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On solution of non-linear $\mathbb{FDE}$ under tempered $\Psi-$Caputo derivative for the first-order and three-point boundary conditions

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In this article, the existence and uniqueness of solutions for non-linear fractional differential equation with Tempered $\Psi-$Caputo derivative with three-point boundary conditions were studied. The existence and uniqueness of the solution were proved by applying the Banach contraction mapping principle and Schaefer's fixed point theorem.

Keywords: fractional differential equations, tempered $\Psi-$Caputo derivative, nonlinear analysis, Schaefer's fixed point theorem; Banach contraction.

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Introduction

Fractional calculus is a strong tool of mathematical analysis that studies derivatives and integrals of a fractional order. Fractional differential equations ($\mathbb{FDE}$s) are used in many fields of engineering and sciences such as physics, mechanics, chemistry, viscoelasticity, electro chemistry, porous media, electromagnetic, for more details see the books [1–3] and applicable papers [4–10].

One of the useful generalizations of a fractional derivative and an integral is associated with a dependent function [11]. Mali *et al.* developed well the theory of tempered fractional integrals and derivatives of a function with respect to another function [12]. This theory combines the tempered fractional calculus with the ψ-fractional calculus, both of which have found applications in topics including continuous time random walks. In [13], Benchohra *et al.*, by means of the Banach fixed point theorem and the nonlinear alternative of Leray-Schauder type, proved the existence of solutions for the first order boundary value problem ($\mathbb{BVP}$) for a $\mathbb{FDE}$

$$\mathcal{D}_C^{\eta}\varkappa(\mathrm{w}) = \mathfrak{h}(\mathrm{w}, \varkappa(\mathrm{w})), \qquad \mathrm{w}, \eta \in \Omega := (0,1), \tag{1}$$

under condition $p\varkappa(0) + q\varkappa(1) = \mathfrak{h}_{\circ}$. In 1996, authors proved existence and uniqueness of problem (1), for $\mathrm{w} \in \Omega$, where $\mathfrak{h} : [0,\top] \times \mathbb{R} \to \mathbb{R}$, $0 < \top < +\infty$ is a given continuous function [14]. Also, the authors in [15] by the Banach contraction principle and Schauder's fixed point theorem investigated the existence of solutions for problem (1) with integral conditions $\varkappa(0) + p\int_0^{\top} \varkappa(\grave{\zeta})\,\mathrm{d}\grave{\zeta} = \varkappa(\top)$. Recently,

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authors have presented very valuable works on the ability of fractional derivatives and fractional q-derivatives with Caputo sense [16–23]. Salim *et al.* concerned some existence and uniqueness results for a class of problems for nonlinear Caputo tempered implicit $\mathbb{FDE}$s

$$\begin{cases} \mathcal{D}^{\eta,\psi}_{C;\kappa_1,\xi}\varkappa(\mathrm{w}) = \mathfrak{h}\left(\mathrm{w},\varkappa(\mathrm{w}),\mathcal{D}^{\eta,\psi}_{C;\kappa_1,\xi}\varkappa(\mathrm{w})\right), & \mathrm{w}\in[\kappa_1,\kappa_2],\ \eta\in\Omega, \\ p_1\varkappa(\kappa_1)+p_2\varkappa(\kappa_2)=p_3\varkappa(\eta)+q, \end{cases} \tag{2}$$

in b-Metric spaces with three-point boundary conditions, where $\mathfrak{h}\in C(\Omega\times\mathbb{R}^2)$, $\kappa_1<\eta<\kappa_2<+\infty$ and $p_i, i=1,2,3,$ q are real constants [24]. For more instance, consider [25–27].

Motivated by the studies [28–33], we characterize an alteration of the $\Psi-$Caputo derivative, the Tempered $\Psi-$Caputo derivative and consider the Cauchy problem for $\mathbb{FDE}$s with this type of a fractional derivative. This derivative incorporates as uncommon cases the Tempered Caputo [30]. In this manner, we study the following ($\mathbb{BVP}$) for a $\mathbb{FDE}$ with the tempered $\Psi-$Caputo fractional derivative type

$$\begin{cases} \mathcal{D}^{\eta,\lambda,\psi}_{C}\varkappa(\mathrm{w}) = \mathfrak{h}(\mathrm{w},\varkappa(\mathrm{w})), & \mathrm{w}\in\bar{\mathrm{J}}=[0,\top],\ \eta\in\Omega, \\ p_1\varkappa(0)+p_2\varkappa(\top_\circ)+p_3\varkappa(\top)=q, \end{cases} \tag{3}$$

where $\mathfrak{h}:\bar{\mathrm{J}}\times\mathbb{R}^2$ is a continuous function, $\mathcal{D}^{\eta,\lambda,\psi}_{C}$ is a Tempered $\Psi-$Caputo fractional derivative, increase function Ψ is a continuously differentiable on $[0,\infty)$ with $\Psi(0)=0$, $\Psi'(\mathrm{w})>0$, for each $\mathrm{w}\in(0,\infty)$, $\lim_{\mathrm{w}\to\infty}\Psi(\mathrm{w})=\infty$, $p_i\,(i=1,2,3)$ are real constants with $\acute{q}=p_1+p_2e^{-\lambda\Psi(\top_\circ)}+p_3e^{-\lambda\Psi(\top)}$, $\acute{q}\neq0$, $0<\top_\circ<\top$.

In Section 2, we give a result, based on Banach (Theorem 2) and Schaefer's (Theorem 3) fixed point theorems. In Section 2.2, a case is given that illustrates the application of our primary comes about. These comes about can be considered as a commitment to this developing field.

1 *Preliminaries*

In this section we present definitions and theorems from fractional calculus theory which are used in this paper. Let $\Psi\in C^n[\top_1,\top_2]$ be an increasing differentiable function for all $\top_1\le\mathrm{w}\le\top_2$. The tempered Ψ-fractional integral of an order $n-1<\eta<n\,(n\in\mathbb{N})$ is present by

$$\mathcal{I}^{\eta,\lambda,\Psi}_{\top_1}\varkappa(\mathrm{w}) = \int_{\top_1}^{\mathrm{w}}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}))^{\eta-1}\frac{e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w})}\Psi'(\grave{\zeta})}{\Gamma(\eta)}\varkappa(\grave{\zeta})\,\mathrm{d}\grave{\zeta}, \qquad \lambda\geqslant0,$$

where $\tilde{\Psi}_{\mathrm{v}}(\mathrm{w})=\Psi(\mathrm{w})-\Psi(\mathrm{v})$. Now, let $\Psi'(\mathrm{w})\neq0$ for all $\mathrm{w}\in[\top_1,\top_2]$. The tempered $\Psi-$Caputo fractional derivative of an order η is defined as

$$\mathcal{D}^{\eta,\lambda,\Psi}_{C;\top_1}\varkappa(\mathrm{w}) = \int_{\top_1}^{\mathrm{w}}\frac{e^{-\lambda\Psi(\mathrm{w})}\Psi'(\mathrm{w})}{\Gamma(n-\eta)}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}))^{n-\eta-1}\varkappa^{[n]}_{\lambda,\Psi}(\grave{\zeta})\,\mathrm{d}\grave{\zeta}, \qquad \lambda\geqslant0,$$

where $\varkappa^{[n]}_{\lambda,\Psi}(\mathrm{w})=\left[\frac{1}{\Psi'(\mathrm{w})}\frac{\mathrm{d}}{\mathrm{dw}}\right]^n\left(e^{\lambda\Psi(\mathrm{w})}\varkappa(\mathrm{w})\right)$. By employing the above assumptions the next theorem is satisfied.

Theorem 1. Let $\Psi\in C^n[\top_1,\top_2]$. Then the following holds (I) $\mathcal{D}^{\eta,\lambda,\Psi}_{C;\top_1}\left[\mathcal{I}^{\eta,\lambda,\Psi}_{\top_1}\varkappa(\mathrm{w})\right]=\varkappa(\mathrm{w})$; (II) $\mathcal{I}^{\eta,\lambda,\Psi}_{\top_1}\left[\mathcal{D}^{\eta,\lambda,\Psi}_{C;\top_1}(\varkappa(\mathrm{w}))\right]=\varkappa(\mathrm{w})-e^{-\lambda\Psi(\mathrm{w})}\sum\limits_{k=0}^{n-1}\mathfrak{h}_k\left[\tilde{\Psi}_{\top_1}(\mathrm{w})\right]^k$ where

$$\mathfrak{h}_k=\frac{\varkappa^{[k]}_{\lambda,\Psi}(\top_1)}{k!}=\frac{1}{k!}\left[\frac{1}{\Psi'(\mathrm{w})}\frac{\mathrm{d}}{\mathrm{dw}}\right]\left(e^{\lambda\Psi(\mathrm{w})}\varkappa(\mathrm{w})\right)\Big|_{\mathrm{w}=\top_1}.$$

2 Main results

In this section, we consider 𝔹𝕍ℙ (3). We consider the norm $|\varkappa|_\infty := \sup\left\{\varkappa(\mathrm{w}) : \mathrm{w} \in \bar{\mathrm{J}}\right\}$ on space $C(\bar{\mathrm{J}})$.

2.1 Existence of solution

Let us start by defining what we mean by a solution of 𝔹𝕍ℙ (3).

Definition 1. A continuous function $\varkappa : \bar{\mathrm{J}} \to \mathbb{R}$ is a solution of the 𝔹𝕍ℙ (3), if $\mathcal{D}_C^{\eta,\lambda,\psi}\varkappa(\mathrm{w})$ exists for all $\mathrm{w} \in \bar{\mathrm{J}}$, continuous on $\bar{\mathrm{J}}$, and $\varkappa(\mathrm{w})$ fulfils equality (3) for all $\mathrm{w} \in \bar{\mathrm{J}}$.

Lemma 1. Let the function $\mathfrak{h} \in C(\bar{\mathrm{J}} \times \mathbb{R})$ be bounded. Then the function $\varkappa(\mathrm{w})$ is a solution of the 𝔹𝕍ℙ (3) defined on the interval $\bar{\mathrm{J}}$ iff it is a solution of the following equation

$$\varkappa(\mathrm{w}) = \tfrac{q}{\acute{q}}e^{-\lambda\Psi(\mathrm{w})} + \int_0^{\top} \mathbb{G}(\mathrm{w}, \grave{\zeta})\tfrac{\Psi'(\grave{\zeta})}{\Gamma(\eta)}\mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta}$$

with $\mathbb{G}(\mathrm{w},\mathrm{v}) = \mathbb{G}_1(\mathrm{w},\mathrm{v})$, whenever $0 \le \mathrm{w} \le \top_\circ$, and $\mathbb{G}(\mathrm{w},\mathrm{v}) = \mathbb{G}_2(\mathrm{w},\mathrm{v})$, whenever $\top_\circ < \mathrm{w} \le \top$, where

$$\mathbb{G}_1(\mathrm{w},\mathrm{v}) = \begin{cases} -\frac{p_2e^{-\lambda\Psi(\mathrm{w})}}{\acute{q}}(\tilde{\Psi}_{\mathrm{v}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top_\circ)} \\ \quad -\frac{p_3e^{-\lambda\Psi(\mathrm{w})}}{\acute{q}}(\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)} + (\tilde{\Psi}_{\mathrm{v}}(\mathrm{w}))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\mathrm{w})}, & 0 \le \mathrm{v} \le \mathrm{w}, \\ -\frac{p_2e^{-\lambda\Psi(\mathrm{w})}}{\acute{q}}(\tilde{\Psi}_{\mathrm{v}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top_\circ)} - \frac{p_3e^{-\lambda\Psi(\mathrm{w})}}{\acute{q}}(\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)}, & \mathrm{w} < \mathrm{v} \le \top, \\ -\frac{p_3e^{-\lambda\Psi(\mathrm{w})}}{\acute{q}}(\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)}, & \top_\circ < \mathrm{v} \le \top, \end{cases}$$

$$\mathbb{G}_2(\mathrm{w},\mathrm{v}) = \begin{cases} -\frac{p_2}{\acute{q}}e^{-\lambda\Psi(\mathrm{w})}(\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)} \\ \quad -\frac{p_3}{\acute{q}}e^{-\lambda\Psi(\mathrm{w})}(\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)} + (\tilde{\Psi}_{\mathrm{v}}(\mathrm{w}))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\mathrm{w})}, & 0 \le \mathrm{v} \le \top_\circ, \\ -\frac{p_3}{\acute{q}}e^{-\lambda\Psi(\mathrm{w})}(\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)} + (\tilde{\Psi}_{\mathrm{v}}(\mathrm{w}))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\mathrm{w})}, & \top_\circ < \mathrm{v} \le \mathrm{w}, \\ -\frac{p_3}{\acute{q}}e^{-\lambda\Psi(\mathrm{w})}(\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)}, & \mathrm{w} < \mathrm{v} \le \top. \end{cases}$$

Proof. By performing the integral $\mathcal{I}_0^{\eta\lambda,\Psi}$ to both of Equation (3) and applying assertion (2) of Theorem 1, we get $\varkappa(\mathrm{w}) = c_0e^{-\lambda\Psi(\mathrm{w})} + \frac{1}{\Gamma(\eta)}\int_0^{\mathrm{w}}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w})}\Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta}$. Using condition (3) we have

$$c_0 = \tfrac{q}{\acute{q}} - \tfrac{p_2}{\acute{q}}\int_0^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}\frac{e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}}{\Gamma(\eta)}\Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta} - \int_0^{\top}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}\frac{p_3e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}}{\acute{q}\Gamma(\eta)}\Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta},$$

then the unique solution of (3) is given by the formula

$$\begin{aligned}\varkappa(\mathrm{w}) = {}&\tfrac{q}{\acute{q}}e^{-\lambda\Psi(\mathrm{w})} - \tfrac{p_2e^{-\lambda\Psi(\mathrm{w})}}{\acute{q}\Gamma(\eta)}\int_0^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}\Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta} \\ &- \tfrac{p_3}{\acute{q}\Gamma(\alpha)}e^{-\lambda\Psi(\mathrm{w})}\int_0^{\top}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta} \\ &+ \int_0^{\mathrm{w}}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}))^{\eta-1}\frac{e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w})}}{\Gamma(\eta)}\Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta}.\end{aligned} \tag{4}$$

Let $0 \leq \mathrm{w} \leq \top_\circ$. Then (4) can be rewritten

$$\begin{aligned}\varkappa(\mathrm{w}) = {} & \tfrac{q}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})} - \tfrac{p_2 e^{-\lambda\Psi(\mathrm{w})}}{\acute{q}\Gamma(\eta)}\Big\{ \int_0^{\mathrm{w}} (\tilde{\Psi}_{\xi}(\top_\circ))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)} \Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta} \\ & + \int_{\mathrm{w}}^{\top_\circ} (\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)} \Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta}\Big\} \\ & - \tfrac{p_3}{\acute{q}\Gamma(\eta)} e^{-\lambda\Psi(\mathrm{w})}\Big\{ \int_0^{\mathrm{w}} (\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)} \Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta} \\ & + \int_{\mathrm{w}}^{\top_\circ} (\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)} \Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta} \\ & + \int_{\top_\circ}^{\top} (\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)} \Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta}\Big\} \\ & + \tfrac{1}{\Gamma(\eta)} \int_0^{\mathrm{w}} (\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w})} \Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta}.\end{aligned}$$

Here grouping the like terms, and then simplifying, we get the new function as follows

$$\mathbb{G}_1(\mathrm{w},\mathrm{v}) = \begin{cases} -\tfrac{p_2}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})}(\tilde{\Psi}_{\mathrm{v}}(\top_\circ))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top_\circ)} & \\ \quad -\tfrac{p_3}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})}(\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)} + (\tilde{\Psi}_{\mathrm{v}}(\mathrm{w}))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\mathrm{w})}, & 0 \leq \mathrm{v} \leq \mathrm{w}, \\ -\tfrac{p_2}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})}(\tilde{\Psi}_{\mathrm{v}}(\top_\circ))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top_\circ)} - \tfrac{p_3}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})}(\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)}, & \mathrm{w} < \mathrm{v} \leq \top, \\ -\tfrac{p_3}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})}(\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)}, & \top_\circ < \mathrm{v} \leq \top, \end{cases}$$

using this equality, relation (4) may be written as an integral equation,

$$\varkappa(\mathrm{w}) = \tfrac{q}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})} + \tfrac{1}{\Gamma(\eta)} \int_0^{\top_\circ} \mathbb{G}_1(\mathrm{w},\grave{\zeta})\Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta},$$

for the case $\mathrm{w} \in [\top_\circ, \top]$ we can write equality (4) in the form

$$\begin{aligned}\varkappa(\mathrm{w}) = {} & \tfrac{q}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})} - \tfrac{p_2 e^{-\lambda\Psi(\mathrm{w})}}{\acute{q}\Gamma(\eta)} \int_0^{\top_\circ} (\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)} \Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta} \\ & - \tfrac{p_3 e^{-\lambda\Psi(\mathrm{w})}}{\acute{q}\Gamma(\eta)}\Big\{ \int_0^{\top_\circ} (\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)} \Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta} \\ & + \int_{\top_\circ}^{\mathrm{w}} (\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)} \Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta} \\ & + \int_{\mathrm{w}}^{\top} (\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)} \Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta}\Big\} \\ & + \tfrac{1}{\Gamma(\eta)}\Big\{ \int_0^{\top_\circ} (\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w})} \Psi'(\eta)\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta}\Big\} \\ & + \int_{\top_\circ}^{\mathrm{w}} (\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w})} \Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta}.\end{aligned}$$

Here we introduce the new function

$$\mathbb{G}_2(\mathrm{w},\mathrm{v}) = \begin{cases} -\tfrac{p_2}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})}(\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)} & \\ -\tfrac{p_3}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})}(\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)} + (\tilde{\Psi}_{\mathrm{v}}(\mathrm{w}))^{eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\mathrm{w})}, & 0 \leq \mathrm{v} \leq \top_\circ, \\ -\tfrac{p_3}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})}(\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)} + (\tilde{\Psi}_{\mathrm{v}}(\mathrm{w}))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\mathrm{w})}, & \top_\circ < \mathrm{v} \leq \mathrm{w}, \\ -\tfrac{p_3}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})}(\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)}, & \mathrm{w} < \mathrm{v} \leq \top. \end{cases}$$

Hence for the case $\mathrm{w} \in [\top_\circ, \top]$, we can write (4) in the form $\varkappa(\mathrm{w}) = \frac{q}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})} + \frac{1}{\Gamma(\eta)} \int_{\top_\circ}^{\top} \mathbb{G}_2(\mathrm{w}, \grave{\zeta}) \Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\, \mathrm{d}\grave{\zeta}$. So, we conclude that the solution of $\mathbb{BVP}$ (3) has the form $\varkappa(\mathrm{w}) = \frac{q}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})} + \int_0^{\top} \mathbb{G}(\mathrm{w}, \grave{\zeta}) \frac{\Psi'(\grave{\zeta})}{\Gamma(\eta)} \mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\, \mathrm{d}\grave{\zeta}$. The proof is completed.

Theorem 2. Assume that

(H_1) There exists a constant $k > 0$ such that $|\mathfrak{h}(\mathrm{w}, \mathrm{v}_1) - \mathfrak{h}(\mathrm{w}, \mathrm{v}_2)| \leqslant k|\mathrm{v}_1 - \mathrm{v}_2|$, for all $\mathrm{w} \in \bar{\mathrm{J}}$, and for each $\mathrm{v}_1, \mathrm{v}_2 \in \mathbb{R}$.

If $k/\lambda^\eta \left(|p_2/\acute{q}| + |p_3/\acute{q}| + 1\right) < 1$, then the $\mathbb{BVP}$ (3) has a unique solution on $\bar{\mathrm{J}}$.

Proof. We transform the problem (3) into a fixed point problem considering the operator $\mathcal{O} : C(\bar{\mathrm{J}}) \to C(\bar{\mathrm{J}})$ defined by

$$\mathcal{O}(\varkappa)(\mathrm{w}) = \frac{q}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})} + \int_0^{\top} \mathbb{G}(\mathrm{w}, \grave{\zeta})\Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\, \mathrm{d}\grave{\zeta}. \tag{5}$$

It isn't troublesome to see that, a fixed point $\mathcal{O}$ is a solution of (3). We might utilize the Banach contraction principle to demonstrate that $\mathcal{O}$ characterized by (3) includes a fixed point and $\mathcal{O}$ is a contraction.

Case 1: Let $\mathrm{w} \in \bar{\mathrm{J}}$, so we have

$$\begin{aligned}
|\mathcal{O}(\varkappa_1)(\mathrm{w}) - \mathcal{O}(\varkappa_2)(\mathrm{w})| &= \left| \int_0^{\top} \mathbb{G}_1(\mathrm{w}, \grave{\zeta}) \tfrac{\Psi'(\grave{\zeta})}{\Gamma(\eta)} \left[\mathfrak{h}(\grave{\zeta}, \varkappa_1(\grave{\zeta})) - \mathfrak{h}(\grave{\zeta}, \varkappa_2(\grave{\zeta})) \right] \mathrm{d}\grave{\zeta} \right| \\
&\le \tfrac{k|\varkappa_1 - \varkappa_2|}{\Gamma(\eta)} \int_0^{\top} |\mathbb{G}_1(\mathrm{w}, \grave{\zeta})| \Psi'(\grave{\zeta})\, \mathrm{d}\grave{\zeta} \\
&\le \tfrac{k|\varkappa_1 - \varkappa_2|}{\Gamma(\eta)} \Bigg(\left|\tfrac{p_2}{\acute{q}}\right| e^{-\lambda\Psi(\mathrm{w})} \Bigg\{ \int_0^{\mathrm{w}} (\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)} \Psi'(\grave{\zeta})\, \mathrm{d}\grave{\zeta} \\
&\quad + \int_0^{\mathrm{w}} (\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)} \Psi'(\grave{\zeta})\, \mathrm{d}\grave{\zeta} \Bigg\} + \left|\tfrac{p_3}{\acute{q}}\right| e^{-\lambda\Psi(\mathrm{w})} \Bigg\{ \int_0^{\mathrm{w}} (\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)} \Psi'(\grave{\zeta})\, \mathrm{d}\grave{\zeta} \\
&\quad + \int_{\mathrm{w}}^{\top_\circ} (\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)} \Psi'(\grave{\zeta})\, \mathrm{d}\grave{\zeta} + \int_{\top_\circ}^{\top} (\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)} \Psi'(\grave{\zeta})\, \mathrm{d}\grave{\zeta} \Bigg\} \\
&\quad + \int_0^{\mathrm{w}} (\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)} \Psi'(\grave{\zeta})\, \mathrm{d}\grave{\zeta} \Bigg) \\
&\le \tfrac{k|\varkappa_1 - \varkappa_2|}{\Gamma(\eta)} \Bigg\{ \left|\tfrac{p_2}{\acute{q}}\right| e^{-\lambda\Psi(\mathrm{w})} \int_0^{\top_\circ} (\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)} \Psi'(\grave{\zeta})\, \mathrm{d}\grave{\zeta} \\
&\quad + \left|\tfrac{p_3}{\acute{q}}\right| e^{-\lambda\Psi(\mathrm{w})} \int_0^{\top} (\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)} \Psi'(\grave{\zeta})\, \mathrm{d}\grave{\zeta} + \int_0^{\mathrm{w}} (\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w})} \Psi'(\grave{\zeta})\, \mathrm{d}\grave{\zeta} \Bigg\} \\
&\leqslant \tfrac{k|\varkappa_1 - \varkappa_2|}{\Gamma(\eta)} \Bigg\{ \left|\tfrac{p_2}{\acute{q}}\right| \int_0^{\Psi(\top_\circ)} \grave{\zeta}^{\eta-1} e^{-\lambda\grave{\zeta}}\, \mathrm{d}\grave{\zeta} + \left|\tfrac{p_3}{\acute{q}}\right| \int_0^{\Psi(\top)} \grave{\zeta}^{\eta-1} e^{-\lambda\grave{\zeta}}\, \mathrm{d}\grave{\zeta} + \int_0^{\Psi(\mathrm{w})} \grave{\zeta}^{\eta-1} e^{-\lambda\grave{\zeta}}\, \mathrm{d}\grave{\zeta} \Bigg\} \\
&\leqslant \tfrac{k|\varkappa_1 - \varkappa_2|}{\Gamma(\eta)} \Bigg\{ \left|\tfrac{p_2}{\acute{q}}\right| \int_0^{\infty} \grave{\zeta}^{\eta-1} e^{-\lambda\grave{\zeta}}\, \mathrm{d}\grave{\zeta} + \left|\tfrac{p_3}{\acute{q}}\right| \int_0^{\infty} \grave{\zeta}^{\eta-1} e^{-\lambda\grave{\zeta}}\, \mathrm{d}\grave{\zeta} + \int_0^{\infty} \grave{\zeta}^{\eta-1} e^{-\lambda\grave{\zeta}}\, \mathrm{d}\grave{\zeta} \Bigg\} \\
&\leqslant \tfrac{k|\varkappa_1 - \varkappa_2|}{\Gamma(\eta)} \left\{ \left|\tfrac{p_2}{\acute{q}}\right| \tfrac{\Gamma(\eta)}{\lambda^\eta} + \left|\tfrac{p_3}{\acute{q}}\right| \tfrac{\Gamma(\eta)}{\lambda^\eta} + \tfrac{\Gamma(\eta)}{\lambda^\eta} \right\} \leqslant \tfrac{k}{\lambda^\eta} \left(\left|\tfrac{p_2}{\acute{q}}\right| + \left|\tfrac{p_3}{\acute{q}}\right| + 1 \right) |\varkappa_1 - \varkappa_2|.
\end{aligned}$$

Case 2: Let w $\in [\top_\circ, \top]$, so we have

$$
\begin{aligned}
|\mathcal{O}(\varkappa_1)(\mathrm{w})-\mathcal{O}(\varkappa_2)(\mathrm{w})| & \left|\frac{1}{\Gamma(\eta)}\int_0^{\top}\mathbb{G}_2(\mathrm{w},\grave{\zeta})\Psi'(\grave{\zeta})\left[\mathfrak{h}(\grave{\zeta},\varkappa)_1(\grave{\zeta}))-\mathfrak{h}(\grave{\zeta},\varkappa_2(\grave{\zeta}))\right]\,\mathrm{d}\grave{\zeta}\right| \\
&\le \frac{k|\varkappa_1-\varkappa_2|}{\Gamma(\eta)}\int_0^{\top}|\mathbb{G}_2(\mathrm{w},\grave{\zeta})|\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta} \\
&\le \frac{k|\varkappa_1-\varkappa_2|}{\Gamma(\eta)}\left(\left|\frac{p_2}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w})}\left\{\int_0^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\right\}\right. \\
&\quad+\left|\frac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w})}\left\{\int_0^{\top}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\right. \\
&\quad\left.+\int_{\top_\circ}^{\mathrm{w}}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}+\int_{\mathrm{w}}^{\top}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\right\} \\
&\quad\left.+\int_0^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w})}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}+\int_{\top_\circ}^{\mathrm{w}}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w})}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\right) \\
&\leqslant \frac{k|\varkappa_1-\varkappa_2|}{\Gamma(\eta)}\left\{\left|\frac{p_2}{\acute{q}}\right|\int_0^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\right. \\
&\quad\left.+\left|\frac{p_3}{\acute{q}}\right|\int_0^{\top}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}+\int_0^{\mathrm{w}}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w})}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\right\} \\
&\leqslant \frac{k|\varkappa_1-\varkappa_2|}{\Gamma(\eta)}\left\{\left|\frac{p_2}{\acute{q}}\right|\int_0^{\Psi(\top_\circ)}\grave{\zeta}^{\eta-1}e^{-\lambda\grave{\zeta}}\,\mathrm{d}\grave{\zeta}+\left|\frac{p_3}{\acute{q}}\right|\int_0^{\Psi(\top)}\grave{\zeta}^{\eta-1}e^{-\lambda\grave{\zeta}}\,\mathrm{d}\grave{\zeta}+\int_0^{\Psi(\mathrm{w})}\grave{\zeta}^{\eta-1}e^{-\lambda\grave{\zeta}}\,\mathrm{d}\grave{\zeta}\right. \\
&\leqslant \frac{k|\varkappa_1-\varkappa_2|}{\Gamma(\eta)}\left\{\left|\frac{p_2}{\acute{q}}\right|\int_0^{\infty}\grave{\zeta}^{\eta-1}e^{-\lambda\grave{\zeta}}\,\mathrm{d}\grave{\zeta}+\left|\frac{p_3}{\acute{q}}\right|\int_0^{\infty}\grave{\zeta}^{\eta-1}e^{-\lambda\grave{\zeta}}\,\mathrm{d}\grave{\zeta}+\int_0^{\infty}\grave{\zeta}^{\eta-1}e^{-\lambda\grave{\zeta}}\,\mathrm{d}\grave{\zeta}\right\} \\
&\leqslant \frac{k|\varkappa_1-\varkappa_2|}{\Gamma(\eta)}\left\{\left|\frac{p_2}{\acute{q}}\right|\frac{\Gamma(\eta)}{\lambda^\eta}+\left|\frac{p_3}{\acute{q}}\right|\frac{\Gamma(\eta)}{\lambda^\eta}+\frac{\Gamma(\eta)}{\lambda^\eta}\right\}\leqslant\frac{k}{\lambda^\eta}\left(\left|\frac{p_2}{\acute{q}}\right|+\left|\frac{p_3}{\acute{q}}\right|+1\right)|\varkappa_1-\varkappa_2|.
\end{aligned}
$$

Thus, for all w $\in \bar{\mathrm{J}}$, $|\mathcal{O}(\varkappa_1)(\mathrm{w})-\mathcal{O}(\varkappa_2)(\mathrm{w})|\leqslant\frac{k}{\lambda^\eta}\left(\left|\frac{p_2}{\acute{q}}\right|+\left|\frac{p_3}{\acute{q}}\right|+1\right)|\varkappa_1-\varkappa_2|$. Consequently by (3), $\mathcal{O}$ is a contraction. As a consequence of Banach fixed point theorem we deduce that $\mathcal{O}$ has a fixed point which is a solution of the problem (3).

The second result is based on Schaefer's fixed point.

Theorem 3. Assume that

(H_2) The function $\mathfrak{h}:\bar{\mathrm{J}}\times\mathbb{R}\to\mathbb{R}$ is continuous;

(H_3) There exists $\breve{\Delta}>0$ such that $|\mathfrak{h}(\mathrm{w},\varkappa)|\le\breve{\Delta}$ for each w $\in\bar{\mathrm{J}}$ and all $\varkappa\in\mathbb{R}$.

Then the $\mathbb{BVP}$ (3) has at least one solution on $\bar{\mathrm{J}}$.

Proof. We shall use Schaefer's fixed point theorem to prove that $\mathcal{O}$ defined by (5) has a fixed point. The proof will be given in several steps.

Step 1: $\mathcal{O}$ is continuous. Let $\{\varkappa_n\}$ be a sequence such that $\varkappa_n\to\varkappa$ in $C(\bar{\mathrm{J}})$. Then for each w $\in\bar{\mathrm{J}}$,

$$
\begin{aligned}
|\mathcal{O}(\varkappa_n)(\mathrm{w})-\mathcal{O}(\varkappa)(\mathrm{w})| &= \int_0^{\top}\mathbb{G}(\mathrm{w},\grave{\zeta})\frac{\Psi'(\grave{\zeta})}{\Gamma(\eta)}\mathfrak{h}(\grave{\zeta},\varkappa_n(\grave{\zeta}))-\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta} \\
&\leqslant \frac{1}{\Gamma(\eta)}\sup_{\mathrm{w}\in\bar{\mathrm{J}}}\Big|\mathfrak{h}(\mathrm{w},\varkappa_n(\mathrm{w}))-\mathfrak{h}(\mathrm{w},\varkappa(\mathrm{w}))\Big|\int_0^{T}\mathbb{G}(\mathrm{w},\grave{\zeta})\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta} \\
&\leqslant \frac{1}{\lambda^\eta}\left|\frac{p_2}{\acute{q}}+\frac{p_3}{\acute{q}}+1\right||\mathfrak{h}(\mathrm{w},\varkappa_n(\mathrm{w}))-\mathfrak{h}(\mathrm{w},\varkappa(\mathrm{w}))|_\infty.
\end{aligned}
$$

Since $\mathfrak{h}$ is a continuous function, we have

$$
|\mathcal{O}(\varkappa_n)(\mathrm{w})-\mathcal{O}(\varkappa)(\mathrm{w})|\leqslant\frac{1}{\lambda^\eta}\left|\frac{p_2}{\acute{q}}+\frac{p_3}{\acute{q}}+1\right||\mathfrak{h}(\mathrm{w},\varkappa_n(\mathrm{w}))-\mathfrak{h}(\mathrm{w},\varkappa(\mathrm{w}))|_\infty\to0,
$$

as $n\longrightarrow\infty$.

Step 2: $\mathcal{O}$ maps bounded sets into bounded sets in $C(\bar{\mathrm{J}})$. Indeed, it is enough to show that for any $r>0$ there exists a positive constant l such that for each $\varkappa \in B_r = \{\varkappa \in C(\bar{\mathrm{J}}) : |\varkappa|_\infty \leqslant r\}$, we have $|\mathcal{O}(\varkappa)|_\infty \leqslant l$. By ($\mathrm{H}_3$) we have for each $\mathrm{w} \in \bar{\mathrm{J}}$,

$$\begin{aligned}|\mathcal{O}(\varkappa)(\mathrm{w})| &\leqslant \left|\tfrac{q}{\acute{q}}\right| + \tfrac{1}{\Gamma(\eta)}\left|\int_0^\top \mathbb{G}(\mathrm{w},\grave{\zeta})\Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta}\right| \\ &\leqslant \left|\tfrac{q}{\acute{q}}\right| + \tfrac{\breve{\Delta}}{\Gamma(\eta)}\int_0^\top \mathbb{G}(\mathrm{w},\grave{\zeta})\Psi'(\grave{\zeta}))\,\mathrm{d}\grave{\zeta} \leqslant \left|\tfrac{q}{\acute{q}}\right| + \tfrac{\breve{\Delta}}{\lambda}^{\eta}\left(\left|\tfrac{p_2}{\acute{q}}\right| + \left|\tfrac{p_3}{\acute{q}}\right| + 1\right) := l.\end{aligned}$$

Step 3: Here we prove that the operator $\mathcal{O}$ maps bounded sets into equicontinuous sets from $C(\bar{\mathrm{J}})$. Let $\mathrm{w}_1,\mathrm{w}_2 \in \bar{\mathrm{J}}$, B_r be a bounded set in $C(\bar{\mathrm{J}})$. As in Step 2 we assume that $\varkappa \in B_r$ and $K_\Psi = \lambda\max\left\{\Psi'(\mathrm{w})e^{-\lambda\Psi(\mathrm{w})} : \mathrm{w} \in \bar{\mathrm{J}}\right\}$. Then the mean value theorem implies that $\left|e^{-\lambda\Psi(\mathrm{w}_2)} - e^{-\lambda\Psi(\mathrm{w}_1)}\right| \leqslant K_\Psi|\mathrm{w}_2-\mathrm{w}_1|$.

Case 1: Let $\mathrm{w}_1,\mathrm{w}_2 \in \bar{\mathrm{J}}$. Then

$$\begin{aligned}|\mathcal{O}(\varkappa)(\mathrm{w}_2) - \mathcal{O}(\varkappa)(\mathrm{w}_1)| &= \Bigg|\tfrac{q}{\acute{q}}\left(e^{-\lambda\Psi(\mathrm{w}_2)} - e^{-\lambda\Psi(\mathrm{w}_1)}\right) \\ &\quad + \int_0^\top \left(\mathbb{G}_1(\mathrm{w}_2,\grave{\zeta}) - \mathbb{G}_1(\mathrm{w}_1,\grave{\zeta})\right)\tfrac{\Psi'(\grave{\zeta})}{\Gamma(\eta)}\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta}\Bigg| \\ &\leqslant \left|\tfrac{q}{\acute{q}}\right|\left(e^{-\lambda\Psi(\mathrm{w}_2)} - e^{-\lambda\Psi(\mathrm{w}_1)}\right) + \int_0^\top \left(\mathbb{G}_1(\mathrm{w}_2,\grave{\zeta}) - \mathbb{G}_1(\mathrm{w}_1,\grave{\zeta})\right)\tfrac{\breve{\Delta}\Psi'(\grave{\zeta})}{\Gamma(\eta)}\,\mathrm{d}\grave{\zeta} \\ &\leqslant \left|\tfrac{q}{\acute{q}}\right|\left(e^{-\lambda\Psi(\mathrm{w}_2)} - e^{-\lambda\Psi(\mathrm{w}_1)}\right) \\ &\quad + \tfrac{\breve{\Delta}}{\Gamma(\eta)}\Bigg\{\Bigg(\int_0^{\mathrm{w}_2}\left|\tfrac{p_2}{\acute{q}}\right| e^{-\lambda\Psi(\mathrm{w}_2)}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta} \\ &\quad + \int_0^{\mathrm{w}_2}\left|\tfrac{p_3}{\acute{q}}\right| e^{-\lambda\Psi(\mathrm{w}_2)}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda(\tilde{\Psi}_{\grave{\zeta}}(\top))}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta} \\ &\quad + \int_0^{\mathrm{w}_2} e^{-\lambda\Psi(\mathrm{w}_2)}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta} \\ &\quad + \int_{\mathrm{w}_2}^{\top_\circ}\left|\tfrac{p_2}{\acute{q}}\right| e^{-\lambda\Psi(\mathrm{w}_2)}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta} \\ &\quad + \int_{\mathrm{w}_2}^{\top_\circ}\left|\tfrac{p_3}{\acute{q}}\right| e^{-\lambda\Psi(\mathrm{w}_2)}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\Bigg) \\ &\quad - \int_0^{\mathrm{w}_1}\mathrm{w}\tfrac{p_2}{\acute{q}}\mathrm{w}e^{-\lambda\Psi(\mathrm{w}_1)}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta} \\ &\quad - \int_0^{\mathrm{w}_1}\left|\tfrac{p_3}{\acute{q}}\right| e^{-\lambda\Psi(\mathrm{w}_1)}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(s)\,\mathrm{d}\grave{\zeta} \\ &\quad - \int_0^{\mathrm{w}_1} e^{-\lambda\Psi(\mathrm{w}_1)}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_1))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_1)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta} \\ &\quad - \int_{\mathrm{w}_1}^{\top_\circ}\left|\tfrac{p_2}{\acute{q}}\right| e^{-\lambda\Psi(\mathrm{w}_1)}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta} \\ &\quad - \int_{\mathrm{w}_1}^{\top_\circ}\left|\tfrac{p_3}{\acute{q}}\right| e^{-\lambda\Psi(\mathrm{w}_1)}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\Bigg\} \\ &\leqslant \left|\tfrac{q}{\acute{q}}\right| K_\Psi|\mathrm{w}_2-\mathrm{w}_1| + \tfrac{\breve{\Delta}}{\Gamma(\eta)}\Bigg\{\left|\tfrac{p_2}{\acute{q}}\right|\left(e^{-\lambda\Psi(\mathrm{w}_2)} - e^{-\lambda\Psi(\mathrm{w}_1)}\right) \\ &\quad \cdot \int_0^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\end{aligned}$$

$$
\begin{aligned}
&+\left|\tfrac{p_3}{\acute{q}}\right|\left(e^{-\lambda\Psi(\mathrm{w}_2)}-e^{-\lambda\Psi(\mathrm{w}_1)}\right)\int_0^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&+\int_0^{\mathrm{w}_2}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}-\int_0^{\mathrm{w}_1}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_1))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_1)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\Bigg\}\\
&\leqslant K_\Psi|\mathrm{w}_2-\mathrm{w}_1|\Bigg\{\left|\tfrac{q}{\acute{q}}\right|+\tfrac{\check{\Delta}}{\Gamma(\eta)}\left|\tfrac{p_2}{\acute{q}}\right|\int_0^{\Psi(\top_\circ)}\grave{\zeta}^{\eta-1}e^{-\lambda\grave{\zeta}}\,\mathrm{d}\grave{\zeta}\\
&+\tfrac{M}{\Gamma(\eta)}\left|\tfrac{p_3}{\acute{q}}\right|\int_{\tilde{\Psi}_{\top_\circ}(\top)}^{\Psi(\top_\circ)}\grave{\zeta}^{\eta-1}e^{-\lambda\grave{\zeta}}\,\mathrm{d}\grave{\zeta}\Bigg\}+\int_{\Psi(\mathrm{w}_1)}^{\Psi(\mathrm{w}_2)}\grave{\zeta}^{\eta-1}e^{-\lambda\eta}\,\mathrm{d}\grave{\zeta}.
\end{aligned}
$$

Case 2: Let $\mathrm{w}_1\in[0,\top_\circ]$, $\mathrm{w}_2\in[\top_\circ,\top]$. Then

$$
\begin{aligned}
|\mathcal{O}(\varkappa)(\mathrm{w}_2)-\mathcal{O}(\varkappa)(\mathrm{w}_1)|&=\Bigg|\tfrac{q}{\acute{q}}\left(e^{-\lambda\Psi(\mathrm{w}_2)}-e^{-\lambda\Psi(\mathrm{w}_1)}\right)\\
&+\tfrac{1}{\Gamma(\eta)}\int_0^{\top}\left(\mathbb{G}_2(\mathrm{w}_2,\grave{\zeta})-\mathbb{G}_1(\mathrm{w}_1,\grave{\zeta})\right)\Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta}\Bigg|\\
&\leqslant\left|\tfrac{q}{\acute{q}}\right|\left(e^{-\lambda\Psi(\mathrm{w}_2)}-e^{-\lambda\Psi(\mathrm{w}_1)}\right)\\
&+\tfrac{\check{\Delta}}{\Gamma(\eta)}\Bigg\{\left|\tfrac{p_2}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_2)}\int_0^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&+\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_2)}\int_0^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&+\int_0^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&+\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_2)}\int_{\top_\circ}^{\mathrm{w}_2}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&+\int_{\top_\circ}^{\mathrm{w}_2}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2))^{\eta-1}e^{-\lambda(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2))}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&+\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_2)}\int_{\mathrm{w}_2}^{\top}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&-\left|\tfrac{p_2}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_1)}\int_0^{\mathrm{w}_1}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&-\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_1)}\int_0^{\mathrm{w}_1}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&-\int_0^{\mathrm{w}_1}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_1))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_1)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&-\left|\tfrac{p_2}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_1)}\int_{\mathrm{w}_1}^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&-\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_1)}\int_{\mathrm{w}_1}^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&-\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_1)}\int_{\top_\circ}^{\top}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&\leqslant K_\Psi|\mathrm{w}_2-\mathrm{w}_1|\Bigg\{\left|\tfrac{q}{\acute{q}}\right|+\tfrac{\check{\Delta}}{\Gamma(\lambda)}\left|\tfrac{p_2}{\acute{q}}\right|\int_0^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&+\tfrac{\check{\Delta}}{\Gamma(\lambda)}\left|\tfrac{p_3}{\acute{q}}\right|\int_0^{\top}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\Bigg\}
\end{aligned}
$$

$$+\int_0^{w_2}(\tilde{\Psi}_{\grave{\zeta}}(w_2))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(w_2)}\Psi'(\grave{\zeta})\,d\grave{\zeta}-\int_0^{w_1}(\tilde{\Psi}_{\grave{\zeta}}(w_1))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(w_1)}\Psi'(\grave{\zeta})\,d\grave{\zeta}$$
$$\leqslant K_\Psi|w_2-w_1|\bigg\{\left|\tfrac{q}{\acute{q}}\right|+\tfrac{\check{\Delta}}{\Gamma(\eta)}\left|\tfrac{p_2}{\acute{q}}\right|\int_0^{\Psi(\top_\circ)}\grave{\zeta}^{\eta-1}e^{-\lambda\grave{\zeta}}\,d\grave{\zeta}$$
$$+\tfrac{\check{\Delta}}{\Gamma(\alpha)}\left|\tfrac{p_3}{\acute{q}}\right|\int_{\tilde{\Psi}_{\grave{\zeta}}(\top)}^{\Psi(\top_\circ)}\grave{\zeta}^{\eta-1}e^{-\lambda\grave{\zeta}}\,d\grave{\zeta}\bigg\}+\int_{\Psi(w_1)}^{\Psi(w_2)}\grave{\zeta}^{\eta-1}e^{-\lambda\grave{\zeta}}\,d\grave{\zeta}.$$

Case 3: Let $w_1, w_2 \in [\top_\circ, \top]$. Then

$$|\mathcal{O}(\varkappa)(w_2)-\mathcal{O}(\varkappa)(w_1)|=\bigg|\tfrac{q}{\acute{q}}\left(e^{-\lambda\Psi(w_2)}-e^{-\lambda\Psi(w_1)}\right)$$
$$+\int_0^{\top}\mathbb{G}_2(w_2,\grave{\zeta})-\mathbb{G}_2(w_1,\grave{\zeta})\tfrac{\Psi'(\grave{\zeta})}{\Gamma(\eta)}\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,d\grave{\zeta}\bigg|$$
$$\leqslant\left|\tfrac{q}{\acute{q}}\right|\left(e^{-\lambda\Psi(w_2)}-e^{-\lambda\Psi(w_1)}\right)$$
$$+\tfrac{\check{\Delta}}{\Gamma(\eta)}\bigg\{\int_{\top_\circ}^{w_2}\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(w_2)}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,d\grave{\zeta}$$
$$+\int_{\top_\circ}^{w_2}(\tilde{\Psi}_{\grave{\zeta}}(w_2))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(w_2)}\Psi'(\grave{\zeta})\,d\grave{\zeta}$$
$$+\int_{w_2}^{\top}\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(w_2)}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,d\grave{\zeta}$$
$$-\int_{\top_\circ}^{w_1}\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(w_1)}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,d\grave{\zeta}$$
$$-\int_{\tau}^{w_1}(\tilde{\Psi}_{\grave{\zeta}}(w_1))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(w_1)}\Psi'(\grave{\zeta})\,d\grave{\zeta}$$
$$-\int_{w_1}^{\top}\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(w_1)}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,d\grave{\zeta}\bigg\}$$
$$\leqslant\left|\tfrac{q}{\acute{q}}\right|\left(e^{-\lambda\Psi(w_2)}-e^{-\lambda\Psi(w_1)}\right)$$
$$+\tfrac{\check{\Delta}}{\Gamma(\eta)}\bigg\{\int_{\top_\circ}^{\top}\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(w_2)}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,d\grave{\zeta}$$
$$-\int_{\top_\circ}^{\top}\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(w_1)}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,d\grave{\zeta}$$
$$+\int_{\top_\circ}^{w_2}(\tilde{\Psi}_{\grave{\zeta}}(w_2))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(w_2)}\Psi'(\grave{\zeta})\,d\grave{\zeta}-\int_{\top_\circ}^{w_1}(\tilde{\Psi}_{\grave{\zeta}}(w_2))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(w_1)}\Psi'(\grave{\zeta})\,d\grave{\zeta}\bigg\}$$
$$\leqslant K_\Psi|w_2-w_1|\bigg\{\left|\tfrac{q}{\acute{q}}\right|+\tfrac{\check{\Delta}}{\Gamma(\eta)}\left|\tfrac{p_3}{\acute{q}}\right|\int_0^{\tilde{\Psi}_{\top_\circ}(\top)}\grave{\zeta}^{\eta-1}e^{-\lambda\grave{\zeta}}\,d\grave{\zeta}\bigg\}+\int_{\tilde{\Psi}_{\top_\circ}(w_2)}^{\tilde{\Psi}_{\top_\circ}(w_2)}\grave{\zeta}^{\eta-1}e^{-\lambda\grave{\zeta}}\,d\grave{\zeta}.$$

The right hand side of the above inequalities for all Cases 1–3 tend for zero by $w_2 \longrightarrow w_1$. From this due to Arzelà-Ascoli theorem and Steps 1–3 follows that the mapping $\mathcal{O} : C(\bar{J}) \to \mathcal{O}(\bar{J})$ is continuous.

Step 4: Here we prove the necessary prior bounds. Indeed we show that the set $\Upsilon = \{\varkappa \in C([0,\mathbb{R}]) : \varkappa = \mu\mathcal{O}(\varkappa)$ for some $\mu \in \Omega\}$, is bounded. Suppose that $\varkappa = \mu\mathcal{O}(\varkappa)$ for some $0 < \mu < 1$. Then for each $w \in \bar{J}$ we can write

$$\varkappa(w)=\mu\bigg\{\tfrac{q}{\acute{q}}+\int_0^{\top}\mathbb{G}(w,\grave{\zeta})\tfrac{\Psi'(s)}{\Gamma(\alpha)}\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,d\grave{\zeta}\bigg\}.$$

This fact in combination with (H_3) shows that for each $\mathrm{w} \in \bar{\mathrm{J}}$,

$$|\mathcal{O}(\varkappa)(\mathrm{w})| \leqslant \left|\frac{q}{\acute{q}}\right| + \frac{\check{\Delta}}{\Gamma(\eta)} \int_0^{\top} \mathbb{G}(\mathrm{w}, \grave{\zeta})\Psi'(\grave{\zeta})\, \mathrm{d}\grave{\zeta} \leqslant \left|\frac{q}{\acute{q}}\right| + \frac{\check{\Delta}}{\Gamma(\eta)} \left(\left|\frac{p_2}{\acute{q}}\right| + \left|\frac{p_3}{\acute{q}}\right| + 1 \right),$$

for each $\mathrm{w} \in \bar{\mathrm{J}}$. Thus the set Υ is bounded and $\mathcal{O}$ has a fixed point by Schaefer's fixed point theorem, that is a solution of problem.

2.2 An illustrative example

In this section we give an example to illustrate the usefulness of our main results. Let us consider $\Psi(\mathrm{w}) = \ln(\mathrm{w}+1)$, and the following $\mathbb{FBVP}$,

$$\mathcal{D}_C^{\eta,\lambda,\Psi}\varkappa(\mathrm{w}) = \frac{e^{-2\mathrm{w}}|\varkappa(\mathrm{w})|}{24|1+\varkappa(\mathrm{w})|}, \qquad \mathrm{w} \in \bar{\mathrm{J}} = [0,1]\,, \ \eta \in (0,1]\,, \tag{6}$$

with

$$\varkappa(0) + \tfrac{3}{2}\varkappa\left(\tfrac{1}{2}\right) + 2\,\varkappa(1) = \tfrac{1}{2}. \tag{7}$$

Put $\mathfrak{h}(\mathrm{w}, \varkappa(\mathrm{w})) = \frac{e^{-2\mathrm{w}}\varkappa(\mathrm{w})}{24(\mathrm{w}+1)}$, $(\mathrm{w}, \varkappa) \in \bar{\mathrm{J}} \times [0, +\infty)$. Let $\varkappa_1, \varkappa_2 \in [0, +\infty)$ and $\mathrm{w} \in \bar{\mathrm{J}}$. Then we have

$$|\mathfrak{h}(\mathrm{w}, \varkappa_1) - \mathfrak{h}(\mathrm{w}, \varkappa_2)| = \frac{e^{-2\mathrm{w}}}{24}\left|\frac{\varkappa_1}{2\varkappa_1+1} - \frac{\varkappa_2}{2\varkappa_2+1}\right| = \frac{e^{-2\mathrm{w}}}{24}\frac{|\varkappa_1-\varkappa_2|}{(\varkappa_1+1)(\varkappa_2+1)} \leqslant \frac{e^{-2\mathrm{w}}}{24}|\varkappa_1-\varkappa_2| \leqslant \frac{1}{24}|\varkappa_1-\varkappa_2|.$$

Hence the condition (H_1) holds with $k = \frac{1}{24}$. We shall check that condition (7) is satisfied for appropriate values of $\eta \in]0,1[$ with $p_1 = 1$, $p_2 = \frac{3}{2}$, $p_3 = 2$, $\top = 1$, $\top_\circ = \frac{1}{2}$ and

$$\acute{q} = p_1 + p_2 e^{-\lambda\Psi(\top_\circ)} + p_3 e^{-\lambda\Psi(\top)} = 1 + \tfrac{3}{2}e^{-\lambda\Psi(1/2)} + 2e^{-\lambda\Psi(1)}. \tag{8}$$

Then by Theorem 2 the problem (6)-(7) has a unique solution on $\bar{\mathrm{J}}$ for values of η and λ satisfying condition (H_1). For example

- If $\lambda = 1$ and for all $\eta \in (0,1)$ then thanks to Eq. (8), we have $\acute{q} = 3$ and

$$\frac{k}{\lambda^\eta}\left(\left|\frac{p_2}{\acute{q}}\right| + \left|\frac{p_3}{\acute{q}}\right| + 1\right) \simeq 0.09027 < 1.$$

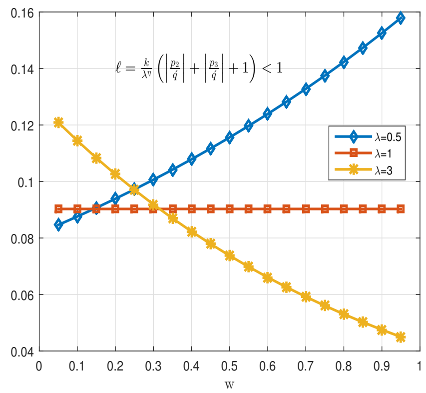

Figure 1. 2D-graph of $\ell < 1$ for the $\mathbb{BVP}$ (6) whenever $\lambda = 0.5, 1, 3$, $\eta, \mathrm{w} \in \Omega$

Fig. 1 shows 2D-graph of ℓ for the $\mathbb{BVP}$ (6) whenever $\lambda = 0.5, 1, 3$ and $\eta, \mathrm{w} \in \Omega$.

- If $\lambda = 3$ and $\eta \in \Omega$, we have

$$\frac{k}{\lambda^{\eta}}\left(\left|\frac{p_2}{\acute{q}}\right| + \left|\frac{p_3}{\acute{q}}\right| + 1\right) = \frac{1}{24\times 3^{\eta}}\left(\left|\frac{1}{2}\right| + \left|\frac{2}{3}\right| + 1\right) < 1. \tag{9}$$

Table 1 presents numerical values of ℓ for the $\mathbb{BVP}$ (6) whenever $\lambda = 0.5, 1, 3$ and $\eta, \mathrm{w} \in \Omega$. Fig. 1 shows 2D-graph of ℓ for the $\mathbb{BVP}$ (6) whenever $\lambda = 0.5, 1, 3$ and $\eta, \mathrm{w} \in \Omega$.

Table 1

Obtained results of $\ell < 1$ in (9) when $\lambda = 0.5, 1, 3$ and $\eta, \mathrm{w} \in \Omega$

w	ℓ		
	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 3$
0.05	0.08463	0.09028	0.12091
0.10	0.08761	0.09028	0.11444
0.15	0.09070	0.09028	0.10833
0.20	0.09390	0.09028	0.10254
0.25	0.09721	0.09028	0.09706
⋮	⋮	⋮	⋮
0.80	0.14232	0.09028	0.05304
0.85	0.14734	0.09028	0.05021
0.90	0.15254	0.09028	0.04752
0.95	0.15792	0.09028	0.04498

2.3 Data comparison

At present, we consider $\lambda = 3$, three values for $\eta = 0.7, 0.8, 0.9$ and four cases for $\Psi(\mathrm{w})$ as $\Psi_1(\mathrm{w}) = 2^{\mathrm{w}}$; $\Psi_2(\mathrm{w}) = \mathrm{w}$ (Caputo derivative); $\Psi_3(\mathrm{w}) = \ln \mathrm{w}$ (Caputo–Hadamard derivative); $\Psi_4(\mathrm{w}) = \sqrt{\mathrm{w}}$ (Katugampola derivative); for the $\mathbb{BVP}$ (6). Tables 2, 3 and 4 show the numerical results for these cases. One can see illustrative results in the Figs. 2, 3 and 4. Therefore, these results guarantee that for all of three different cases by terms of the order η and four standard fractional derivatives Ψ, the $\mathbb{BVP}$ admits at least a solution on $\bar{J}$.

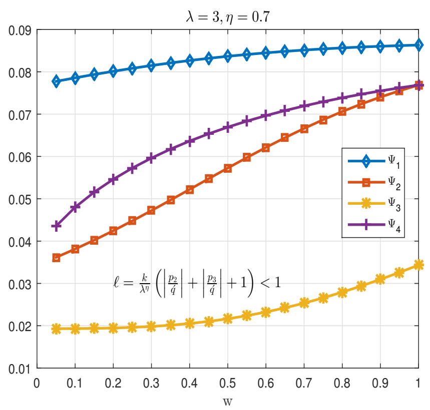

Figure 2. 2D plot of ℓ in $\mathbb{BVP}$ (6) when $\lambda = 3$, $\Psi(\mathrm{w}) \in \{2^{\mathrm{w}}, \mathrm{w}, \ln \mathrm{w}, \sqrt{\mathrm{w}}\}$ and $\eta = 0.7$ for $\mathrm{w} \in \Omega$

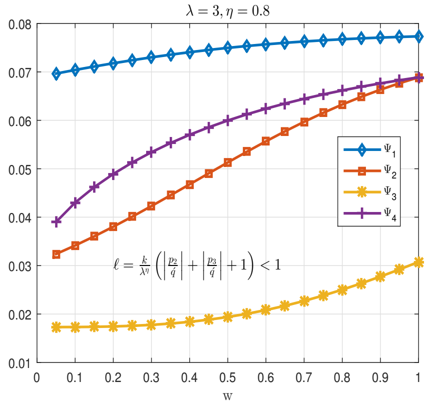

Figure 3. 2D plot of ℓ in $\mathbb{BVP}$ (6) when $\lambda = 3$, $\Psi(\mathrm{w}) \in \{2^{\mathrm{w}}, \mathrm{w}, \ln \mathrm{w}, \sqrt{\mathrm{w}}\}$ and $\eta = 0.8$ for $\mathrm{w} \in \Omega$

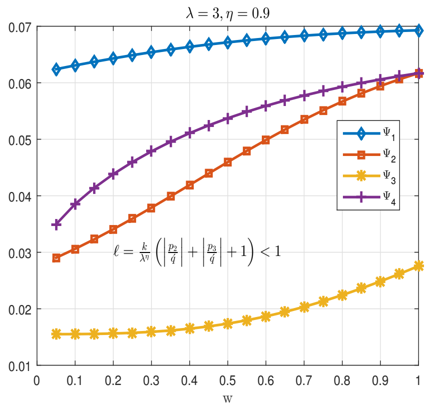

Figure 4. 2D plot of ℓ in $\mathbb{BVP}$ (6) when $\lambda = 3$, $\Psi(\mathrm{w}) \in \{2^{\mathrm{w}}, \mathrm{w}, \ln \mathrm{w}, \sqrt{\mathrm{w}}\}$ and $\eta = 0.9$ for $\mathrm{w} \in \Omega$

Table 2

Obtained results of $\ell < 1$ in $\mathbb{BVP}$ (6) when $\lambda = 3$, $\Psi(\mathrm{w}) \in \{2^{\mathrm{w}}, \mathrm{w}, \ln \mathrm{w}, \sqrt{\mathrm{w}}\}$ and $\eta = 0.7$ for $\mathrm{w} \in \Omega$

w	$\Psi_1(\mathrm{w}) = 2^{\mathrm{w}}$		$\Psi_2(\mathrm{w}) = \mathrm{w}$		$\Psi_3(\mathrm{w}) = \ln \mathrm{w}$		$\Psi_4(\mathrm{w}) = \sqrt{\mathrm{w}}$	
	$\acute{q}$	ℓ	$\acute{q}$	ℓ	$\acute{q}$	ℓ	$\acute{q}$	ℓ
0.05	1.157	0.078	4.012	0.036	28001.000	0.019	2.790	0.044
0.10	1.140	0.079	3.593	0.038	3501.000	0.019	2.355	0.048
0.15	1.125	0.079	3.232	0.040	1038.037	0.019	2.095	0.052
0.20	1.112	0.080	2.921	0.042	438.500	0.019	1.915	0.055
0.25	1.099	0.081	2.653	0.045	225.000	0.020	1.781	0.057
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.70	1.027	0.085	1.429	0.067	11.204	0.025	1.284	0.072
0.75	1.023	0.085	1.369	0.069	9.296	0.027	1.260	0.073
0.80	1.019	0.086	1.318	0.071	7.836	0.028	1.239	0.074
0.85	1.016	0.086	1.273	0.072	6.699	0.029	1.220	0.075
0.90	1.013	0.086	1.235	0.074	5.801	0.031	1.203	0.075
0.95	1.011	0.086	1.202	0.076	5.082	0.033	1.188	0.076
1.00	1.009	0.086	1.174	0.077	4.500	0.034	1.174	0.077

Table 3

Obtained results of $\ell < 1$ in $\mathbb{BVP}$ (6) when $\lambda = 3$, $\Psi(\mathrm{w}) \in \{2^{\mathrm{w}}, \mathrm{w}, \ln \mathrm{w}, \sqrt{\mathrm{w}}\}$ and $\eta = 0.8$ for $\mathrm{w} \in \Omega$

w	$\Psi_1(\mathrm{w}) = 2^{\mathrm{w}}$		$\Psi_2(\mathrm{w}) = \mathrm{w}$		$\Psi_3(\mathrm{w}) = \ln \mathrm{w}$		$\Psi_4(\mathrm{w}) = \sqrt{\mathrm{w}}$	
	$\acute{q}$	ℓ	$\acute{q}$	ℓ	$\acute{q}$	ℓ	$\acute{q}$	ℓ
0.05	1.157	0.070	4.012	0.032	28001.000	0.017	2.790	0.039
0.10	1.140	0.070	3.593	0.034	3501.000	0.017	2.355	0.043
0.15	1.125	0.071	3.232	0.036	1038.037	0.017	2.095	0.046
0.20	1.112	0.072	2.921	0.038	438.500	0.017	1.915	0.049
0.25	1.099	0.072	2.653	0.040	225.000	0.018	1.781	0.051
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.75	1.023	0.077	1.369	0.062	9.296	0.024	1.260	0.065
0.80	1.019	0.077	1.318	0.063	7.836	0.025	1.239	0.066
0.85	1.016	0.077	1.273	0.065	6.699	0.026	1.220	0.067
0.90	1.013	0.077	1.235	0.066	5.801	0.028	1.203	0.068
0.95	1.011	0.077	1.202	0.068	5.082	0.029	1.188	0.068
1.00	1.009	0.077	1.174	0.069	4.500	0.031	1.174	0.069

Table 4

Obtained results of $\ell < 1$ in $\mathbb{BVP}$ (6) when $\lambda = 3$, $\Psi(\mathrm{w}) \in \{2^{\mathrm{w}}, \mathrm{w}, \ln \mathrm{w}, \sqrt{\mathrm{w}}\}$ and $\eta = 0.9$ for $\mathrm{w} \in \Omega$

w	$\Psi_1(\mathrm{w}) = 2^{\mathrm{w}}$		$\Psi_2(\mathrm{w}) = \mathrm{w}$		$\Psi_3(\mathrm{w}) = \ln \mathrm{w}$		$\Psi_4(\mathrm{w}) = \sqrt{\mathrm{w}}$	
	$\acute{q}$	ℓ	$\acute{q}$	ℓ	$\acute{q}$	ℓ	$\acute{q}$	ℓ
0.05	1.157	0.062	4.012	0.029	28001.000	0.016	2.790	0.035
0.10	1.140	0.063	3.593	0.031	3501.000	0.016	2.355	0.039
0.15	1.125	0.064	3.232	0.032	1038.037	0.016	2.095	0.041
0.20	1.112	0.064	2.921	0.034	438.500	0.016	1.915	0.044
0.25	1.099	0.065	2.653	0.036	225.000	0.016	1.781	0.046
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.75	1.023	0.069	1.369	0.055	9.296	0.021	1.260	0.059
0.80	1.019	0.069	1.318	0.057	7.836	0.022	1.239	0.059
0.85	1.016	0.069	1.273	0.058	6.699	0.024	1.220	0.060
0.90	1.013	0.069	1.235	0.059	5.801	0.025	1.203	0.061
0.95	1.011	0.069	1.202	0.061	5.082	0.026	1.188	0.061
1.00	1.009	0.069	1.174	0.062	4.500	0.028	1.174	0.062

Conclusion

This paper contains a new fractional mathematical model of a $\mathbb{BVP}$ consisting of the Tempered $\Psi-$Caputo derivative in the framework of the generalized sequential G-operators. We turned to the investigation of the qualitative behaviors of its solutions including existence and uniqueness. To confirm the existence criterion, we used the Banach contraction mapping principle and Schaefer's fixed point theorem. Comparison of data obtained by choosing several types of fractional derivatives is of great importance.

Author Contributions

K. Bensassa: Actualization, formal analysis, methodology, initial draft, validation, investigation and was a major contributor in writing the manuscript. M. Benbachir: Methodology, actualization, validation, investigation, formal analysis and initial draft. M.E. Samei: Validation, actualization, formal analysis, methodology, investigation, simulation, initial draft, software and was a major contributor in writing the manuscript. S. Salahshour: Methodology, actualization, validation, investigation, formal analysis and initial draft. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare that they have no competing interests.

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