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A modified Jacobi elliptic functions method for optical soliton solutions of a conformable nonlinear Schrödinger equation

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In this paper, we study precise and exact traveling wave solutions of the conformable differential nonlinear Schrödinger equation. Then, we transform the given equation into an integer order differential equation by utilizing the wave transformation and the characteristics of the conformable derivative. To extract optical soliton solutions, we divide the wave profile into amplitude and phase components. Further, we introduce a new extension of a modified Jacobi elliptic functions method to the conformable differential nonlinear Schrödinger equation with group velocity dispersion and coefficients of second-order spatiotemporal dispersion.

Keywords: Non-linear Schrödinger equation, Conformable fractional derivative, Modified Jacobi elliptic functions method, Extracting optical solitons-solutions.

2020 Mathematics Subject Classification: 26A33, 34A25, 35R11.

Introduction

Fractional partial and ordinary differential equations (FPDEs and FODEs) are a type of differential equation that involve fractional derivatives. They have been extensively used in many areas of science and engineering, including physics, biology, and finance. One important aspect of fractional calculus is the ability to model complex systems with memory, where the behavior of the system depends on past history [1–8]. The conformable fractional sense is a new approach to fractional calculus that has gained significant attention in recent years. It provides a more accurate representation of non-local effects and has been used to model various physical and biological systems. Conformable PDEs are a type of FPDEs that utilize the conformable fractional derivative, and they have been used to provide a more realistic representation of a wide range of real-world phenomena, such as diffusion and wave propagation. As such, the study of FPDEs and FODEs in the conformable fractional sense is an active and exciting area of research with broad applications [9, 10].

The behavior of conformable PDEs has gained significant attention in recent years due to their wide-ranging applications in various domains, including physics, biology, and engineering. However, understanding the behavior of solutions to conformable PDEs is a complex task, making it challenging to determine accurate answers. To overcome this challenge, several approaches have been suggested to discover analytical solutions for conformable PDEs. These approaches include integral transform methods, numerical methods, and special function techniques, among others. Each of these methods

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has its strengths and weaknesses, making it essential to select the appropriate approach based on the specific characteristics of the conformable PDE being studied. A comprehensive reference list of these approaches can be found in [11], which can aid researchers in selecting the most appropriate approach to solve a particular conformable PDE.

In recent years, finding accurate traveling wave solutions of various nonlinear conformable PDEs has been the primary focus of many researchers. Numerous methods have been proposed to tackle this challenge and provide more general solutions to nonlinear PDEs. These methods include the Kudryashov method [12], the improved tan-expansion method [13], the Sine-Gordon Expansion method [14], the exponential rational function method [15], the sub-equation method [16], the tanh method [16], the auxiliary equation method [17], the Exp-function Method [18], the Jacobi elliptic function expansion method [19], the extended direct algebraic method [20], the first integral method [21] and the improved Bernoulli subequation function method [22]. Among these methods, the modified Jacobi elliptic functions approach [23] is the most crucial method for solitary wave solutions in optics, which has been widely used to give more general solutions to nonlinear PDEs. These approaches have enabled researchers to better understand the behavior of conformable PDEs and their solutions, leading to more accurate predictions and improved modeling of various physical systems.

The focus of this paper is on obtaining an accurate solution to a 1D conformable differential nonlinear Schrödinger equation (CNLSE). To accomplish this, the paper proposes using a second-order nonlinear ODE with a sixth-degree nonlinear component, which extends the elliptic equation. Additionally, the paper aims to develop solutions to the CNLSE using Jacobi's elliptic functions (JEFs). In doing so, optical solitons and other solutions can be observed in the limiting situation of the modulus of ellipticity. The remainder of the paper is dedicated to utilizing a modified auxiliary equation approach to identify all soliton solutions in terms of JEFs and providing soliton solutions with suitable limiting values of the modulus of ellipticity. Ultimately, the results of this paper will contribute to advancing the understanding and application of the CNLSE. However, this paper is structured as follows:

- Section 1 presents the modified auxiliary method (modified Jacobi elliptic functions (MJEFs) method) for obtaining solitary solutions of CNLSE, including a demonstration that some solutions from a previous paper are particular to our model.
- Section 2 discusses the obtained results and their novelty compared to previous methods.
- Section 2 presents the study's results.

1 *Mathematical analysis*

In this paper, we apply a MJEFs approach to obtain exact wave solutions for CNLSE having group velocity dispersion GVD and second order spatiotemporal dispersion coefficients provided $\overline{\omega} = 1$ and $b_2 = 0$ (see [24]). The governing model is written as follows:

$$\begin{cases} i\frac{\partial q}{\partial x} + i\rho\frac{\partial^{\overline{\omega}} q}{\partial t^{\overline{\omega}}} + +\beta\frac{\partial^{2\overline{\omega}} q}{\partial t^{2\overline{\omega}}} + \gamma\frac{\partial^2 q}{\partial t^2} + b_2|q|^2 q = 0, \\ \\ t > 0, \overline{\omega} > 1, \end{cases} \tag{1}$$

where $q(x,t)$ denotes the macroscopic complex-valued wave profile, x and t are, respectively, the spatial and temporal variables. The numbers β and γ denote the coefficients of the GVD and spatial dispersion, respectively. Whereas, ρ is proportional to the group speed ratio and b_2 is nonzero real-valued constant coefficient which coefficient constitute the nonlinearity component. For extracting optical solitons-solutions, the wave profile is split into amplitude and phase components as

$$q(x,t) = u(\xi)e^{i\psi}, \tag{2}$$

where

$$\xi = x - \nu \frac{t^{\overline{\omega}}}{\overline{\omega}}, \tag{3}$$

ν being a real constant and $u(\xi)$ the amplitude components of the wave profiles. The phase factor is

$$\psi = -cx + \omega \tfrac{t^{\overline{\omega}}}{\overline{\omega}} + \theta_0, \tag{4}$$

where c is the frequency of the solitons and ω is the wave number and θ_0 is the phase constant. We reduce NLPDE (1) into one-dimensional ODE; if we take the necessary of (2) with (3) for (1), we get the following expressions:

$$\begin{cases} q_x = u'e^{i\psi} - icue^{i\psi}, \\ q_{xx} = u''e^{i\psi} - 2icu'e^{i\psi} - c^2ue^{i\psi}, \\ \frac{\partial^{\overline{\omega}} q}{\partial t^{\overline{\omega}}} = -\nu u'e^{i\psi} + i\omega ue^{i\psi}, \\ \frac{\partial^{2\overline{\omega}} q}{\partial t^{2\overline{\omega}}} = \nu^2 u''e^{i\psi} - 2i\nu\omega u'e^{i\psi} - \omega^2 ue^{i\psi}, |q|^2q = u^3e^{i\psi}. \end{cases} \tag{5}$$

By using (2), (4) and (5) in (1), CNLSE (1) turns into an ODE that we decompose into real and imaginary parts. The imaginary part yields a relation which is constraint between the soliton parameters as

$$\nu = \tfrac{1-2c\gamma}{\rho+2\omega\beta}. \tag{6}$$

The real part of the equation in (6) is

$$(c - \rho\omega - \beta\omega^2 - \gamma c^2)u + (\beta\nu^2 + \gamma)u'' + b_2u^3 = 0. \tag{7}$$

The balance rule detailed in [25] gives $N = 1$. Solitons emerges from the limiting process are presenting in the next section.

1.1 Solitons-solutions

Applying the modified auxiliary equation method to the CNLSE (1) and using the balance rule of [25] (when $N = 1$),we get to write the solution of (7) as follows:

$$u(\xi) = \textstyle\sum_{i=0}^{N=1} a_iF^i(\xi) = a_0 + a_1F(\xi), \tag{8}$$

where, a_0, a_1 are arbitrary constants such that $a_1 \neq 0$ and $F(\xi)$ is a Jacobian elliptic function (see [23]), when $F(\xi)$ satisfying the following:

$$(F'(\xi))^2 = A_2F^2(\xi) + A_4F^4(\xi) + A_6F^6(\xi), \tag{9}$$

where A_2, A_4 and A_6 are arbitrary constants determined by Jacobi elliptic functions JEFs method [23]. Substituting (8) and the derivative of (9) in (7), while collecting all terms with the same power and setting them to zero, we have the following of algebraic equations:

$$\begin{cases} 3(\beta\nu^2 + \gamma)A_6a_1 = 0, \\ 2(\beta\nu^2 + \gamma)A_4 + b_2a_1^2 = 0, \\ 3b_2a_0a_1^2 = 0, \\ (\beta v^2 + \gamma)A_2 + 3b_2a_0^2 + (c - \rho\omega - \beta\omega^2 - \gamma c^2) = 0, \\ b_2a_0^3 + (c - \rho\omega - \beta\omega^2 - \gamma c^2)a_0 = 0. \end{cases} \tag{10}$$

Form the first equation of system (10), we take $A_6 = 0$ according to [23], we deduce existence of one modulus ($0 \leq k_1 \leq 1$ and $k_2 = 0$, see [23]).

Solving algebraic equation (10) by using any computer software (Matlab, Maple, Wolfram, Mathematica, ...) yields three cases of solutions and according with [23] as follow:

$$a_0 = 0, a_1^2 = -2(\beta\nu^2 + \gamma)\frac{A_4}{b_2} > 0, c = \frac{1}{2\gamma}(1 + \sqrt{4\beta\gamma\nu^2 A_2 - 4\beta\gamma\omega^2 + 4\beta\gamma A_2 - 4\gamma\omega\rho + 1}) \geq 0, A_6 = 0,$$

with

$$4\beta\gamma\nu^2 A_2 - 4\beta\gamma\omega^2 + 4\beta\gamma A_2 - 4\gamma\omega\rho + 1 \geq 0.$$

Case 1: When $A_2 = -(1+k_1^2)$ and $A_4 = k_1^2 > 0$ with $(\beta\nu^2+\gamma)\frac{A_4}{b_2} < 0$, we can acquire the following new complex Jacobi sine function solution for equation (1):

$$q_1(x,t,k_1) = \pm\sqrt{-2(\beta\nu^2+\gamma)\frac{k_1^2}{b_2}}e^{i\left(\frac{-1}{2\gamma}\left(1\pm\sqrt{-4\beta\gamma\nu^2(1+k_1^2)-4\beta\gamma\omega^2-4\beta\gamma(1+k_1^2)-4\gamma\omega\rho+1}\right)x+\omega\frac{t^{\varpi}}{\varpi}+\theta_0\right)}sn(x-\nu\frac{t^{\varpi}}{\varpi},k_1). \tag{11}$$

Case 2: When $A_2 = 2k_1^2 - 1$ and $A_4 = -k_1^2 < 0$ with $(\beta\nu^2+\gamma)\frac{A_4}{b_2} > 0$, we can acquire the following new complex Jacobi cosine function solution for equation (1):

$$q_2(x,t,k_1) = \pm\sqrt{2(\beta\nu^2+\gamma)\frac{k_1^2}{b_2}}e^{i\left(\frac{-1}{2\gamma}\left(1\pm\sqrt{4\beta\gamma\nu^2(2k_1^2-1)-4\beta\gamma\omega^2+4\gamma(2k_1^2-1)-4\gamma\omega\rho+1}\right)x+\omega\frac{t^{\varpi}}{\varpi}+\theta_0\right)}cn(x-\nu\frac{t^{\varpi}}{\varpi},k_1). \tag{12}$$

Case 3: When $A_2 = 2 - k_1^2$ and $A_4 = -1 < 0$ with $(\beta\nu^2+\gamma)\frac{A_4}{b_2} > 0$, we can acquire the following new complex Jacobi function solution of the third kind for equation (1):

$$q_3(x,t,k_1) = \pm\sqrt{\frac{2(\beta\nu^2+\gamma)}{b_2}}e^{i\left(\frac{-1}{2\gamma}\left(1\pm\sqrt{4\beta\gamma\nu^2(2-k_1^2)-4\beta\gamma\omega^2+4\gamma(2-k_1^2)-4\gamma\omega\rho+1}\right)x+\omega\frac{t^{\varpi}}{\varpi}+\theta_0\right)}dn(x-\nu\frac{t^{\varpi}}{\varpi},k_1). \tag{13}$$

1.2 Particular cases

When $k_1 \to 0$, the JEFs (11)-(12)-(13) degenerate to the triangular functions, that is,

$$sn\xi \to \sin\xi, cn\xi \to \cos\xi, dn\xi \to 1. \tag{14}$$

When $k_1 \to 0$, the JEFs (11)-(12)-(13) degenerate to the hyperbolic functions, that is,

$$sn\xi \to \tan\xi, cn\xi \to sech\xi, dn\xi \to sech\xi. \tag{15}$$

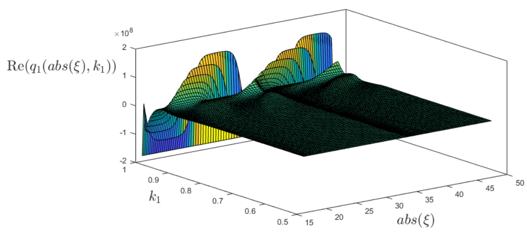
In [11], several specific solutions from (11)-(12)-(13) with (14)-(15) are described.

2 Physical interpretation and discussion

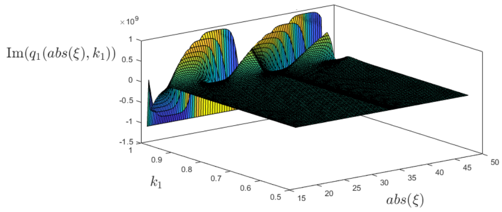
In a specific example of the constants when setting the variables $\gamma = 1$, $\beta = 3$ and $\varpi = 0.5$, it would be extremely helpful if we had real figures that visually illustrated some of the new solutions to equation 1 that were achieved, corresponding to case 1 (Fig. 1), case 2 (Fig. 2) and case 3 (Fig. 3). In this study, key characteristics of the modified JEFs approach were employed to provide a physical explanation for several complex and Jacobi elliptic solutions that were derived for an equation 1. The modified JEFs technique is more broad than the classical methods (such a tanh method, sin-cos method, simplest equation method [26] and the expansion method [18]) because it can discover additional types of analytical solutions that cannot be found by using Bäckland method [11] as an example. In order to acquire additional analytical answers, a better knowledge of engineering and physical challenges, and new physical predictions, the process described in [27] will help.

In section (1.1), we demonstrate that the JEFs solutions to (1) only have one k_1 modulus $0 \leq k_1 \leq 1$ according to [23]. As far as we are aware, this is the first place in the literature where the new solutions (11)-(12)-(13) of (1) may have been found.

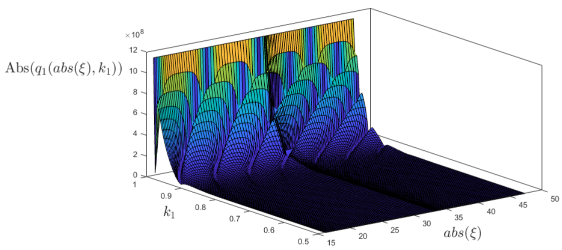
In regards to figures, surfaces have been plotted by taking into account the appropriate values for the parameters. When we verify every analytical solution produced in this study using the modified JEFs approach, we find that Figures 1, 2, and 3 have three-dimensional surfaces. As a result, it may be claimed that they are physically plausible because nearly every figure demonstrates similar wave behaviors given the appropriate parameter values.



(a)

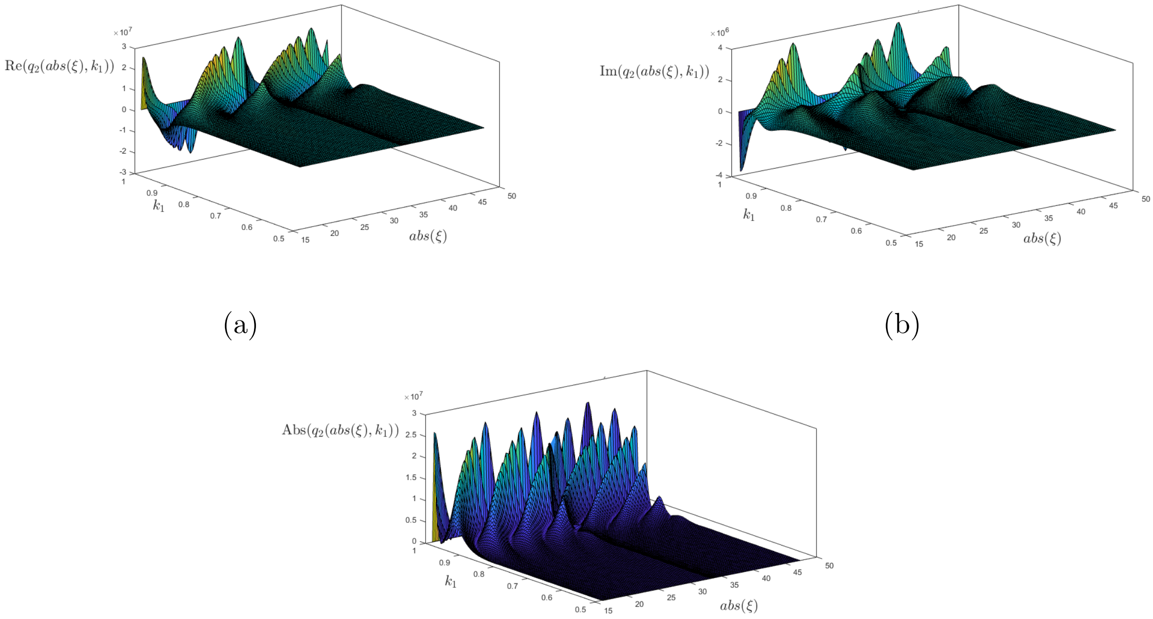


(b)



(c)

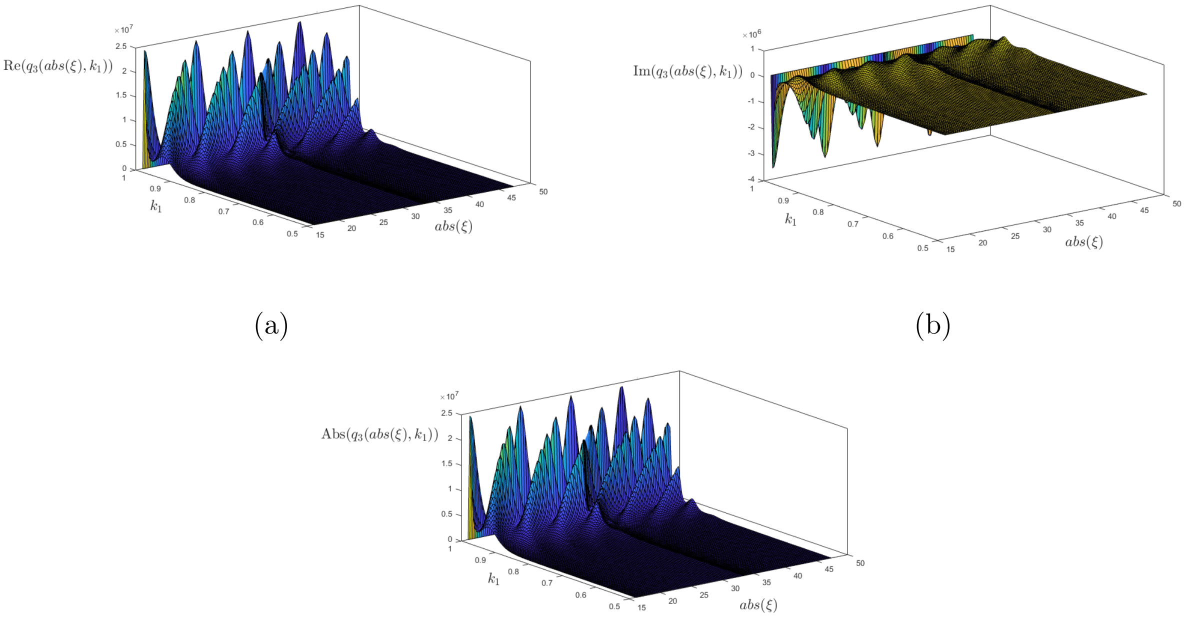
Figure 1. (a) Real, (b) imaginary and (c) absolute plots in 3D sketches of equation (11), respectively, when $v = 2$, $b_2 = -2$, $\omega = 5$, $\theta_0 = 2$ and $\rho = 2$



(a) (b)

(c)

Figure 2. (a) Real, (b) imaginary and (c) absolute plots in 3D sketches of equation (12), respectively, when $v = 2$, $b_2 = -2$, $\omega = 5$, $\theta_0 = 2$ and $\rho = 2$



(a) (b)

(c)

Figure 3. (a) Real, (b) imaginary and (c) absolute plots in 3D sketches of equation (13), respectively, when $v = 2$, $b_2 = -2$, $\omega = 5$, $\theta_0 = 2$ and $\rho = 2$

Conclusion

The current work employs the MJEFs approach to address the CNLSE, incorporating fractional derivatives featuring second-order spatiotemporal and GVD coefficients via wave transformation and conformable derivatives. A diverse array of optical solitons-solutions are constructed for the governing equation. Figures 1, 2, and 3 present a viewpoint of the resulting solitons solutions with respect to distinct parameters. Our novel MJEFs technique generates a new set of solutions (with one modulus) that are exclusively presented in this work. These unrestricted parameter solutions hold significant importance in elucidating various physical interpretations. The outcomes demonstrate the capability of our approach to be used in a variety of CPDEs and offer numerous precise solutions for CPDEs.

Author Contributions

Aicha Boussaha collected and analyzed data, and led manuscript preparation. B. Semmar and N. Djeddi assisted in data collection and analysis. M. Al-Smadi and S. Al-Omari served as the principal investigator of the research grant and supervised the research process. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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