\ N-1$ be the discretization error, so that

$$E=[e_1,e_2,...,e_{N-1}]^T=U-u.$$

Subtracting Eq. (14) from Eq. (13), then the error equation is

$$(\mathbb{D}+\mathbb{P})\,E=T(h). \tag{15}$$

Let $|a_i|\le K_1$, $|b_i|\le K_2$ so that if $A_{i,j}$ is the (i,j) th element of matrix $\widehat{\mathcal{P}}$, then

$$|A_{i,i+1}|=|w_i|\le\varepsilon+h\,(\alpha_1+\beta_1+3\gamma_1)\,K_1+h^2\alpha_1 K_2,\ i=1,2,\dots,N-2,$$

$$|A_{i,\ i-1}|=|u_i|\le\varepsilon+h\,(3\alpha_1+\beta_1+\gamma_1)\,K_1+h^2\alpha_1 K_2,\ i=2,3\dots,N-1.$$

Thus, for sufficiently small $h (h\to 0)$, it observes that $|A_{i,i+1}|<\varepsilon$, for $i=1,2,\dots,N-2$ and $|A_{i,i-1}|<\varepsilon$, for $i=2,3,\dots,N-1$. Hence $(\mathbb{D}+\mathbb{P})$ is irreducible [29].

Let $\mathbb{S}_i$ be the sum of i^{th} row elements of the matrix $(\mathbb{D}+\mathbb{P})$, then

$$\mathbb{S}_i=-\varepsilon\sigma+\frac{3\alpha_1 h a_{i-1}}{2}+\frac{\beta_1 h a_i}{2}-\frac{\gamma_1 h a_{i+1}}{2}+h^2\left(\gamma_1 b_{i+1}+\beta_1 b_i\right)\ \text{for } i=1,$$
$$\mathbb{S}_i=h^2\left(\alpha_1 b_{\ i-1}+\beta_1 b_i+\gamma_1 b_{i+1}\right)\ \text{for } i=2,3,\dots,N-2,$$
$$\mathbb{S}_i=-\varepsilon\sigma+\frac{\alpha_1 h a_{i-1}}{2}-\frac{\beta_1 h a_i}{2}-\frac{3\gamma_1 h a_{i+1}}{2}+h^2\left(\ \alpha_1 b_{i-1}+\beta_1 b_i\right)\ \text{for } i=N-1.$$

Let $K_{1*}=\min_{1\le i\le N-1}|a_i|\,,K_1{}^*=\max_{1\le i\le N}|a_i|\,,$
$K_{2*}=\min_{1\le i\le N-1}|b_i|\,,K_2{}^*=\max_{1\le i\le N}|b_i|\,,$ then

$$0\le K_{1*}\le K_1\le K_1{}^*,\ \ 0\le K_{2*}\le K_2\le K_2{}^*,$$

for sufficiently small h, $(\mathbb{D}+\mathbb{P})$ is monotone [30–32]. Hence $(\mathbb{D}+\mathbb{P})^{-1}$ exists and $(\mathbb{D}+\mathbb{P})^{-1}\ge 0$.

Thus, from Eq. (15), it has

$$\|E\| \leq \left\|(\mathbb{D}+\mathbb{P})^{-1}\right\| \|T\|. \tag{16}$$

For sufficiently small h, let $(\mathbb{D}+\mathbb{P})^{-1}_{i,k}$ be the $(i,k)^{th}$ element of $(\mathbb{D}+\mathbb{P})^{-1}$ and define

$$\left\|(\mathbb{D}+\mathbb{P})^{-1}\right\| = \max_{1\leq i\leq N-1} \sum_{k=1}^{N-1} (\mathbb{D}+\mathbb{P})^{-1}_{i,k} \quad \text{and} \quad \|T(h)\| = \max_{1\leq i\leq N-1} |T_i|.$$

Since $(\mathbb{D}+\mathbb{P})^{-1}_{i,k} \geq 0$ and $\sum_{k=1}^{N-1} (\mathbb{D}+\mathbb{P})^{-1}_{i,k} \mathbb{S}_k = 1$ for $i = 1,2,...,N-1$, we have

$$(\mathbb{D}+\mathbb{P})^{-1}_{i,k} \leq \frac{1}{\mathbb{S}_i} < \frac{1}{h^2 K_2} \quad \text{for} \quad i = N-1.$$

Furthermore,

$$\sum_{k=1}^{N-1} (\mathbb{D}+\mathbb{P})^{-1}_{i,k} \leq \frac{1}{\min_{2\leq i\leq N-2} \mathbb{S}_i} < \frac{1}{h^2 K_2}. \tag{17}$$

By the help of Eqs. (17) and using Eq. (16), it becomes

$$\|E\| \leq O\left(h^2\right).$$

This illustrates the proposed finite difference scheme Eq. (12) reaches a maximum of second order convergence at certain stage for $(\alpha_1, \beta_1, \gamma_1) = \left(\frac{1}{12}, \frac{10}{12}, \frac{1}{12}\right)$.

5 Numerical Examples

Four example problems are chosen and presented the numerical results in terms of the maximum absolute errors (MAE) with the computed rates of convergence in the tables to demonstrate the accuracy and efficiency of the proposed method. These maximum absolute errors for different values of N and ε are obtained using the relation. Wherever exact solutions are not known, the MAEs are calculated using the double mesh principle given by

$$E_N = \max_{0\leq i\leq N} \left|u_i{}^N - u_{2i}{}^{2N}\right|,$$

where $u_i{}^N$ and $u_{2i}{}^{2N}$ are the numerical solutions of the problem for N and $2N$ mesh points respectively. Further, formula is used to determine the numerical rate of convergence

$$R_N = log_2 \left|\frac{E_N}{E_{2N}}\right|.$$

The exact solution of the considered singularly perturbed differential-difference equation with constant coefficients, say $p(v) = p$, $q(v) = q$, $r(v) = r$, $s(v) = s$, $f(v) = f$, $\phi(v) = \phi$, $\psi(v) = \psi$ in Eq. (1) and Eq. (2), then

$$\varepsilon u''(v) + pu'(v) + qu(v-\delta) + ru(v) + su(v+\eta) = f, \; 0 < v < 1,$$

with respect to the interval and boundary conditions $u(v) = \phi$, $-\delta \leq v \leq 0$ and $u(\,v) = \psi$, $1 \leq v \leq 1+\eta$ is given by

$$y(x) = C_1 e^{m_1(x)} + C_2 e^{m_2(x)} + \frac{f}{c},$$

where

$$C_1 = \frac{\left[e^{m_2}\left(f - c\phi\right) - f + \psi c\right]}{c\left(e^{m_1} - e^{m_2}\right)},$$

$$C_2 = \frac{\left[e^{m_1}\left(-f + c\phi\right) - f + \psi c\right]}{c\left(e^{m_1} - e^{m_2}\right)}.$$

$$m_1 = \frac{-\left(p - q\delta + s\eta\right) + \sqrt{\left(p - q\delta + s\eta\right)^2 - 4c\varepsilon}}{2\varepsilon}, m_2 = \frac{-\left(p - q\delta + s\eta\right) - \sqrt{\left(p - q\delta + s\eta\right)^2 - 4c\varepsilon}}{2\varepsilon},$$

with $c = q + r + s$.

Example 1. Consider a boundary value problem $\varepsilon u''(v) + u'(v) - u(v-\delta) + u(v) - u(v+\eta) = -1$, with boundary constraints $u(v) = 1;\ -\delta \le v \le 0,\ u(v) = 1,\ 1 \le v \le 1+\eta$.

Example 2. Consider a boundary value problem $\varepsilon u''(v) + 2.5u'(v) - 2exp(v)\,u(v-\delta) - u(v) - vu(v+\eta) = 1$, with boundary constraints $u(v) = 1;\ -\delta \le v \le 0,\ u(v) = 1,\ 1 \le v \le 1+\eta$.

Example 3. Consider a boundary value problem $\varepsilon u''(v) - \left(1 + exp\left(-v^2\right)\right) u'(v) - vu(v-\delta) - v^2 u(v) - (1.5 - exp(-v))\,u(v+\eta) = 1$, with boundary constraints $u(v) = 1; -\delta \le v \le 0$, $u(v) = 1,\ 1 \le v \le 1+\eta$.

Example 4. Consider a boundary value problem $\varepsilon u''(v) - \left(1 + exp\left(v^2\right)\right) u'(v) - vu(v-\delta) + v^2 u(v) - (1 - exp(-v))\,u(v+\eta) = 1$, with boundary constraints $u(v) = 1\ ;\ -\delta \le v \le 0,\ u(v) = -1$, $1 \le v \le 1+\eta$.

Table 1

The MAEs in Example 1 for various values of ε

$\varepsilon \downarrow \quad N \rightarrow$	2^3	2^4	2^5	2^6	2^7	2^8
Present method $\eta = \delta = 0.5\varepsilon$						
10^{-1}	2.405(-03)	6.367(-04)	1.615(-04)	4.053(-05)	1.014(-05)	2.553(-06)
	1.9174	1.9788	1.9948	1.9988	1.9898	
10^{-2}	9.363(-03)	4.724(-03)	1.765(-03)	5.110(-04)	1.331(-04)	3.365(-05)
	0.9870	1.4203	1.7882	1.9408	1.9834	
10^{-3}	9.438(-03)	5.180(-03)	2.724(-03)	1.397(-03)	6.805(-04)	2.679(-04)
	0.8655	0.9272	0.9633	1.0376	1.3449	
10^{-4}	9.441(-03)	5.183(-03)	2.725(-03)	1.398(-03)	7.087(-04)	3.568(-04)
	0.8651	0.9275	0.9628	0.9801	0.9900	
10^{-5}	9.442(-03)	5.183(-03)	2.725(-03)	1.398(-03)	7.088(-04)	3.568(-04)
	0.8334	0.9075	0.9516	0.9752	0.9874	
10^{-6}	9.442(-03)	5.183(-03)	2.725(-03)	1.398(-03)	7.088(-04)	3.568(-04)
	0.8653	0.9075	0.9516	0.9752	0.9874	
Results in [22]						
10^{-1}	3.658(-03)	9.595(-04)	2.409(-04)	6.759(-05)	1.776(-05)	1.232(-05)
10^{-2}	1.695(-02)	7.297(-03)	2.486(-03)	6.964(-04)	1.776(-04)	2.616(-05)
10^{-3}	2.0208(-02)	1.047(-02)	5.210(-03)	2.461(-03)	1.057(-03)	3.771(-04)
10^{-4}	2.052(-02)	1.079(-02)	5.520(-03)	2.769(-03)	1.363(-03)	6.539(-04)
10^{-5}	2.061(-02)	1.088(-02)	5.608(-03)	2.858(-03)	1.453(-03)	7.417(-04)
10^{-6}	1.951(-02)	9.783(-03)	4.513(-03)	1.762(-03)	3.577(-04)	3.729(-04)

Table 2

MAEs of Example 2 with various values of ε

$\varepsilon\downarrow \quad N\rightarrow$	10^1	10^2	10^3	10^4
Present method $\delta = 0.7\varepsilon, \eta = 0.5$				
10^{-1}	1.429(-02)	1.798(-04)	1.803(-06)	1.803(-08)
10^{-2}	2.692(-02)	1.852(-03)	2.111(-05)	2.111(-07)
10^{-3}	2.718(-02)	3.371(-03)	1.918(-04)	2.161(-06)
10^{-4}	2.721(-02)	3.375(-03)	3.461(-04)	1.925(-05)
Results in [22]				
10^{-1}	1.533(-02)	1.917(-04)	1.921(-06)	1.917(-08)
10^{-2}	2.817(-02)	1.865(-03)	2.024(-05)	2.026(-07)
10^{-3}	2.853(-02)	3.389(-03)	1.919(-04)	2.162(-06)
10^{-4}	2.857(-02)	3.395(-03)	3.463(-04)	1.925(-05)

Table 3

MAEs and rate of convergence of Example 2 with various values of ε

$\varepsilon\downarrow \quad N\rightarrow$	2^5	2^6	2^7	2^8	2^9	2^{10}
Present method $\eta = \delta = 0.5\varepsilon$						
2^{-3}	1.338(-03)	3.389(-04)	8.501(-05)	2.127(-05)	5.318(-06)	1.329(-06)
	1.9811	1.9951	1.9987	1.9988	2.000	
2^{-4}	2.827(-03)	7.339(-04)	1.852(-04)	4.642(-05)	1.161(-05)	2.904(-06)
	1.9456	1.9864	1.9962	1.9993	1.9992	
2^{-5}	5.401(-03)	1.511(-03)	3.898(-04)	9.822(-05)	2.460(-05)	6.154(-06)
	1.8377	1.9546	1.9886	1.9973	1.9990	
2^{-6}	8.369(-03)	2.842(-03)	7.859(-04)	2.019(-04)	5.084(-05)	1.273(-05)
	1.5581	1.8544	1.9607	1.9896	1.9977	
2^{-7}	9.770(-03)	4.381(-03)	1.461(-03)	4.014(-04)	1.029(-04)	2.590(-05)
	1.1570	1.5843	1.8638	1.9637	1.9902	
2^{-8}	9.930(-03)	5.106(-03)	2.244(-03)	7.411(-04)	2.029(-04)	5.200(-05)
	0.9596	1.1861	1.5983	1.8688	1.9641	
2^{-9}	9.946(-03)	5.183(-03)	2.613(-03)	1.136(-03)	3.733(-04)	1.020(-04)
	0.9191	0.9770	1.1965	1.6038	1.8703	
Results in [22]						
2^{-3}	1.378(-03)	3.486(-04)	8.742(-05)	2.187(-05)	5.469(-06)	1.367(-06)
2^{-4}	2.880(-03)	7.458(-04)	1.881(-04)	4.714(-05)	1.179(-05)	2.948(-06)
2^{-5}	5.477(-03)	1.526(-03)	3.930(-04)	9.902(-05)	2.480(-05)	6.204(-06)
2^{-6}	8.487(-03)	2.862(-03)	7.898(-04)	2.028(-04)	5.105(-05)	1.278(-05)
2^{-7}	9.922(-03)	4.413(-03)	1.466(-03)	4.024(-04)	1.031(-04)	2.596(-05)
2^{-8}	1.009(-02)	5.148(-03)	2.252(-03)	7.424(-04)	2.032(-04)	5.206(-05)
2^{-9}	1.011(-02)	5.228(-03)	2.624(-03)	1.138(-03)	3.736(-04)	1.021(-04)

Table 4

MAEs and rate of convergence of Example 3 with various values of ε

$\varepsilon\downarrow\quad N\rightarrow$	2^5	2^6	2^7	2^8	2^9	2^{10}
Present method $\eta=\delta=0.5\varepsilon$						
2^{-3}	3.025(-04)	7.465(-05)	1.899(-05)	4.693(-06)	1.183e-06	2.924e-07
	1.9943	1.9846	1.9940	2.0000	1.9998	
2^{-4}	6.723(-04)	1.674(-04)	4.258(-05)	1.034(-05)	2.609(-06)	6.524(-07)
	1.9979	1.9950	1.9962	1.9991	1.9996	
2^{-5}	1.524e-03	3.658e-04	9.013e-05	2.143e-05	5.612e-06	1.404e-06
	1.0934	1.9846	1.9200	1.9970	1.9989	
2^{-6}	2.230(-03)	7.967(-04)	3.222(-04)	1.852(-04)	1.174(-05)	2.940(-06)
	1.4849	1.3060	0.7988	0.979	1.9975	
2^{-7}	2.298(-03)	1.157(-03)	4.099(-04)	9.466(-05)	2.429(-05)	6.034e-06
	0.9899	1.4973	1.9933	1.9626	1.0984	
2^{-8}	2.294(-03)	1.181(-03)	5.905(-04)	2.085(-04)	4.798(-05)	1.232(-05)
	0.9578	1.0000	1.5046	1.9781	1.9584	
2^{-9}	2.298(-03)	1.177(-03)	5.998(-04)	2.985(-04)	1.059(-04)	2.506(-05)
	0.9652	0.9725	1.0060	1.5087	2.0000	
Results in [22]						
2^{-3}	8.434(-04)	2.112(-04)	5.284(-05)	1.321(-05)	3.303(-06)	8.260(-07)
2^{-4}	4.172(-03)	1.047(-03)	2.640(-04)	6.602(-05)	1.650(-05)	4.127(-06)
2^{-5}	1.858(-02)	4.743(-03)	1.190(-03)	2.980(-04)	7.452(-05)	1.864(-05)
2^{-6}	6.074(-02)	1.988(-02)	5.080(-03)	1.275(-03)	3.192(-04)	7.981(-05)
2^{-7}	1.111(-01)	6.451(-02)	2.061(-02)	5.270(-03)	1.323(-03)	3.311(-04)
2^{-8}	1.297(-01)	1.176(-01)	6.658(-02)	2.101(-02)	5.372(-03)	1.349(-03)
2^{-9}	1.310(-01)	1.372(-01)	1.212(-01)	6.766(-02)	2.122(-02)	5.425(-03)

Table 5

MAEs and rate of convergence of Example 4 with various values of ε

$\varepsilon\downarrow\quad N\rightarrow$	2^5	2^6	2^7	2^8	2^9
Present method $\eta=\delta=0.5\varepsilon$					
2^{-3}	1.826(-03)	4.086(-04)	1.002(-04)	2.543(-05)	6.246(-06)
	1.9982	1.9994	1.9989	2.0000	
2^{-4}	4.576(-03)	9.745(-04)	2.165(-04)	5.245(-05)	1.310(-05)
	1.9567	1.9988	1.9909	1.0099	
2^{-5}	9.233(-03)	2.431(-03)	4.959(-04)	1.104(-04)	2.678(-05)
	1.9359	1.9989	1.9846	1.9999	
2^{-6}	1.229(-02)	4.704(-03)	1.225(-03)	2.420(-04)	5.598(-05)
	1.3855	1.9411	2.0000	1.9994	
2^{-7}	1.275(-02)	6.203(-03)	2.374(-03)	6.166(-04)	1.264(-04)
	1.0394	1.3856	1.9430	1.9998	
2^{-8}	1.275(-02)	6.417(-03)	3.115(-03)	1.192(-03)	3.098(-04)
	0.8131	1.0426	1.3858	1.9439	
Results in [33]					
2^{-3}	8.354(-03)	2.013(-03)	4.986(-04)	1.249(-04)	3.121(-05)
2^{-4}	1.719(-02)	4.378(-03)	1.041(-03)	2.571(-04)	6.429(-05)
2^{-5}	2.517(-02)	8.889(-03)	2.238(-03)	5.290(-04)	1.303(-04)
2^{-6}	3.154(-02)	1.294(-02)	4.516(-03)	1.131(-03)	2.664(-04)
2^{-7}	4.478(-02)	1.622(-02)	6.559(-03)	2.276(-03)	5.686(-04)
2^{-8}	7.878(-02)	2.317(-02)	8.224(-03)	3.301(-03)	1.142(-03)

Table 6

MAE of Example 4 with $\varepsilon = 0.1$

$\delta \downarrow N \rightarrow$	2^3	2^5	2^7	2^9
Present method with $\eta = 0.5 * \varepsilon$				
0.00	3.889(-02)	2.576(-03)	1.305(-04)	8.065(-06)
0.05	3.850(-02)	2.527(-03)	1.286(-04)	7.948(-06)
0.09	3.818(-02)	2.489(-03)	1.272(-04)	7.853(-06)
Results in [33]				
0.00	9.109(-02)	1.112(-02)	6.382(-04)	4.004(-05)
0.05	9.047(-02)	1.095(-02)	6.306(-04)	3.950(-05)
0.09	8.996(-02)	1.082(-02)	6.244(-04)	3.906(-05)
$\eta \downarrow N \rightarrow$	2^3	2^5	2^7	2^9
Present Method $\delta = 0.5 * \varepsilon$				
0.00	3.835(-02)	2.502(-03)	1.277(-04)	7.888(-06)
0.05	3.850(-02)	2.527(-03)	1.286(-04)	7.948(-06)
0.09	3.862(-02)	2.548(-03)	1.294(-04)	7.995(-06)
Results in [33]				
0.00	9.604(-02)	1.116(-02)	6.458(-04)	3.924(-05)
0.05	9.621(-02)	1.124(-02)	6.494(-04)	3.952(-05)
0.09	9.634(-02)	1.131(-02)	6.522(-04)	3.970(-05)

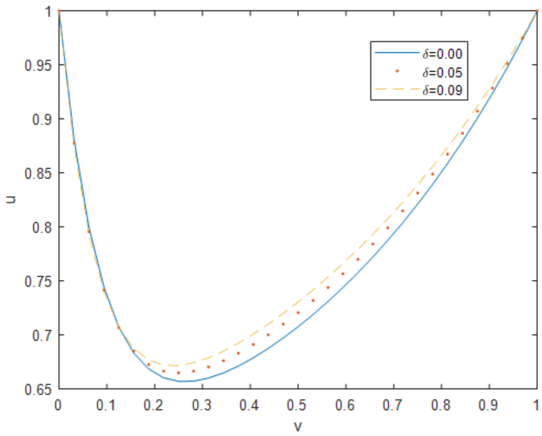

Figure 1. Numerical solution of Ex. 1 for various values of δ with N=2^5, $\varepsilon = 10^{-1}$ and $\eta = 0.05$

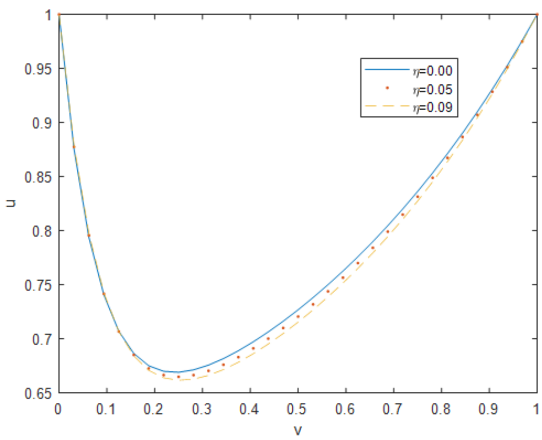

Figure 2. Numerical solution of Ex. 1 for various values of η with N=2^6, $\varepsilon = 10^{-2}$ and δ=0.05

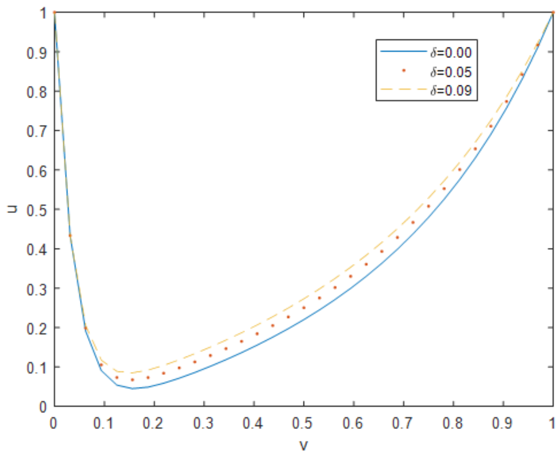

Figure 3. Numerical solution of Ex. 2 for various values of δ with N=2^5, $\varepsilon = 10^{-1}$ and η=0.05

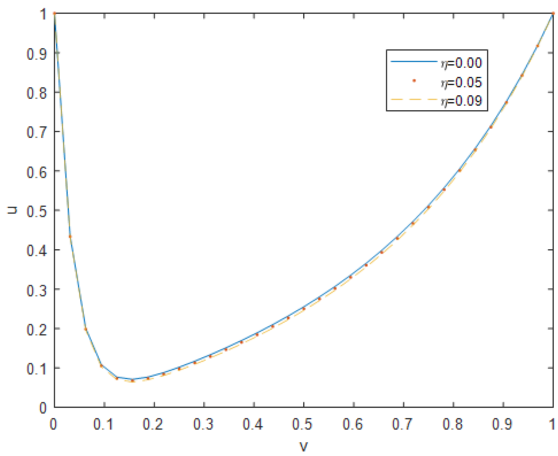

Figure 4. Numerical solution of Ex. 2 for various values of η with N=2^5, $\varepsilon = 10^{-1}$ and δ=0.05

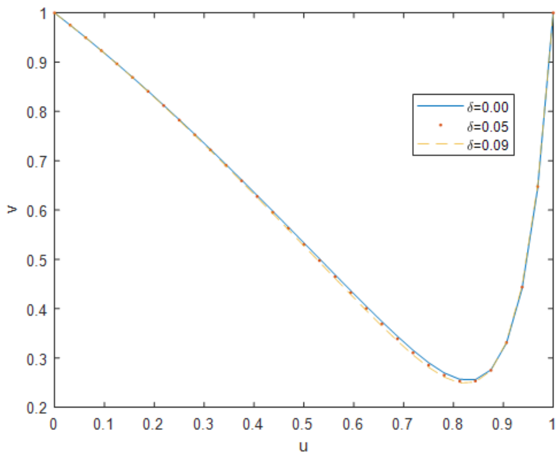

Figure 5. Numerical solution of Ex. 3 for various values of δ with N=2^5, $\varepsilon = 10^{-1}$ and η=0.05

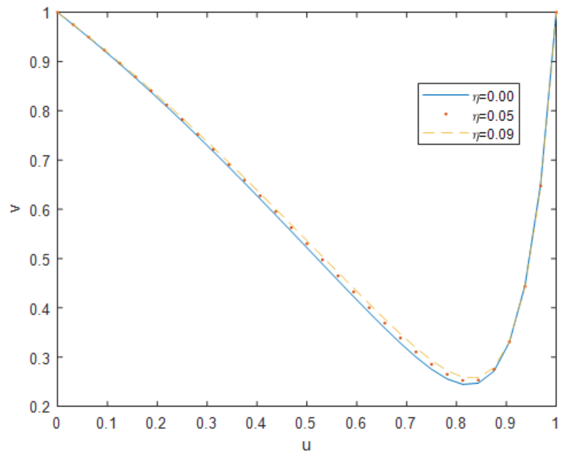

Figure 6. Numerical solution of Ex. 3 for various values of η with N=2^5, $\varepsilon = 10^{-1}$ and δ=0.05

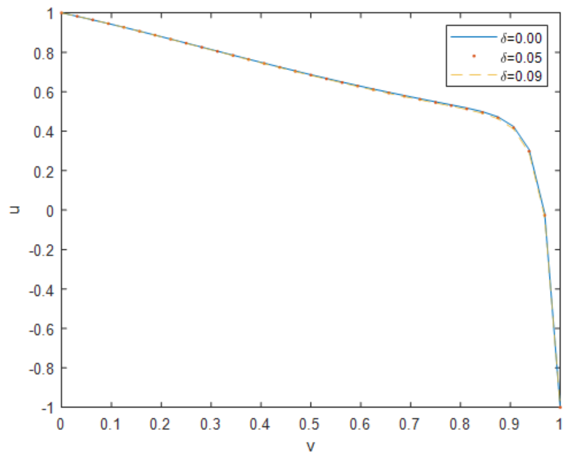

Figure 7. Numerical solution of Ex. 4 for various values of δ with N=2^5, $\varepsilon = 10^{-1}$ and η=0.05

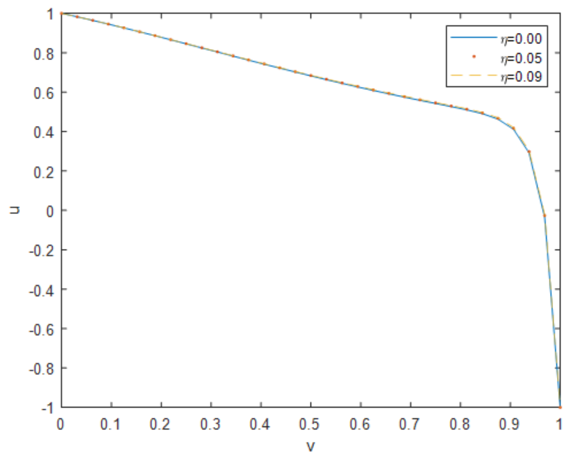

Figure 8. Numerical solution of Ex. 4 for various values of η with N=2^5, $\varepsilon = 10^{-1}$ and δ=0.05

Conclusion

For solving SPDDEs of second order with mixed shifts and boundary layers at the left or right end of the underlying interval using a non-polynomial cubic spline with fitting factor, a novel finite difference algorithm is recommended. To illustrate the accuracy and effectiveness of the approach, four example problems are tested for different values of N and with $\delta = \eta = 0.5\varepsilon$ and presented the numerical results in terms of maximum absolute errors and numerical rates of convergence. Using MATLAB, the MAEs in the solutions listed in comparison to the method given in [22] in Tables 1, 2, 3 and 4. Tables 5 and 6 give the MAEs in the solution of Example 4 to compare the method given in [33]. The detailed examination of the solution graphs plotted in Figs. 1, 2, 3, 4, 5, 6, 7 and 8 reveals that the mixed shifts have no significant impact on the boundary layer behaviour of the solution for problems with boundary layers at the left-end points of the given interval (Figs. 1, 2, 3, and 4), whereas these parameters affect the solution for problems with boundary layers at the right-end points of the given interval (Figs. 5, 6, 7, and 8). According to the results, the thickness of the layer increases as the delay parameter size increases and it decreases as the advance parameter size increases. The proposed method is simple and can be easily implemented on a computer.

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Author Contributions

Swarnakar Dornala selected the problem and analyzed it, and led the manuscript preparation. Kumar Ragula analyzed the problem and developed a method to solve it. Ganesh Kumar Vadla made a MATLAB programme to find the numerical solutions of the problem using the proposed method. BSL Soujanya G supervised the manuscript without spelling mistakes and grammatical errors. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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