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Darboux transformation and solution of the modified Korteweg–de Vries equation

Darboux transformation and a comprehensive approach to construct exact solutions of the nonlinear differential equation are counted. It is applied to construct the explicit solutions of the (2+1)-dimensional modified Korteweg–de Vries (KdV) equation. In this work we derive one flod Darboux transformation of the modified KdV equation. Using the obtained Darboux transformation, the one-soliton solution is built from the «seed» solution. Further, we will construct other explicit solutions for this equation.

Key words: Darboux transformation, «seed» solutions, (2+1)-dimensional modified Korteweg-de Vries equation, soliton solutions.

At the present time for explaining the mystery presence in the challenges of science, nonlinear phenomena has been emerged as a powerful subject. Interplay between dispersion and nonlinearity gives rise to several important phenomena in optical fibers among nonlinear evolution phenomena. It includes wavelength, conversion, modulation instability, parametric amplification, soliton propagation and so on. Solitons, positons and rogue wave has been studied extensively in several areas, such as optics and lasers, hydrodynamics, Bose-Einstein condensate, plasma physics, super fluid, finance, atmosphere and so forth. Inverse scattering transformation, Hirota method, bilinear method, dressing method, Backlund transformation and etc.are the methods to solve the integrable nonlinear equations. Among all the methods, Darboux transformation (see [1–21]) is an efficient way to find different solutions, such as soliton, positon, breathear, rogue wave solutions and etc.

In this work we use this method. Soliton solutions are received by Darboux transformation for the (2+1)-dimensional modified Korteweg—de Vries equation. Furthermore, the pictures for soliton solutions are constructed.

The (2+1)-dimensional modified KdV equation. In this paper we consider the nonlinear differential equation, namely so called the (2+1)-dimensional modified Korteweg–de Vries equation (KdV) in the form:

$$\begin{aligned} & iq_t + iq_{xxy} - vq + i(wq)_x - 2ip = 0; \\ & v_x - 2i\delta(q^*_{xy}q - q^*q_{xy}) = 0; \\ & w_x - 2\delta(|q|^2)_y = 0; \\ & p_x - 2i\mu p - 2\eta q = 0; \\ & \eta_x + \delta(q^*p + p^*q) = 0, \end{aligned} \tag{1}$$

where subscripts x, y, t denote partial derivatives with respect to the variables, the asterisk symbol denotes the complex conjugate. In system (1) q, p are complex functions, v, w, η are real functions, δ is real constant and λ, μ are spectral parameters. Our functions from equation (1) are depending on variables x, y, t. This system (1) is integrable by the Inverse Scattering Method. We can take the corresponding Lax replesentation equation (1) in the form

$$\begin{aligned} \Psi_x &= A\Psi; \\ \Psi_t &= 4\lambda^2\Psi_y + B\Psi, \end{aligned} \tag{2}$$

where A, B are the matrix valued functions with polinomial dependence on the spectral parameter λ:

$$\begin{aligned} A &= -i\lambda\sigma_3 + A_0; \\ B &= \lambda B_1 + B_0 + \frac{i}{\lambda+\mu}B_{-1}. \end{aligned} \tag{3}$$

Here σ_3, A_0, B_1, B_0, B_{-1} are $.2\times 2$. matrices and taking the form

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} i\mu & q \\ -r & 0 \end{pmatrix};$$

$$B_0 = \begin{pmatrix} -\frac{i}{2}v & -q_{xy} - wq \\ r_{xy} + wr & \frac{i}{2}v \end{pmatrix}, \quad B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}. \tag{4}$$

In the system (2) function Ψ is a vector:

$$\Psi = \Psi(x, y, t, \lambda) = \begin{pmatrix} \psi_1(x,y,t,\lambda) \\ \psi_2(x,y,t,\lambda) \end{pmatrix}.$$

In the formula (4) functions $r = \delta q^*$, $k = \delta p^*$, where δ $\delta = \pm 1$.

One-fold DT for the (2+1)-dimensional modified KdV equation. Darboux transformations are a very important tool in the theory of integrable system [15–30]. In this section, considering the particularity of the Lax representation, we construct the Darboux transformation of the (2+1)-dimensional modified KdV equation (1). Furthermore, we will find some solutions of the (2+1)-dimensional modified KdV equation using its Darboux transformation.

We consider the following gauge transformation of linear function $\Psi^{[1]}(x,y,t,\lambda)$

$$\Psi^{[1]}(x,y,t,\lambda) = T(x,y,t,\lambda)\Psi(x,y,t,\lambda),$$

where $T(x,y,t,\lambda)$ is a 2×2 matrix

$$T(x,y,t,\lambda) = \lambda I - M(x,y,t,\lambda),$$

where I, M are matrices and taking the form

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

and M matrix is unknown. The new function $\Psi^{[1]}$ satisfies the next system

$$\begin{aligned} \Psi_x^{[1]} &= A^{[1]}\Psi^{[1]}, \quad A^{[1]} = \left(T_x + TA\right)T^{-1}; \\ \Psi_t^{[1]} &= 4\lambda^2\Psi_y^{[1]} + B^{[1]}\Psi^{[1]}, \quad B^{[1]} = \left(T_t + TB - 4\lambda^2 T_y\right)T^{-1}, \end{aligned} \tag{5}$$

where $A^{[1]}$ and $B^{[1]}$ depend on $q^{[1]}$, $v^{[1]}$, . $p^{[1]}$., $\eta^{[1]}$, $w^{[1]}$ and λ. Then the matrix T can be proven to satisfy following equations

$$T_x + TA = A^{[1]}T; \tag{6.1}$$

$$T_t + TB = 4\lambda^2 T_y + B^{[1]}T. \tag{6.2}$$

From the above identities, after simplifications and comparing coefficients of λ^i from equation (6.1) we get

$$\begin{aligned} q^{[1]} &= q - 2im_{12}; \\ q^{*[1]} &= q^* - 2im_{21}. \end{aligned} \tag{7}$$

After this system we derive condition $m_{21} = -m_{12}^*$ for attractive interaction case which is «if $\delta = +1$«. From system (6) we are taking $m_{22} = m_{11}^*$, then comparing different power coefficients of λ^i of the two sides of the equation (6.2) gives us

$$\begin{aligned} v^{[1]} &= v + 4(m_{12}q_y^* + m_{12}^* q_y + 2im_{11}m_{11y} - 2im_{12}^* m_{12y}); \\ w^{[1]} &= w - 4im_{11y} = w + 4im_{22y}; \end{aligned} \tag{8}$$

$$\begin{aligned} \eta^{[1]} &= \frac{(|\mu + m_{11}|^2 - |m_{12}|^2)\eta - pm_{12}^*(\mu + m_{11}) - p^* m_{12}(\mu + m_{11}^*)}{W}; \\ p^{[1]} &= \frac{p(\mu + m_{11})^2 - p^* m_{12}^2 + 2\eta m_{12}(\mu + m_{11})}{W}; \\ p^{*[1]} &= \frac{p^*(\mu + m_{11}^*)^2 - pm_{12}^{*2} + 2\eta m_{12}^*(\mu + m_{11}^*)}{W}, \end{aligned} \tag{9}$$

where

$$W = \mu^2 + \mu(m_{11} + m_{11}^*) + |m_{11}|^2 + |m_{12}|^2 .$$

One-soliton solutions. As far as we got Darboux transformation, our next aim is to construct the one-soliton solution of modified KdV equation (1). We are taking the trivial «seed» solutions as condition $q=0$, $v=0$, $p=0$, $w=0$, $\eta=1$. Hence, the corresponding associated «seed» solution for functions Ψ_i takes the form

$$\Psi_1 = e^{\theta_1 + i\chi_1};$$
$$\Psi_2 = e^{\theta_2 + i\chi_2},$$

where

$$\theta_1 = \beta x - \nu y - [8\alpha\beta\eta + 4\nu(\alpha^2 - \beta^2) - \frac{\beta}{(\alpha+\mu)^2 + \beta^2}]t + \delta_1;$$
$$\chi_1 = -\alpha x + \eta y + [4\eta(\alpha^2 - \beta^2) - 8\alpha\beta\nu + \frac{\alpha+\mu}{(\alpha+\mu)^2 + \beta^2}]t + \delta_2,$$

here θ_2, χ_2 functions suppose

$$\theta_2 = -\theta_1;$$
$$\chi_2 = -\chi_1 + \delta_0.$$

We have one-soliton solution of the (2+1)-dimensional modified KdV equation (1):

$$q^{[1]} = \frac{2\beta e^{i\chi_1 - i\chi_2}}{ch 2\theta_1};$$
$$v^{[1]} = 16\frac{\alpha\beta\nu}{ch^2 2\theta_1} + 16\frac{\beta^2\eta}{ch^2 2\theta_1};$$
$$w^{[1]} = -\frac{8\beta\nu}{ch^2 2\theta_1};$$
$$p^{[1]} = \frac{2m_{12}(\mu + m_{11})}{W};$$
$$\eta^{[1]} = \frac{|\mu + m_{11}|^2 - |m_{12}|^2}{W},$$

where

$$m_{11} = \alpha + i\beta th 2\theta_1;$$
$$m_{12} = \frac{i\beta e^{i\chi_1 - i\chi_2}}{ch 2\theta_1};$$
$$W = \mu^2 + 2\alpha\mu + \alpha^2 + \beta^2 = (\mu+\alpha)^2 + \beta^2 .$$

One-soliton solutions of the (2+1)-dimensional modified KdV equation (1) has the form Figure:

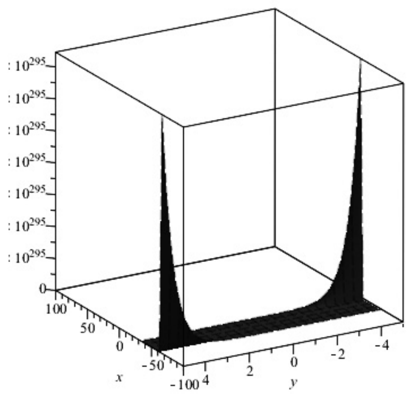

Figure. One-soliton solutions

Furthermore, we can use the above presented for two-fold and N-fold Darboux transformations, in the same way we can get the N-soliton, periodic, positon, breather and rogue wave solutions of the (2+1)-dimensional modified KdV equation (1).

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Дарбу түрлендіруі және модифицирленген Кортевег-де Фриз теңдеуінің солитондық шешімдері

Дарбу түрлендіруі сызықты емес дифференциалдық теңдеулер дәл шешімдер табу үшін кешенді тәсіл болып есептелінеді. Бұл түрлендіру (2+1)-өлшемді модификацияланған Кортевег-де Фриз (КдФ) теңдеуінің нақты шешімін құру үшін пайдаланылады. Мақалада модификацияланған КдФ теңдеуіне Дарбу түрлендіруін бір рет қолданамыз, сондай-ақ бір солитондық шешімді КдФ теңдеуінен аламыз. Болашақта осы теңдеудің «seed» шешімдерінен басқа нақты шешімдерін құрамыз.

Г.Кемелбекова, К.Есмаханова, С.Тапеева, Д.Тунгушбаева

Преобразование Дарбу и солитонные решения модифицированного уравнения Кортевега-де Фриза

Преобразование Дарбу — комплексный подход для построения точных решений нелинейного дифференциального уравнения. Оно применяется для построения явных решений (2+1)-мерного модифицированного уравнения Кортевега-де Фриза (КдФ). В статье получили однократное преобразование Дарбу для модифицированного уравнения КдФ. Используя полученное преобразование Дарбу, построили односолитонное решение с помощью «seed» решения. Далее будут построены другие явные решения для этого уравнения.