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Fixed point results in C^*-algebra valued fuzzy metric space with applications to boundary value problem and control theory

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In this paper, we derive some new fixed point results in C^*-algebra valued fuzzy metric space with the help of subadditive altering distance function with respect to a t-norm. Our results generalize some existing fixed point results in the literature. A common fixed point result is also derived for a pair of mappings on complete C^*-algebra valued fuzzy metric space. The results are supported by suitable examples along with the graphical demonstration of the used conditions. As application, we establish the solvability of a second order boundary value problem. Moreover, the results are also applied in control theory to study the possibility of optimally controlling the solution of an ordinary differential equation in dynamic programming.

Keywords: C^*-algebra valued metric space, fuzzy metric space, fixed point, boundary value problem, control theory.

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Introduction

The concept of fuzzy metric was introduced by Kramosil and Michalek [1] in 1975 and the study of fixed points in fuzzy metric space was given by Grabiec [2] in 1988. Fixed point theory has emerging applications in various domains including applied analysis, physics, mechanics, medical science etc. During recent years, several researchers ([3–10]) have done the study of fixed point theory by introducing different types of mappings as well as considering different spaces along with various applications.

In 1984, Khan et al. [11] introduced the concept of altering distance function between two points and again in 2011, Shen et al. [12] defined the same by introducing a new condition and derived many fixed point results in fuzzy metric space. After that Roldán-López-de-Hierro et al. [3] established some results on common fixed point theorems for weakly compatible mappings in fuzzy metric spaces with new contractive conditions. In 2018, Shoaib et al. [13] derived some fixed point results in dislocated complete b-metric space and gave some examples as well as applications relating the results to common fixed points for multivalued mappings. Using the altering distance function, Patir et al. [5,6,8] derived some fixed point results using different types of mappings and gave examples as well as applications to boundary value problem and integral equations.

The concept of C^*-algebra valued metric space was given by Ma et al. [14] by replacing the set of non negative real numbers with a (unital) C^*-algebra. In 2020, Madadi et al. [15] introduced the concept of C^*-algebra valued fuzzy metric space and derived some topological properties of the same. After that in 2021, Khaofong et al. [16] gave a new definition of C^*-algebra valued fuzzy metric space by replacing $[0,1]$ by $[0_{\mathbb{A}}, 1_{\mathbb{A}}]$, where $0_{\mathbb{A}}$ and $1_{\mathbb{A}}$ are the zero element and the unit element of an algebra $\mathbb{A}$ respectively in the sense of George and Veeramani [17], and established some results by introducing C^*-algebra valued contraction mapping with application to integral equations.

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Motivated by these, in this paper, we establish some fixed point results for complete C^*-algebra valued fuzzy metric space using subadditive altering distance function with respect to some t-norm. We also derive a common fixed point result for a pair of mappings on complete C^*-algebra valued fuzzy metric space. Some of our results extend the works of Shoaib et al. [13] and Patir et al. [5, 6, 8, 18] in the setting of C^*-algebra valued fuzzy metric space. In the third section we give an application of our established result to second order boundary value problem. In view of the vast application of control theory in present times in different technological fields viz., spacecraft control, robot technology, smart fluid technology, etc., the section four of our paper is devoted to the study of control theory via our derived result. Here we apply our results to study the possibility of optimally controlling the solution of an ordinary differential equation in dynamic programming.

1 Preliminaries

Throughout the paper, $\mathbb{A}$ denotes a unital C^*-algebra with unity $1_{\mathbb{A}}$. A complex algebra $\mathbb{A}$ is called a complex *-algebra if there is an involution $* : \mathbb{A} \to \mathbb{A}$ defined on it by $u \to u^*$, where u^* is called the adjoint of u and having the properties that for all $u, v \in \mathbb{A}$, $(\lambda u + v)^* = \bar{\lambda} u^* + v^*$, $(uv)^* = v^* u^*$ and $(u^*)^* = u$, where $\bar{\lambda}$ denotes the conjugate of $\lambda \in \mathbb{C}$. A complete unital *-algebra is called a Banach *-algebra with the norm satisfying $||u^*|| = ||u||$ for all $u \in \mathbb{A}$. Moreover, a Banach *-algebra is a C^*-algebra if $||u^* u|| = ||u||^2$ for all $u \in \mathbb{A}$.

An element $\xi \in \mathbb{A}$ is called a positive element of $\mathbb{A}$ and denoted by $0_{\mathbb{A}} \preccurlyeq \xi$ ($0_{\mathbb{A}}$ being the zero element of $\mathbb{A}$) if $\xi \in \mathbb{A}_h$ and $\sigma(\xi) \subset [0, \infty)$, where $\mathbb{A}_h = \{\xi \in \mathbb{A} : \xi^* = \xi\}$ and $\sigma(\xi)$ is the spectrum of ξ. A partial ordering on $\mathbb{A}$ is defined by $\xi \preccurlyeq \eta$ (or, $\eta \succcurlyeq \xi$) if and only if $0_{\mathbb{A}} \preccurlyeq \eta - \xi$ (or, $\eta - \xi \succcurlyeq 0_{\mathbb{A}}$). When $\xi - \eta$ is positive and non-zero, we call $\xi - \eta$ as strictly positive and denote it by $\xi - \eta \succ 0_{\mathbb{A}}$ (or, $\xi \succ \eta$). The set $\{\xi \in \mathbb{A} : 0_{\mathbb{A}} \preccurlyeq \xi\}$ is denoted by $\mathbb{A}^+$ and we denote $(\xi^* \xi)^{\frac{1}{2}}$ as $|\xi|$ and for invertible η, $\xi \eta^{-1}$ as $\frac{\xi}{\eta}$. Let $\mathbb{A}'$ be the set $\{\xi \in \mathbb{A}^+ : \xi\eta = \eta\xi \text{ for all } \eta \in \mathbb{A}\}$. Moreover, $[0_{\mathbb{A}}, 1_{\mathbb{A}}]$ denotes the set $\{\xi \in \mathbb{A} : 0_{\mathbb{A}} \preccurlyeq \xi \preccurlyeq 1_{\mathbb{A}}\}$.

Definition 1. [14] Let X be a nonempty set and $\mathbb{A}$ be a C^*-algebra. Suppose that a mapping $d : X \times X \to \mathbb{A}^+$ satisfies:

(i) $d(\xi, \eta) = 0_{\mathbb{A}}$ if and only if $\xi = \eta$ for all $\xi, \eta \in X$,

(ii) $d(\xi, \eta) = d(\eta, \xi)$ for all $\xi, \eta \in X$,

(iii) $d(\xi, \eta) \preccurlyeq d(\xi, \zeta) + d(\zeta, \eta)$ for all $\xi, \eta, \zeta \in X$.

Then d is called a C^*-algebra valued metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra valued metric space.

Example 1. [19] Let $X = [0, 1]$ and $\mathbb{A} = \mathbb{M}_2(\mathbb{R})$, the set of all bounded linear operators on the Hilbert space $\mathbb{R}^2$. Define $d : X \times X \to \mathbb{A}^+$ by

$$d(\xi, \eta) = \begin{bmatrix} |\xi - \eta| & 0 \\ 0 & 2|\xi - \eta| \end{bmatrix},$$

where $\xi, \eta \in X$. Then, $(X, \mathbb{A}, d)$ is a C^*-algebra valued metric space.

Lemma 1. [20, 21] Suppose that $\mathbb{A}$ is a unital C^*-algebra with unit element $1_{\mathbb{A}}$.

(i) For any $\xi \in \mathbb{A}^+$, $\xi \preccurlyeq 1_{\mathbb{A}}$ if and only if $||\xi|| \leq 1$.

(ii) If $u \in \mathbb{A}^+$ with $||u|| \leq \frac{1}{2}$, then $1_{\mathbb{A}} - u$ is invertible and $||u(1_{\mathbb{A}} - u)^{-1}|| < 1$.

(iii) Suppose that $u, v \in \mathbb{A}$ with $0_{\mathbb{A}} \preccurlyeq u, v$ and $uv = vu$ then $0_{\mathbb{A}} \preccurlyeq uv$.

(iv) Suppose that $\tilde{\mathbb{A}} = \{u \in \mathbb{A} : uv = vu \text{ for all } v \in \mathbb{A}\}$. Let $u \in \tilde{\mathbb{A}}$, if $v, w \in \tilde{\mathbb{A}}$ with $0_{\mathbb{A}} \preccurlyeq w \preccurlyeq v$ and $1_{\mathbb{A}} - u$ is a positive element in $\tilde{\mathbb{A}}$ then $(1_{\mathbb{A}} - u)^{-1} w \preccurlyeq (1_{\mathbb{A}} - u)^{-1} v$.

Lemma 2. [22, 23] Let $\mathbb{A}$ be a C^*-algebra with unit element $1_{\mathbb{A}}$ and let $u, v \in \mathbb{A}$.

(i) If u is self-adjoint, then $u \preccurlyeq ||u|| 1_{\mathbb{A}}$.

(ii) If $0_{\mathbb{A}} \preccurlyeq u \preccurlyeq v$, then $||u|| \leq ||v||$.
(iii) If $u \in \mathbb{A}$, then $1_{\mathbb{A}} + uu^*$ is invertible in $\mathbb{A}$.
(iv) If $u \in \mathbb{A}^+$, then $u = \xi^*\xi$ for some $\xi \in \mathbb{A}$.

Madadi et al. [15] defined the triangular norm or t-norm as follows:

Definition 2. Let $\mathbb{A}$ be a C^*-algebra with unit element $1_{\mathbb{A}}$. A mapping $\tau : \mathbb{A}^+ \times \mathbb{A}^+ \to \mathbb{A}^+$ is called a t-norm if
(i) $\tau(a, 1_{\mathbb{A}}) = a$ for all $a \in \mathbb{A}^+$,
(ii) $\tau(a, b) = \tau(b, a)$ for all $a, b \in \mathbb{A}^+$,
(iii) $a \preccurlyeq a'$, $b \preccurlyeq b' \implies \tau(a, b) \preccurlyeq \tau(a', b')$ for all $a, b, c, d \in \mathbb{A}^+$,
(iv) $\tau(a, \tau(b, c)) = \tau(\tau(a, b), c)$ for all $a, b, c \in \mathbb{A}^+$.

Definition 3. [16] Let $\mathbb{A}$ be a C^*-algebra with unit element $1_{\mathbb{A}}$. For an arbitrary set X, let τ be a continuous t-norm on $\mathbb{A}^+$ and $M_{\mathbb{A}}$ be a fuzzy set from $X \times X \times (0, \infty) \to [0_{\mathbb{A}}, 1_{\mathbb{A}}]$. Then $(X, M_{\mathbb{A}}, \tau)$ is called a C^*-algebra valued fuzzy metric space, if it satisfies the following conditions, for each $\xi, \eta, \rho \in X$ and $t, s > 0$,
(i) $M_{\mathbb{A}}(\xi, \eta, t) \succ 0_{\mathbb{A}}$,
(ii) $M_{\mathbb{A}}(\xi, \eta, t) = 1_{\mathbb{A}}$ if and only if $\xi = \eta$ for all $t > 0$,
(iii) $M_{\mathbb{A}}(\xi, \eta, t) = M_{\mathbb{A}}(\eta, \xi, t)$,
(iv) $\tau(M_{\mathbb{A}}(\xi, \eta, s), M_{\mathbb{A}}(\eta, \rho, t)) \preccurlyeq M_{\mathbb{A}}(\xi, \rho, s + t)$,
(v) $M_{\mathbb{A}}(\xi, \eta) : (0, \infty) \to [0_{\mathbb{A}}, 1_{\mathbb{A}}]$ is continuous.

As in [12], we define the altering distance function in C^*-algebra valued fuzzy metric space as follows.

Definition 4. Let $(X, M_{\mathbb{A}}, \tau)$ be a C^*-algebra fuzzy metric space with unit element $1_{\mathbb{A}}$. Let $\phi : \mathbb{A}^+ \to \mathbb{A}^+$ be a mapping. Then ϕ is called an altering distance function if
(i) ϕ is strictly decreasing and left continuous,
(ii) $\phi(k) = 0_{\mathbb{A}}$ if and only if $k = 1_{\mathbb{A}}$, i.e., $\lim_{k \to 1_{\mathbb{A}}^-} \phi(k) = 0_{\mathbb{A}}$.

Using the subadditivity condition with respect to a t-norm τ, we give the following definition of subadditive altering distance function with respect to τ.

Definition 5. Let $(X, M_{\mathbb{A}}, \tau)$ be a C^*-algebra valued fuzzy metric space. An altering distance function ϕ is said to be subadditive with respect to the t-norm τ if $\phi(\tau(a, b)) \preccurlyeq \phi(a) + \phi(b)$, $a, b \in \{M_{\mathbb{A}}(\xi, \eta, t) : \xi, \eta \in X, t > 0\}$.

In the same line as Grabiec [2; Lemma 4], we can prove the following lemma in the setting of C*-algebra valued fuzzy metric space.

Lemma 3. Let $(X, M_{\mathbb{A}}, \tau)$ is a C^*-algebra valued fuzzy metric space. Then $M_{\mathbb{A}}(\xi, \eta, t) \preccurlyeq M_{\mathbb{A}}(\xi, \eta, kt)$, where $k \in \mathbb{N}$, $\xi, \eta \in X$ and $t > 0$.

Proof. Let $t, s > 0$ with $t < s$. Suppose that for all $\xi, \eta \in X$, $M_{\mathbb{A}}(\xi, \eta, t) \succ M_{\mathbb{A}}(\xi, \eta, s)$. Now, by condition (iv) of Definition 3,

$$\begin{aligned} \tau(M_{\mathbb{A}}(\xi, \eta, t), M_{\mathbb{A}}(\eta, \eta, s - t)) &\preccurlyeq M_{\mathbb{A}}(\xi, \eta, s) \\ &\prec M_{\mathbb{A}}(\xi, \eta, t), \\ \tau(M_{\mathbb{A}}(\xi, \eta, t), 1_{\mathbb{A}}) &\prec M_{\mathbb{A}}(\xi, \eta, t), \\ M_{\mathbb{A}}(\xi, \eta, t) &\prec M_{\mathbb{A}}(\xi, \eta, t), \end{aligned}$$

which is a contradiction. So, $M_{\mathbb{A}}(\xi, \eta, t) \preccurlyeq M_{\mathbb{A}}(\xi, \eta, s)$ when $t < s$.

Thus, $M_{\mathbb{A}}(\xi, \eta, t)$ is non-decreasing with respect to t for all $\xi, \eta \in X$ and hence the lemma easily follows.

Following the definition of Cauchy sequence in fuzzy metric space by George and Veeramani [17], the Cauchy sequence in C^*-algebra valued fuzzy metric space can be defined in a similar way.

Definition 6. Let $(X, M_{\mathbb{A}}, \tau)$ be a C^*-algebra valued fuzzy metric space. A sequence $\{\xi_n\}$ in X is said to be a Cauchy sequence if for all $\epsilon_{\mathbb{A}} \in (0_{\mathbb{A}}, 1_{\mathbb{A}})$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m > n > n_0$, $M_{\mathbb{A}}(\xi_m, \xi_n, t) \succcurlyeq 1_{\mathbb{A}} - \epsilon_{\mathbb{A}}$ or equivalently, $\lim_{m,n\to\infty} M_{\mathbb{A}}(\xi_m, \xi_n, t) = 1_{\mathbb{A}}$.

The sequence $\{\xi_n\}$ is said to be convergent to ξ, if $\lim_{n\to\infty} M_{\mathbb{A}}(\xi_n, \xi, t) = 1_{\mathbb{A}}$. If every Cauchy sequence in $(X, M_{\mathbb{A}}, \tau)$ is convergent, then $(X, M_{\mathbb{A}}, \tau)$ is called a complete C^*-algebra valued fuzzy metric space.

2 Main Results

In this section, we derive some fixed point results considering a subadditive altering distance function with respect to a t-norm.

Theorem 1. Let $(X, M_{\mathbf{A}}, \tau)$ be a complete C^*-algebra valued fuzzy metric space. Let ϕ be a subadditive altering distance function with respect to the t-norm τ and let T be a self mapping on X such that

$$\begin{aligned}\phi(M_{\mathbb{A}}(T\xi, T\eta, t)) \preccurlyeq\ & a_1^*\phi(M_{\mathbb{A}}(\xi, T\xi, t))a_1 + a_2^*\phi(M_{\mathbb{A}}(\eta, T\eta, t))a_2 + a_3^*\phi(M_{\mathbb{A}}(\xi, T\eta, 2t))a_3 \\ & + a_4^*\phi(M_{\mathbb{A}}(\eta, T\xi, t))a_4 + a_5^*\phi(M_{\mathbb{A}}(\xi, \eta, t))a_5 \\ & + a_6^*(\phi(M_{\mathbb{A}}(\xi, T\xi, t)) + \phi(M_{\mathbb{A}}(\eta, T\eta, t)))\frac{(1_{\mathbb{A}} + \phi(M_{\mathbb{A}}(\xi, T\xi, t)))}{(1_{\mathbb{A}} + \phi(M_{\mathbb{A}}(\xi, \eta, t)))}a_6, \end{aligned} \tag{1}$$

where $a_i \in \mathbb{A}'$ for $i = 1, ..., 6$ with $\sum_{i=1}^{6} ||a_i||^2 + ||a_3||^2 + ||a_6||^2 < 1$. Then T has a unique fixed point in X.

Proof. For $\xi_0 \in X$, we consider the Picard sequence $\xi_{n+1} = T\xi_n$, $n \in \mathbb{N} \cup \{0\}$. Now,

$$\begin{aligned} & \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) \\ & \preccurlyeq a_1^*\phi(M_{\mathbb{A}}(\xi_{j-1}, T\xi_{j-1}, t))a_1 + a_2^*\phi(M_{\mathbb{A}}(\xi_j, T\xi_j, t))a_2 + a_3^*\phi(M_{\mathbb{A}}(\xi_{j-1}, T\xi_j, 2t))a_3 \\ & + a_4^*\phi(M_{\mathbb{A}}(\xi_j, T\xi_{j-1}, t))a_4 + a_5^*\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t))a_5 \\ & + a_6^*(\phi(M_{\mathbb{A}}(\xi_{j-1}, T\xi_{j-1}, t)) + \phi(M_{\mathbb{A}}(\xi_j, T\xi_j, t)))\frac{(1_{\mathbb{A}} + \phi(M_{\mathbb{A}}(\xi_{j-1}, T\xi_{j-1}, t)))}{(1_{\mathbb{A}} + \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)))}a_6 \\ & \preccurlyeq ||a_1||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t))) + ||a_2||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) + ||a_3||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_{j+1}, 2t)) \\ & + ||a_5||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) + ||a_6||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) + ||a_6||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)). \end{aligned} \tag{2}$$

Using the property of t-norm and the altering distance function ϕ, we have

$$\begin{aligned} M_{\mathbb{A}}(\xi_{j-1}, \xi_{j+1}, 2t) & \succcurlyeq \tau(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t), M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)), \\ \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_{j+1}, 2t)) & \preccurlyeq \phi(\tau(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t), M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t))) \\ & \preccurlyeq \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) + \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)). \end{aligned}$$

So, from (2), we get

$$\begin{aligned}\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) \preccurlyeq\ & ||a_1||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) + ||a_2||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) + ||a_3||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) \\ & + ||a_3||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) + ||a_5||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) + ||a_6||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) \\ & + ||a_6||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)). \end{aligned}$$

Then the above equation becomes

$$\begin{aligned}(1-\|a_2\|^2-\|a_3\|^2-\|a_6\|^2)1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t)) &\preccurlyeq (\|a_1\|^2+\|a_3\|^2+\|a_5\|^2+\|a_6\|^2)1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_{j-1},\xi_j,t)),\\ \phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t)) &\preccurlyeq \frac{(\|a_1\|^2+\|a_3\|^2+\|a_5\|^2+\|a_6\|^2)}{(1-(\|a_2\|^2+\|a_3\|^2+\|a_6\|^2))}1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_{j-1},\xi_j,t)),\\ \phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t)) &\preccurlyeq \gamma\phi(M_{\mathbb{A}}(\xi_{j-1},\xi_j,t))\\ &\preccurlyeq \gamma^j\phi(M_{\mathbb{A}}(\xi_0,\xi_1,t)), \qquad (3)\end{aligned}$$

where $\gamma=\frac{(\|a_1\|^2+\|a_3\|^2+\|a_5\|^2+\|a_6\|^2)}{(1-(\|a_2\|^2+\|a_3\|^2+\|a_6\|^2))}1_{\mathbb{A}}$. Taking norm on both sides of the equation (3), we get

$$\|\phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t))\| \preccurlyeq \|\gamma\|^j\|\phi(M_{\mathbb{A}}(\xi_0,\xi_1,t))\|.$$

Taking the limit as $j\to\infty$, since $\|\gamma\|<1$, from the above equation, we get

$$\begin{aligned}\lim_{j\to\infty}\phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t)) &= 0_{\mathbb{A}},\\ \lim_{j\to\infty}M_{\mathbb{A}}(\xi_j,\xi_{j+1},t) &= 1_{\mathbb{A}}. \qquad (4)\end{aligned}$$

Next we show that $\{\xi_j\}$ is a Cauchy sequence. If not, then there exists $0_{\mathbb{A}}\succ\epsilon_{\mathbb{A}}\succ 1_{\mathbb{A}}$, for which we can find two subsequence $\{\xi_{r(j)}\}$ and $\{\xi_{s(j)}\}$ of $\{\xi_j\}$ with $r(j)>s(j)>j$, $j\in\mathbb{N}\cup\{0\}$ such that

$$M_{\mathbb{A}}(\xi_{r(j)},\xi_{s(j)},t)\preccurlyeq 1_{\mathbb{A}}-\epsilon_{\mathbb{A}}. \qquad (5)$$

Now, without loss of generality, we can choose $r(j)$ as the smallest positive integer satisfying $r(j)>s(j)$ in (5). Then,

$$M_{\mathbb{A}}(\xi_{r(j)-1},\xi_{s(j)},t)\succ 1_{\mathbb{A}}-\epsilon_A. \qquad (6)$$

Now,

$$\begin{aligned}M_{\mathbb{A}}(\xi_{r(j)-1},\xi_{s(j)-1},t) &\succcurlyeq \tau(M_{\mathbb{A}}(\xi_{r(j)-1},\xi_{s(j)},\frac{t}{2}),M_{\mathbb{A}}(\xi_{s(j)},\xi_{s(j)-1},\frac{t}{2})),\ j\in\mathbb{N}\\ &\succcurlyeq \tau(1_{\mathbb{A}}-\epsilon_{\mathbb{A}},M_{\mathbb{A}}(\xi_{s(j)},\xi_{s(j)-1},\frac{t}{2}))\ (by\ (6)),\\ \lim_{j\to\infty}M_{\mathbb{A}}(\xi_{r(j)-1},\xi_{s(j)-1},t) &\succcurlyeq \tau(1_{\mathbb{A}}-\epsilon_A,1_{\mathbb{A}})=1_{\mathbb{A}}-\epsilon_{\mathbb{A}}\ (by\ (4)),\\ \lim_{j\to\infty}M_{\mathbb{A}}(\xi_{r(j)-1},\xi_{s(j)-1},t) &\succcurlyeq 1_{\mathbb{A}}-\epsilon_{\mathbb{A}}. \qquad (7)\end{aligned}$$

Again, from (5),

$$\begin{aligned}1_{\mathbb{A}}-\epsilon_{\mathbb{A}} &\succcurlyeq M_{\mathbb{A}}(\xi_{r(j)},\xi_{s(j)},4t)\\ &\succcurlyeq \tau(M_{\mathbb{A}}(\xi_{r(j)},\xi_{r(j)-1},2t),\tau(M_{\mathbb{A}}(\xi_{r(j)-1},\xi_{s(j)-1},t),M_{\mathbb{A}}(\xi_{s(j)},\xi_{s(j)-1},t)))\\ &\succcurlyeq \tau(1_{\mathbb{A}},\tau(\lim_{j\to\infty}M_{\mathbb{A}}(\xi_{r(j)-1},\xi_{s(j)-1},t),1_{\mathbb{A}}))\ (by\ (4)),\\ 1_{\mathbb{A}}-\epsilon_{\mathbb{A}} &\succcurlyeq \lim_{j\to\infty}M_{\mathbb{A}}(\xi_{r(j)-1},\xi_{s(j)-1},t). \qquad (8)\end{aligned}$$

Hence, from (7) and (8), we get

$$\lim_{j\to\infty}M_{\mathbb{A}}(\xi_{r(j)-1},\xi_{s(j)-1},t)=1_{\mathbb{A}}-\epsilon_{\mathbb{A}}.$$

By (5),

$$\begin{aligned}
\phi(1_{\mathbb{A}}-\epsilon_{\mathbb{A}}) &\preccurlyeq \phi(M_{\mathbb{A}}(\xi_{r(j)},\xi_{s(j)},t)) \\
&\preccurlyeq a_1^*\phi(M_{\mathbb{A}}(\xi_{r(j)-1},T\xi_{r(j)-1},t))a_1 + a_2^*\phi(M_{\mathbb{A}}(\xi_{s(j)-1},T\xi_{s(j)-1},t))a_2 \\
&+ a_3^*\phi(M_{\mathbb{A}}(\xi_{r(j)-1},T\xi_{s(j)-1},2t))a_3 + a_4^*\phi(M_{\mathbb{A}}(T\xi_{s(j)-1},T\xi_{r(j)-1},t))a_4 \\
&+ a_5^*\phi(M_{\mathbb{A}}(\xi_{r(j)-1},\xi_{s(j)-1},t))a_5 + a_6^*(\phi(M_{\mathbb{A}}(\xi_{r(j)-1},T\xi_{r(j)-1},t)) \\
&+ \phi(M_{\mathbb{A}}(\xi_{s(j)-1},T\xi_{s(j)-1},t)))\frac{(1_{\mathbb{A}}+\phi(M_{\mathbb{A}}(\xi_{r(j)-1},T\xi_{r(j)-1},t)))}{(1_{\mathbb{A}}+\phi(M_{\mathbb{A}}(\xi_{r(j)-1},\xi_{s(j)-1},t)))}a_6.
\end{aligned}$$

By taking the limit as $j\to\infty$ the above expression becomes

$$\begin{aligned}
\phi(1_{\mathbb{A}}-\epsilon_{\mathbb{A}}) &\preccurlyeq ||a_1||^2 1_{\mathbb{A}}\phi(1_{\mathbb{A}}) + ||a_2||^2 1_{\mathbb{A}}\phi(1_{\mathbb{A}}) + ||a_3||^2 1_{\mathbb{A}}\phi(1_{\mathbb{A}}-\epsilon_{\mathbb{A}}) + ||a_4||^2 1_{\mathbb{A}}\phi(1_{\mathbb{A}}-\epsilon_{\mathbb{A}}) \\
&+ ||a_5||^2 1_{\mathbb{A}}\phi(1_{\mathbb{A}}-\epsilon_{\mathbb{A}}) + ||a_6||^2(\phi(1_{\mathbb{A}})+\phi(1_{\mathbb{A}}))\frac{(1_{\mathbb{A}}+\phi(1_{\mathbb{A}}))}{(1_{\mathbb{A}}+\phi(1_{\mathbb{A}}-\epsilon_{\mathbb{A}}))}, \\
\phi(1_{\mathbb{A}}-\epsilon_{\mathbb{A}}) &\preccurlyeq ||a_3||^2\phi(1_{\mathbb{A}}-\epsilon_{\mathbb{A}}) + ||a_4||^2\phi(1_{\mathbb{A}}-\epsilon_{\mathbb{A}}) + ||a_5||^2\phi(1_{\mathbb{A}}-\epsilon_{\mathbb{A}}), \\
\phi(1_{\mathbb{A}}-\epsilon_{\mathbb{A}}) &\preccurlyeq 0_{\mathbb{A}} \implies 1_{\mathbb{A}}-\epsilon_{\mathbb{A}} = 1_{\mathbb{A}} \implies \epsilon_{\mathbb{A}} = 0_{\mathbb{A}},
\end{aligned}$$

which is a contradiction. Therefore, $\{\xi_j\}$ is a Cauchy sequence. Then there exists a point z in X such that $\xi_n\to z$. Now,

$$\begin{aligned}
\phi(M_{\mathbb{A}}(T\xi_n,Tz,t)) &\preccurlyeq a_1^*\phi(M_{\mathbb{A}}(\xi_n,T\xi_n,t))a_1 + a_2^*\phi(M_{\mathbb{A}}(z,Tz,t))a_2 \\
&+ a_3^*\phi(M_{\mathbb{A}}(\xi_n,Tz,2t))a_3 + a_4^*\phi(M_{\mathbb{A}}(z,T\xi_n,t))a_4 + a_5^*\phi(M_{\mathbb{A}}(\xi_n,z,t))a_5 \\
&+ a_6^*(\phi(M_{\mathbb{A}}(\xi_n,T\xi_n,t)) + \phi(M_{\mathbb{A}}(z,Tz,t)))\frac{(1_{\mathbb{A}}+\phi(M_{\mathbb{A}}(\xi_n,T\xi_n,t)))}{(1_{\mathbb{A}}+\phi(M_{\mathbb{A}}(\xi_n,z,t)))}a_6.
\end{aligned}$$

Taking the limit as $n\to\infty$ and by Lemma 3, from the above equation, we get

$$\begin{aligned}
&\phi(M_{\mathbb{A}}(z,Tz,t)) \preccurlyeq ||a_2||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(z,Tz,t)) + ||a_3||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(z,Tz,t)) + ||a_6||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(z,Tz,t)), \\
&(1-||a_2||^2-||a_3||^2-||a_6||^2)1_{\mathbb{A}}\phi(M_{\mathbb{A}}(z,Tz,t)) \preccurlyeq 0_{\mathbb{A}}.
\end{aligned}$$

Since the left hand side of the above expression is positive and $a_2,a_3,a_6\in\mathbb{A}'$, using Lemma 1, we get

$$\phi(M_{\mathbb{A}}(z,Tz,t)) = 0_{\mathbb{A}} \implies M_{\mathbb{A}}(z,Tz,t) = 1_{\mathbb{A}} \implies z = Tz.$$

Uniqueness of the fixed point can be proved easily by (1).

Remark 1. The above theorem generalizes the results given by [13] and [5] if we consider C^*-algebra valued fuzzy metric space in place of b-metric space and fuzzy metric space respectively.

We present the following example to demonstrate the above theorem.

Example 2. Let $X=\mathbb{A}=[0,1]$ and $d(\xi,\eta)=|\xi-\eta|$ for all $\xi,\eta\in X$. Let $M_{\mathbb{A}}$ be a fuzzy set from $X^2\times(0,\infty)$ to $[0,1]$ such that $M_{\mathbb{A}}(\xi,\eta,t)=\frac{1}{1+d(\xi,\eta)}$. Then $(X,M_{\mathbb{A}},\tau)$ is a complete C^*-algebra valued fuzzy metric space with respect to the t-norm, $\tau(a,b)=\min\{a,b\}$, $a,b\in[0,1]$. Let $T:X\to X$ be defined by $T(\xi)=\frac{\xi}{7}$ for all $\xi\in X$ and $\phi(\lambda)=1-\lambda$, $\lambda\in[0,1]$. Let $a_i=\frac{1}{3}$ for $i=1,...,6$.
Now,

$$\phi(M_{\mathbb{A}}(T\xi,T\eta,t)) = \phi\left(\frac{1}{1+|T\xi-T\eta|}\right) = 1-\frac{1}{1+|\frac{\xi}{7}-\frac{\eta}{7}|}. \tag{9}$$

Again,

$$
\begin{aligned}
& a_1^*\phi(M_{\mathbb{A}}(\xi,T\xi,t))a_1 + a_2^*\phi(M_{\mathbb{A}}(\eta,T\eta,t))a_2 + a_3^*\phi(M_{\mathbb{A}}(\xi,T\eta,2t))a_3 + a_4^*\phi(M_{\mathbb{A}}(\eta,T\xi,t))a_4 \\
& + a_5^*\phi(M_{\mathbb{A}}(\xi,\eta,t))a_5 + a_6^*(\phi(M_{\mathbb{A}}(\xi,T\xi,t)) + \phi(M_{\mathbb{A}}(\eta,T\eta,t)))\frac{(1_{\mathbb{A}} + \phi(M_{\mathbb{A}}(\xi,T\xi,t)))}{(1_{\mathbb{A}} + \phi(M_{\mathbb{A}}(\xi,\eta,t)))}a_6 \\
& = \frac{1}{9}\left[5 - \left\{\frac{1}{1+|\xi-\frac{\xi}{7}|} + \frac{1}{1+|\eta-\frac{\eta}{7}|} + \frac{1}{1+|\xi-\frac{\eta}{7}|} + \frac{1}{1+|\eta-\frac{\xi}{7}|} + \frac{1}{1+|\xi-\eta|}\right\}\right] \\
& + \frac{1}{9}\left(2 - \frac{1}{1+|\xi-T\xi|} - \frac{1}{1+|\eta-T\eta|}\right)\frac{\left(2-\frac{1}{1+|\xi-T\xi|}\right)}{\left(2-\frac{1}{1+|\xi-\eta|}\right)}.
\end{aligned} \tag{10}
$$

We represent the equations (9) and (10) in the following figure.

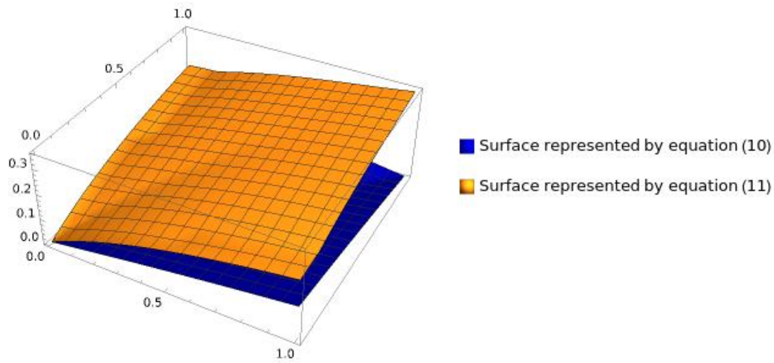

Figure 1. Demonstration of the condition of Theorem 1 by mapping T

In Figure 1, the yellow surface represents the equation (10) and the blue surface represents the equation (9), where the values of ξ, η are between 0 and 1. Clearly, for all values of ξ, η, the value of (10) is greater than the value of (9). Hence, the condition of Theorem 1 is satisfied. Clearly, 0 is the fixed point of T here.

Example 3. Let $X = \{(1,1),(2,1),(2,7)\} \subseteq \mathbb{R}^2$ and $\mathbb{A}$, $M_{\mathbb{A}}$, τ and ϕ be as in Example 2. Let $T : X \to X$ be defined by $T(1,1) = T(2,1) = (1,1)$ and $T(2,7) = (2,1)$ and $a_1 = a_2 = \sqrt{\frac{15}{100}}$, $a_4 = a_5 = \sqrt{\frac{22}{100}}$ and $a_3 = a_6 = 0$. Then for $\xi = (2,1)$ and $\eta = (2,7)$,

$$\phi(M_{\mathbb{A}}(T\xi,T\eta,t)) = 1 - \frac{1}{1+d((1,1),(2,1))} = 1 - \frac{1}{2} = \frac{1}{2}$$

and

$$
\begin{aligned}
& a_1^*\phi(M_{\mathbb{A}}(\xi,T\xi,t))a_1 + a_2^*\phi(M_{\mathbb{A}}(\eta,T\eta,t))a_2 + a_3^*\phi(M_{\mathbb{A}}(\xi,T\eta,2t))a_3 + a_4^*\phi(M_{\mathbb{A}}(\eta,T\xi,t))a_4 \\
& + a_5^*\phi(M_{\mathbb{A}}(\xi,\eta,t))a_5 + a_6^*(\phi(M_{\mathbb{A}}(\xi,T\xi,t)) + \phi(M_{\mathbb{A}}(\eta,T\eta,t)))\frac{(1_{\mathbb{A}} + \phi(M_{\mathbb{A}}(\xi,T\xi,t)))}{(1_{\mathbb{A}} + \phi(M_{\mathbb{A}}(\xi,\eta,t)))}a_6 \\
& = 0.15(1 - \frac{1}{2} + 1 - \frac{1}{7}) + 0.22(1 - \frac{1}{1+\sqrt{37}} + 1 - \frac{1}{6}) = 0.5734 > \frac{1}{2}.
\end{aligned}
$$

Therefore, the condition of Theorem 1 is satisfied. Clearly, $(1,1)$ is the fixed point of T in X.

In the following result, we use minimum and maximum conditions to prove the existence of fixed point. We note that for $a_i \in [0_{\mathbb{A}}, 1_{\mathbb{A}}]$, $i = 1, 2, ..., n$, $n \in \mathbb{N}$, $MIN(a_i)$ denotes an element a_k, $1 \leq k \leq n$ such that $a_k \preccurlyeq a_i$ for each i, $1 \leq i \leq n$. Similarly, $MAX(a_i)$ denotes an element a_k, $1 \leq k \leq n$ such that $a_k \succcurlyeq a_i$ for each i, $1 \leq i \leq n$.

Theorem 2. Let $(X, M_{\mathbf{A}}, \tau)$ be a complete C^*-algebra valued fuzzy metric space. Let ϕ be a subadditive altering distance function with respect to the t-norm τ and let T be a self mapping on X such that

$$\begin{aligned}\phi(M_{\mathbb{A}}(T\xi, T\eta, t)) \preccurlyeq\ & a_1^* MIN\{\phi(M_{\mathbb{A}}(\xi, \eta, t)), \phi(M_{\mathbb{A}}(\xi, T\xi, t)), \phi(M_{\mathbb{A}}(\xi, T\eta, 2t)), \phi(M_{\mathbb{A}}(\eta, T\xi, t)),\\ & \phi(M_{\mathbb{A}}(\eta, T\eta, t))\} a_1 + a_2^* MAX\{\phi(M_{\mathbb{A}}(\xi, \eta, t)), \phi(M_{\mathbb{A}}(\xi, T\xi, t)),\\ & \phi(M_{\mathbb{A}}(\xi, T\eta, 2t)), \phi(M_{\mathbb{A}}(\eta, T\xi, t)), \phi(M_{\mathbb{A}}(\eta, T\eta, t))\} a_2, \end{aligned} \tag{11}$$

where $a_1, a_2 \in \mathbb{A}'$ with $||a_1||^2 + 2||a_2||^2 < 1$. Then T has a unique fixed point in X.

Proof. For $\xi_0 \in X$, let $\xi_{n+1} = T\xi_n$, $n \in \mathbb{N} \cup \{0\}$. Now,

$$\begin{aligned}\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) \preccurlyeq\ & a_1^* MIN\{\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)), \phi(M_{\mathbb{A}}(\xi_{j-1}, T\xi_{j-1}, t)), \phi(M_{\mathbb{A}}(\xi_{j-1}, T\xi_j, 2t)),\\ & \phi(M_{\mathbb{A}}(\xi_j, T\xi_{j-1}, t)), \phi(M_{\mathbb{A}}(\xi_j, T\xi_j, t))\} a_1 + a_2^* MAX\{\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)),\\ & \phi(M_{\mathbb{A}}(\xi_{j-1}, T\xi_{j-1}, t)), \phi(M_{\mathbb{A}}(\xi_{j-1}, T\xi_j, 2t)), \phi(M_{\mathbb{A}}(\xi_j, T\xi_{j-1}, t)), \phi(M_{\mathbb{A}}(\xi_j, T\xi_j, t))\} a_2\\ \preccurlyeq\ & ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)), \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_{j+1}, 2t)), \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t))\}\\ & + ||a_2||^2 1_{\mathbb{A}} MAX\{\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)), \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_{j+1}, 2t)), \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t))\}\\ \preccurlyeq\ & ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)), \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)), \phi(\tau(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t), M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)))\}\\ & + ||a_2||^2 1_{\mathbb{A}} MAX\{\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)), \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)), \phi(\tau(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t), M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)))\}. \end{aligned}$$

Again, $\phi(a) \preccurlyeq \phi(\tau(a, b))$ and $\phi(b) \preccurlyeq \phi(\tau(a, b))$. So, $MIN\{\phi(a), \phi(b), \phi(\tau(a, b))\} = MIN\{\phi(a), \phi(b)\}$ for all $a, b \in [0_{\mathbb{A}}, 1_{\mathbb{A}}]$. Hence,

$$\begin{aligned}\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) \preccurlyeq\ & ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)), \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t))\}\\ & + ||a_2||^2 1_{\mathbb{A}} \phi(\tau(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t), M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)))\\ \preccurlyeq\ & ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)), \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t))\}\\ & + ||a_2||^2 1_{\mathbb{A}} (\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) + \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t))). \end{aligned} \tag{12}$$

If $MIN\{\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)), \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t))\} = \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t))$, then

$$\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) \preccurlyeq ||a_1||^2 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) + ||a_2||^2 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) + ||a_2||^2 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)),$$

$$\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) \preccurlyeq \frac{(||a_1||^2 + ||a_2||^2)}{(1 - ||a_2||^2)} 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) = \gamma_1 \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)).$$

Again, if $MIN\{\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)), \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t))\} = \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t))$, then from (12), we get

$$\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) \preccurlyeq ||a_1||^2 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) + ||a_2||^2 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) + ||a_2||^2 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)),$$

$$\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) \preccurlyeq \Big(\frac{||a_2||^2}{1 - ||a_1||^2 - ||a_2||^2}\Big) 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) = \gamma_2 \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)),$$

where $\gamma_1 = \big(\frac{||a_1||^2 + ||a_2||^2}{1 - ||a_2||^2}\big) 1_{\mathbb{A}}$ and $\gamma_2 = \big(\frac{||a_2||^2}{1 - ||a_1||^2 - ||a_2||^2}\big) 1_{\mathbb{A}}$ are positive elements in $\mathbb{A}$ and strictly less than $1_{\mathbb{A}}$. Proceeding as in Theorem 1, we can easily show that the sequence $\{\xi_n\}$ is a Cauchy sequence.

Let $\xi_n \to z$. Now,

$$\begin{aligned}\phi(M_{\mathbb{A}}(T\xi_n,Tz,t)) \preccurlyeq\ & a_1^* MIN\{\phi(M_{\mathbb{A}}(\xi_n,z,t)),\phi(M_{\mathbb{A}}(\xi_n,T\xi_n,t)),\phi(M_{\mathbb{A}}(\xi_n,Tz,2t)),\phi(M_{\mathbb{A}}(z,T\xi_n,t)),\\ & \phi(M_{\mathbb{A}}(z,Tz,t))\}a_1 + a_2^* MAX\{\phi(M_{\mathbb{A}}(\xi_n,z,t)),\phi(M_{\mathbb{A}}(\xi_n,T\xi_n,t)),\phi(M_{\mathbb{A}}(\xi_n,Tz,2t)),\\ & \phi(M_{\mathbb{A}}(z,T\xi_n,t)),\phi(M_{\mathbb{A}}(z,Tz,t))\}a_2\\ & \preccurlyeq a_1^* MIN\{\phi(M_{\mathbb{A}}(\xi_n,z,t)),\phi(M_{\mathbb{A}}(\xi_n,\xi_{n+1},t)),\phi(M_{\mathbb{A}}(\xi_n,Tz,2t)),\phi(M_{\mathbb{A}}(z,\xi_{n+1},t)),\\ & \phi(M_{\mathbb{A}}(z,Tz,t))\}a_1 + a_2^* MAX\{\phi(M_{\mathbb{A}}(\xi_n,z,t)),\phi(M_{\mathbb{A}}(\xi_n,\xi_{n+1},t)),\phi(M_{\mathbb{A}}(\xi_n,Tz,2t)),\\ & \phi(M_{\mathbb{A}}(z,\xi_{n+1},t)),\phi(M_{\mathbb{A}}(z,Tz,t))\}a_2.\end{aligned}$$

By taking the limit as $n \to \infty$, the above equation becomes

$$\begin{aligned}\phi(M_{\mathbb{A}}(z,Tz,t)) &\preccurlyeq ||a_1||^2\phi(M_{\mathbb{A}}(z,Tz,t)) + ||a_2||^2\phi(M_{\mathbb{A}}(z,Tz,t)),\\ (1-||a_1||^2-||a_2||^2)1_{\mathbb{A}}\phi(M_{\mathbb{A}}(z,Tz,t)) &\preccurlyeq 0_{\mathbb{A}},\end{aligned}$$

which gives $z = Tz$. Clearly, by using (11), the fixed point is unique.

Remark 2. The above theorem can be taken as a generalization of Theorem 2.11 of [5] and Theorem 2.1 of [24] in the setting of C^*-algebra valued fuzzy metric space.

It may be noted here that the mapping we have considered is not necessarily continuous, which can be seen from the following example.

Example 4. We consider $(X, M_{\mathbb{A}}, \tau)$ and ϕ as in Example 3. Let $T : X \to X$ be defined by

$$T(\xi) = \begin{cases} \frac{1}{6}\ if\ \xi \in [0,\frac{1}{2}) \\ \frac{1}{12}\ if \xi \in [\frac{1}{2},1]. \end{cases}$$

Let $a_1 = 0$ and $a_2 = \frac{7}{10}$. Now, three cases will arise:

Case 1. If $\xi,\eta \in [0,\frac{1}{2})$, then $\phi(M_{\mathbb{A}}(T\xi,T\eta,t)) = 1 - \frac{1}{1+d(\frac{1}{6},\frac{1}{6})} = 0$. So, condition (11) is trivially true.

Case 2. If $\xi,\eta \in [\frac{1}{2},1]$, this is similar to Case 1.

Case 3. If $\xi \in [0,\frac{1}{2})$ and $\eta \in [\frac{1}{2},1]$, then

$$\phi(M_{\mathbb{A}}(T\xi,T\eta,t)) = 1 - \frac{1}{1+|\frac{1}{6}-\frac{1}{12}|} = \frac{1}{13} \tag{13}$$

and

$$\begin{aligned}& a_1^* MIN\{\phi(M_{\mathbb{A}}(\xi,\eta,t)),\phi(M_{\mathbb{A}}(\xi,T\xi,t)),\phi(M_{\mathbb{A}}(\xi,T\eta,2t)),\phi(M_{\mathbb{A}}(\eta,T\xi,t)),\phi(M_{\mathbb{A}}(\eta,T\eta,t))\}a_1\\ & + a_2^* MAX\{\phi(M_{\mathbb{A}}(\xi,\eta,t)),\phi(M_{\mathbb{A}}(\xi,T\xi,t)),\phi(M_{\mathbb{A}}(\xi,T\eta,2t)),\phi(M_{\mathbb{A}}(\eta,T\xi,t)),\phi(M_{\mathbb{A}}(\eta,T\eta,t))\}a_2\\ & = \frac{49}{100}MAX\left\{1-\frac{1}{1+|\xi-\frac{1}{6}|},1-\frac{1}{1+|\eta-\frac{1}{12}|},1-\frac{1}{1+|\xi-\frac{1}{12}|},1-\frac{1}{1+|\eta-\frac{1}{6}|},1-\frac{1}{1+|\xi-\eta|}\right\}.\end{aligned} \tag{14}$$

Figure 2 describes equations (13) and (14). Here, the yellow surface represents the equation (14) and the blue surface represents the equation (13). From Figure 2, it is clear that for all $\xi \in [0,\frac{1}{2})$ and $\eta \in [\frac{1}{2},1]$, the value of (14) is greater than the value of (13). Thus, the condition of Theorem 2 is satisfied. Clearly, $\frac{1}{6}$ is a fixed point of T.

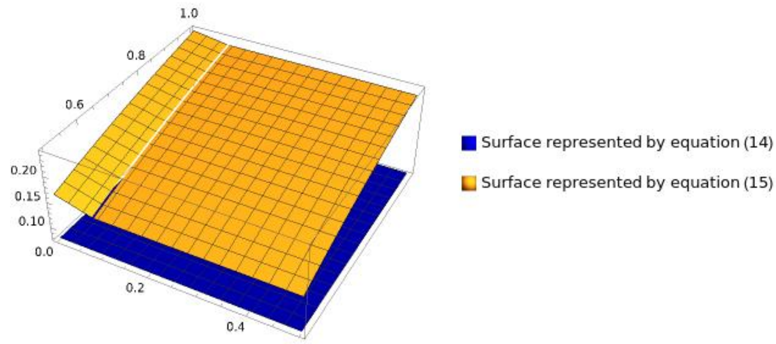

Figure 2. Demonstration of the condition of Theorem 2 by mapping T

Next we derive the following common fixed point theorem.

Theorem 3. Let $(X, M_{\mathbf{A}}, \tau)$ be a complete C^*-algebra valued fuzzy metric space. Let ϕ be a subadditive altering distance function with respect to the t-norm τ and let $T, S : X \to X$ be such that

$$\begin{aligned}\phi(M_{\mathbb{A}}(T\xi, S\eta, t)) \preccurlyeq\ & a_1^* MIN\{\phi(M_{\mathbb{A}}(\xi, \eta, t)), \phi(M_{\mathbb{A}}(\xi, T\xi, t)), \phi(M_{\mathbb{A}}(\xi, S\eta, 2t)), \phi(M_{\mathbb{A}}(\eta, T\xi, 2t)),\\ & \phi(M_{\mathbb{A}}(\eta, S\eta, t))\}a_1 + a_2^* MAX\{\phi(M_{\mathbb{A}}(\xi, \eta, t)), \phi(M_{\mathbb{A}}(\xi, T\xi, t)), \phi(M_{\mathbb{A}}(\xi, S\eta, 2t)),\\ & \phi(M_{\mathbb{A}}(\eta, T\xi, 2t)), \phi(M_{\mathbb{A}}(\eta, S\eta, t))\}a_2,\end{aligned}$$

where $a_1, a_2 \in \mathbb{A}'$ with $||a_1||^2 + 2||a_2||^2 < 1$. Then T and S have a unique common fixed point.

Proof. For $\xi_0 \in X$, let $\xi_{2i+1} = T\xi_{2i}$ and $\xi_{2i+2} = S\xi_{2i+1}$, $i \in \mathbb{N} \cup \{0\}$. Now,

$$\begin{aligned}\phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t)) &= \phi(M_{\mathbb{A}}(T\xi_{2i}, S\xi_{2i+1}, t))\\ &\preccurlyeq a_1^* MIN\{\phi(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)), \phi(M_{\mathbb{A}}(\xi_{2i}, T\xi_{2i}, t)), \phi(M_{\mathbb{A}}(\xi_{2i}, S\xi_{2i+1}, 2t)),\\ &\phi(M_{\mathbb{A}}(\xi_{2i+1}, T\xi_{2i}, 2t)), \phi(M_{\mathbb{A}}(\xi_{2i+1}, S\xi_{2i+1}, t))\}a_1\\ &+ a_2^* MAX\{\phi(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)), \phi(M_{\mathbb{A}}(\xi_{2i}, T\xi_{2i}, t)), \phi(M_{\mathbb{A}}(\xi_{2i}, S\xi_{2i+1}, 2t)),\\ &\phi(M_{\mathbb{A}}(\xi_{2i+1}, T\xi_{2i}, 2t)), \phi(M_{\mathbb{A}}(\xi_{2i+1}, S\xi_{2i+1}, t))\}a_2\\ &= ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)), \phi(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+2}, 2t)), \phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t))\}\\ &+ ||a_2||^2 1_{\mathbb{A}} MAX\{\phi(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)), \phi(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+2}, 2t)), \phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t))\}\\ &\preccurlyeq ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)), \phi(\tau(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)), \phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t))),\\ &\phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t))\}\\ &+ ||a_2||^2 1_{\mathbb{A}} MAX\{\phi(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)), \phi(\tau(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)), \phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t))),\\ &\phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t))\}.\end{aligned}$$

Since $MIN\{\phi(a), \phi(b), \phi(\tau(a, b))\} = MIN\{\phi(a), \phi(b)\}$ for all $a, b \in [0_{\mathbb{A}}, 1_{\mathbb{A}}]$, we have

$$\begin{aligned}\phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t)) &\preccurlyeq ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)), \phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t))\}\\ &+ ||a_2||^2 1_{\mathbb{A}} \phi(\tau(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)), \phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t)))\\ &\preccurlyeq ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)), \phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t))\}\\ &+ ||a_2||^2 1_{\mathbb{A}} (\phi(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)) + \phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t))). \qquad (15)\end{aligned}$$

Similarly,

$$\begin{aligned}\phi(M_{\mathbb{A}}(\xi_{2i+2},\xi_{2i+3},t)) &= \phi(M_{\mathbb{A}}(S\xi_{2i+1},T\xi_{2i+2},t))\\
&\preccurlyeq a_1^* MIN\{\phi(M_{\mathbb{A}}(\xi_{2i+1},\xi_{2i+2},t)),\phi(M_{\mathbb{A}}(\xi_{2i+1},S\xi_{2i+1},t)),\phi(M_{\mathbb{A}}(\xi_{2i+1},T\xi_{2i+2},2t)),\\
&\phi(M_{\mathbb{A}}(\xi_{2i+2},S\xi_{2i+1},2t)),\phi(M_{\mathbb{A}}(\xi_{2i+2},T\xi_{2i+2},t))\}a_1\\
&+ a_2^* MAX\{\phi(M_{\mathbb{A}}(\xi_{2i+1},\xi_{2i+2},t)),\phi(M_{\mathbb{A}}(\xi_{2i+1},S\xi_{2i+1},t)),\phi(M_{\mathbb{A}}(\xi_{2i+1},T\xi_{2i+2},2t)),\\
&\phi(M_{\mathbb{A}}(\xi_{2i+2},S\xi_{2i+1},2t)),\phi(M_{\mathbb{A}}(\xi_{2i+2},T\xi_{2i+2},t))\}a_2\\
&\preccurlyeq ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{2i+1},\xi_{2i+2},t)),\phi(\tau(M_{\mathbb{A}}(\xi_{2i+1},\xi_{2i+2},t)),\phi(M_{\mathbb{A}}(\xi_{2i+2},\xi_{2i+3},t))),\\
&\phi(M_{\mathbb{A}}(\xi_{2i+2},\xi_{2i+3},t))\} + ||a_2||^2 1_{\mathbb{A}} MAX\{\phi(M_{\mathbb{A}}(\xi_{2i+1},\xi_{2i+2},t)),\phi(\tau(M_{\mathbb{A}}(\xi_{2i+1},\xi_{2i+2},t)),\\
&\phi(M_{\mathbb{A}}(\xi_{2i+2},\xi_{2i+3},t))),\phi(M_{\mathbb{A}}(\xi_{2i+2},\xi_{2i+3},t))\}\\
&\preccurlyeq ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{2i+1},\xi_{2i+2},t)),\phi(M_{\mathbb{A}}(\xi_{2i+2},\xi_{2i+3},t))\}\\
&+ ||a_2||^2 1_{\mathbb{A}} \phi(\tau(M_{\mathbb{A}}(\xi_{2i+1},\xi_{2i+2},t)),\phi(M_{\mathbb{A}}(\xi_{2i+2},\xi_{2i+3},t)))\\
&\preccurlyeq ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{2i+1},\xi_{2i+2},t)),\phi(M_{\mathbb{A}}(\xi_{2i+2},\xi_{2i+3},t))\}\\
&+ ||a_2||^2 1_{\mathbb{A}} (\phi(M_{\mathbb{A}}(\xi_{2i+1},\xi_{2i+2},t)) + \phi(M_{\mathbb{A}}(\xi_{2i+2},\xi_{2i+3},t))). \quad (16)\end{aligned}$$

Putting $j = 2i+1$, $i = 0,1,2,...$, from (15) and (16), we get

$$\begin{aligned}\phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t)) \preccurlyeq ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{j-1},\xi_j,t)),\phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t))\}\\
+ ||a_2||^2 1_{\mathbb{A}} (\phi(M_{\mathbb{A}}(\xi_{j-1},\xi_j,t)) + \phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t))). \quad (17)\end{aligned}$$

If $\min\{\phi(M_{\mathbb{A}}(\xi_{j-1},\xi_j,t)),\phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t))\} = \phi(M_{\mathbb{A}}(\xi_{j-1},\xi_j,t))$, then from (17), we get

$$\phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t)) \preccurlyeq \big(\frac{||a_1||^2 + ||a_2||^2}{1 - ||a_2||^2}\big) 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t)). \quad (18)$$

Again, if $MIN\{\phi(M_{\mathbb{A}}(\xi_{j-1},\xi_j,t)),\phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t))\} = \phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t))$, then

$$\phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t)) \preccurlyeq \big(\frac{||a_2||^2}{1 - ||a_1||^2 - ||a_2||^2}\big) 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(\xi_{j-1},\xi_j,t)). \quad (19)$$

Proceeding as Theorem 1, from (18) and (19) we can easily show that $\{\xi_n\}$ is a Cauchy sequence and let $\lim_{n\to\infty} \xi_n = z$. Then,

$$\begin{aligned}\phi(M_{\mathbb{A}}(z,Sz,t)) &\preccurlyeq \phi(M_{\mathbb{A}}(z,\xi_{2n+1},t)) + \phi(M_{\mathbb{A}}(\xi_{2n+1},Sz,t))\\
&= \phi(M_{\mathbb{A}}(z,\xi_{2n+1},t)) + \phi(M_{\mathbb{A}}(T\xi_{2n},Sz,t))\\
&\preccurlyeq \phi(M_{\mathbb{A}}(z,\xi_{2n+1},t)) + a_1^* MIN\{\phi(M_{\mathbb{A}}(\xi_{2n},z,t)),\phi(M_{\mathbb{A}}(\xi_{2n},T\xi_{2n},t)),\\
&\phi(M_{\mathbb{A}}(\xi_{2n},Sz,2t)),\phi(M_{\mathbb{A}}(z,T\xi_{2n},2t)),\phi(M_{\mathbb{A}}(z,Sz,t))\}a_1\\
&+ a_2^* MAX\{\phi(M_{\mathbb{A}}(\xi_{2n},z,t)),\phi(M_{\mathbb{A}}(\xi_{2n},T\xi_{2n},t)),\\
&\phi(M_{\mathbb{A}}(\xi_{2n},Sz,2t)),\phi(M_{\mathbb{A}}(z,T\xi_{2n},2t)),\phi(M_{\mathbb{A}}(z,Sz,t))\}a_2.\end{aligned}$$

Taking the limit as $n \to \infty$ and by Lemma 3, from the above equation, we get

$$\begin{aligned}\phi(M_{\mathbb{A}}(z,Sz,t)) \preccurlyeq ||a_1||^2 \phi(M_{\mathbb{A}}(z,Sz,t)) + ||a_2||^2 \phi(M_{\mathbb{A}}(z,Sz,t)),\\
(1 - ||a_1||^2 - ||a_2||^2) 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(z,Sz,t)) \preccurlyeq 0_{\mathbb{A}},\end{aligned}$$

which gives $z = Sz$. Similarly, we can show that z is also a fixed point of T.

Theorem 4. Let $(X, M_{\mathbf{A}}, \tau)$ be a complete C^*-algebra valued fuzzy metric space. Let ϕ be a subadditive altering distance function with respect to the t-norm τ and let T, S be two self mappings on X such that

$$\begin{aligned}\phi(M_{\mathbb{A}}(T\xi, S\eta, t)) \preccurlyeq\, & a_1^*\phi(M_{\mathbb{A}}(\xi, T\xi, t))a_1 + a_2^*\phi(M_{\mathbb{A}}(\eta, S\eta, t))a_2 + a_3^*\phi(M_{\mathbb{A}}(\xi, S\eta, t))a_3 \\ & + a_4^*\phi(M_{\mathbb{A}}(\eta, T\xi, t))a_4 + a_5^*\phi(M_{\mathbb{A}}(\xi, \eta, t))a_5,\end{aligned}$$

where $a_i \in \mathbb{A}'$ for $i = 1$ to 5 with $\sum_{i=1}^{5} ||a_i||^2 < 1$. Then T and S have a unique common fixed point.

The proof is similar to Theorem 3.
For $T = S$, the above theorem reduces to the following fixed point theorem.

Theorem 5. Let $(X, M_{\mathbb{A}}, \tau)$ be a complete C^*-algebra valued fuzzy metric space. Let ϕ be a subadditive altering distance function with respect to the t-norm τ and let T be a self mapping on X such that

$$\begin{aligned}\phi(M_{\mathbb{A}}(T\xi, T\eta, t)) \preccurlyeq\, & a_1^*\phi(M_{\mathbb{A}}(\xi, T\xi, t))a_1 + a_2^*\phi(M_{\mathbb{A}}(\eta, T\eta, t))a_2 + a_3^*\phi(M_{\mathbb{A}}(\xi, T\eta, t))a_3 \\ & + a_4^*\phi(M_{\mathbb{A}}(\eta, T\xi, t))a_4 + a_5^*\phi(M_{\mathbb{A}}(\xi, \eta, t))a_5,\end{aligned}$$

where $a_i \in \mathbb{A}'$ for $i = 1$ to 5 with $||a_1||^2 + ||a_2||^2 + 2||a_3||^2 + ||a_5||^2 < 1$. Then T has a unique fixed point.

3 Application to boundary value problem

We consider the following boundary value problem:

$$x^2y'' + xy' - y = f(t, y(t)),\ 0 < x < 1,\ t \in I = [0, 1], \tag{20}$$

(where f is a function from $I \times \mathbb{R}$ to $\mathbb{R}$), with the boundary conditions: $y(x)$ is bounded as $x \to 0$ and $y(1) = 0$. This boundary value problem is equivalent to the integral equation:

$$u(t) = \int_0^1 G(t, s)f(s, u(s))ds,\ 0 < t, s < 1,$$

where

$$G(t, s) = \begin{cases} \frac{t}{2}(1 - \frac{1}{s^2}), & s > t \\ \frac{1}{2}(t - \frac{1}{t}), & s < t \end{cases}$$

is the Green's function.

Let $C(I, \mathbb{R})$ denote the set of all continuous functions $f : I \to \mathbb{R}$ such that for $x, y \in C(I, \mathbb{R})$, $|x(t) - y(t)| < k$ for some $k > 0$ and for all $t \in I$.

Theorem 6. For the above problem (20), we consider f as a continuous function from $I \times \mathbb{R} \to \mathbb{R}$ satisfying the following condition:

$$f(s, u(s)) - f(s, v(s)) \leq \frac{1}{9}|u(s) - v(s)|, \text{ for all } u, v \in C(I, \mathbb{R}),\ s \in I.$$

Then the problem (20) has a unique solution.

Proof. Let $T : C(I, \mathbb{R}) \to C(I, \mathbb{R})$ be defined by $Tu(t) = \int_0^1 G(t, s)f(s, u(s))ds,\ u \in C(I, \mathbb{R})$. Let $\mathbb{A} = [0, 1]$ with the usual norm on $\mathbb{R}$. Let $X = C(I, \mathbb{R})$ with $d(x, y) = \sup_{t \in I} |x(t) - y(t)|$, $x, y \in C(I, \mathbb{R})$. Here $0_{\mathbb{A}} = 0$ and $1_{\mathbb{A}} = 1$. We consider $M_{\mathbb{A}} : X \times X \times (0, \infty) \to [0, 1]$ given by $M_{\mathbb{A}}(x, y, t) = 1 - \frac{d(x,y)}{k}$, $x, y \in X$, $t > 0$. Then $(X, M_{\mathbb{A}}, \tau)$ is a complete C^*-algebra valued fuzzy metric space with respect to the t-norm $\tau(x,y)=\max\{x+y-1,0\}$, $x,y\in[0,1]$. Also let $\phi(t)=1-t$, $t\in[0,1]$ be the subadditive altering distance function. Now, for $u,v\in X$ and $t_1>0$,

$$\begin{aligned}\phi(M_{\mathbb{A}}(Tu,Tv,t_1))=\frac{d(Tu,Tv)}{k}&=\frac{1}{k}\sup_{t\in I}|Tu(t)-Tv(t)|\\&=\frac{1}{k}\sup_{t\in I}\left|\int_0^1 G(t,s)f(s,u(s))ds-\int_0^1 G(t,s)f(s,v(s))ds\right|\\&=\frac{1}{k}\sup_{t\in I}\left|\int_0^1 G(t,s)(f(s,u(s))-f(s,v(s)))ds\right|\\&\le\frac{1}{k}\sup_{t\in I}\left|\int_0^1 G(t,s)\frac{1}{9}|u(s)-v(s)|ds\right|\\&\le\frac{1}{9}\frac{d(u,v)}{k}\sup_{t\in I}\left|\int_0^1 G(t,s)ds\right|\\&=\frac{1}{9}\frac{d(u,v)}{k}\sup_{t\in I}\left|\int_0^t\frac{1}{2}(t-\frac{1}{t})ds+\int_t^1\frac{t}{2}(1-\frac{1}{s^2})ds\right|\\&=\frac{1}{9}\frac{d(u,v)}{k}\sup_{t\in I}|t-1|=\frac{1}{3}\phi(M_{\mathbb{A}}(u,v,t_1))\frac{1}{3},\end{aligned}$$

where $a_i=0$ for $i=1$ to 4 and $a_5=\frac{1}{3}$. Then all the conditions of Theorem 5 are satisfied. Hence the boundary value problem has a unique solution.

4 Application to control theory

In [25], Pathak et al. and in [26] Rhoades et al. investigated the possibility of optimally controlling the solution of ordinary differential equation via dynamic programming. Inspired by their work, we give an application to solve such ordinary differential equations in control theory using C^*-algebra valued metric space.

Let K be a compact subset of $\mathbb{R}^n$ with the Euclidean distance which we denote here by $|.|$. Let $T_a:\mathbb{R}^n\to\mathbb{R}^n$ be a mapping such that $T_a(\xi)=f(\xi,a)$ for each $a\in K$ and for all $\xi\in\mathbb{R}^n$, where $f:\mathbb{R}^n\times K\to\mathbb{R}^n$ is a bounded continuous function such that

$$|f(\xi,a)|\le C \text{ for some } C>0 \tag{21}$$

and for $t_1>0$, $\xi,\eta\in\mathbb{R}^n$,

$$\begin{aligned}\frac{t_1}{t_1+|f(\xi,a)-f(\eta,a)|}\le a_1^*\frac{t_1}{t+|\xi-f(\xi,a)|}a_1+a_2^*\frac{t_1}{t_1+|\eta-f(\eta,a)|}a_2+a_3^*\frac{t_1}{t_1+|\xi-f(\eta,a)|}a_3\\+a_4^*\frac{t_1}{t_1+|\eta-f(\xi,a)|}a_4+a_5^*\frac{t_1}{t_1+|\xi-\eta|}a_5,\end{aligned} \tag{22}$$

where $||a_1||^2+||a_2||^2+2||a_3||^2+||a_5||^2<1$. For $X=\mathbb{R}^n$ and $\mathbb{A}=\mathbb{R}$, $\tau(\xi,\eta)=\min\{\xi,\eta\}$, $\xi,\eta\in\mathbb{R}^+$, we define $M_{\mathbb{A}}(\xi,\eta,t_1)=\frac{t_1}{t_1+|\xi-\eta|}$. We take ϕ as the identity mapping on $\mathbb{A}^+$.

Now, we study the possibility of optimally controlling the solution $\xi(.)$ of the ordinary differential equation:

$$\begin{cases}\xi'(s)=f(\xi(s),\alpha(s)), & t<s<T,\\ \xi(t)=\xi,\end{cases} \tag{23}$$

where $\xi \in \mathbb{R}^n$ is a given initial point, taken by $\xi(.)$ at the initial time $t \geq 0$, and $T > 0$ is a fixed terminal time and $\xi'(s) = \frac{d\xi(s)}{ds}$. Here $\alpha(.)$ is a control function which is some appropriate scheme for adjusting parameters from the compact set K as time progresses thereby affecting the dynamics of the system modelled by (23). We assume that

$$K' = \{\alpha : [0, T] \to K, \alpha(.) \text{ is measurable}\}$$

denotes the set of admissible controls. Since $T_a(\xi) = f(\xi, a)$ for all $\xi \in \mathbb{R}^n$, $a \in K$, from (21) and (22) we have

$$\frac{t_1}{t_1 + |T_a(\xi) - T_a(\eta)|} \leq a_1^* \frac{t_1}{t + |\xi - T_a(\xi)|} a_1 + a_2^* \frac{t_1}{t_1 + |\eta - T_a(\eta)|} a_2 + a_3^* \frac{t_1}{t_1 + |\xi - T_a(\eta)|} a_3 + a_4^* \frac{t_1}{t_1 + |\eta - T_a(\xi)|} a_4 + a_5^* \frac{t_1}{t_1 + |\xi - \eta|} a_5$$

for all $\xi, \eta \in \mathbb{R}^n$, $t_1 > 0$ and $a \in K$. Now, applying Theorem 5, we deduce that for each control $\alpha(.) \in K'$, the ordinary differential equation (23) has a unique continuous solution $\xi = \xi^{\alpha(.)}(.)$, existing on the time interval $[t, T]$. Solving the ordinary differential equation for almost everywhere time $t < s < T$, we say that $\xi(.)$ is the response of that system to the control $\alpha(.)$, and $\xi(s)$ is the state of the system at a particular time s.

To find a function $\alpha^*(.)$ which can control the system, the following cost criterion is introduced for each admissible control $\alpha(.) \in K'$ (refer to [27]).

$$\Omega_{\xi,t}(\alpha(.)) = \int_t^T p(\xi(s), \alpha(s))ds + q(\xi(T)), \tag{24}$$

where $\xi = \xi^{\alpha(.)}(.)$ is a solution of (23) and $p : \mathbb{R}^n \times K \to \mathbb{R},\ q : \mathbb{R}^n \to \mathbb{R}$ are given functions, where p is the running cost per unit time and q is the terminal cost. Suppose that,

$$\begin{cases} \max\{|P_a(\xi)|, |q(\xi)|\} & \leq C \text{ for some } C > 0 \\ \frac{t_1}{t_1+|P_a(\xi)-P_a(\eta)|} & \leq a_1^* \frac{t_1}{t+|\xi-P_a(\xi)|} a_1 + a_2^* \frac{t_1}{t_1+|\eta-P_a(\eta)|} a_2 + a_3^* \frac{t_1}{t_1+|\xi-P_a(\eta)|} a_3 \\ & + a_4^* \frac{t_1}{t_1+|\eta-P_a(\xi)|} a_4 + a_5^* \frac{t_1}{t_1+|\xi-\eta|} a_5 \\ \frac{t_1}{t_1+|q(\xi)-q(\eta)|} & \leq a_1^* \frac{t_1}{t+|\xi-q(\xi)|} a_1 + a_2^* \frac{t_1}{t_1+|\eta-q(\eta)|} a_2 + a_3^* \frac{t_1}{t_1+|\xi-q(\eta)|} a_3 \\ & + a_4^* \frac{t_1}{t_1+|\eta-q(\xi)|} a_4 + a_5^* \frac{t_1}{t_1+|\xi-\eta|} a_5,\ for\ all\ \xi, \eta \in \mathbb{R}^n,\ a \in K, \end{cases}$$

where $P_a : \mathbb{R}^n \to \mathbb{R}^n$ is a mapping such that $P_a(\xi) = p(\xi, a)$ for all $\xi \in \mathbb{R}^n$. For given $\xi \in \mathbb{R}^n$ and $0 < t < T$, we are to find if possible a control $\alpha^*(.)$ which minimizes the cost functional (24) among all other admissible controls.

For the solution of the above problem we now apply the dynamic programming as described in [27], where the value function $u(\xi, t)$ is defined by

$$u(\xi, t) = \inf_{\alpha(.) \in K'} \Omega_{\xi,t}(\alpha(.))\ \xi \in \mathbb{R}^n,\ 0 \leq t \leq T.$$

Here $u(\xi, t)$ is the least cost for the position ξ at time t.

For fixed $\xi \in \mathbb{R}^n$ and $0 \leq t \leq T$, proceeding as in [27; 554], the following theorem gives the optimality conditions:

Theorem 7. For each $\zeta > 0$ small enough such that $t + \zeta \leq T$,

$$u(\xi, t) = \inf_{\alpha(.) \in K'} \left\{ \int_t^{t+\zeta} p(\xi(s), \alpha(s))ds + u(\xi(t + \zeta), t + \zeta) \right\},$$

where $\xi = \xi^{\alpha(.)}$ solves the ODE (23) for the control $\alpha(.)$.

Proof. The proof follows from Theorem 5 and [27; 554].

Conclusions and Future Works

In this paper, we have obtained some fixed point and common fixed point results for some generalized mappings in C^*-algebra valued fuzzy metric space. Moreover, the results are applied to boundary value problem and control theory. Some open problems concerning our results are as follows.

In Theorems 1, 2 and 3, we have considered complete C^*-algebra valued fuzzy metric space. The investigation of the existence of fixed point via our defined contractive conditions in case of incomplete C^*-algebra valued fuzzy metric space is a problem of further study.

In [28] and [29], the authors obtained some important results in fuzzy bipolar metric space. The analogous study in case of bipolar C^*-algebra valued fuzzy metric space for the mappings defined in this paper is a scope for future research.

In 2024, Gnanaprakasam et al. [30] applied fixed point techniques to discuss solvability of fractional integro-differential equation in orthogonal complete metric space. In this regard, we can extend our study to investigate solvability of fractional integro-differential equation.

Further investigation can be done considering coupled fixed point, best proximity point, coupled best proximity point, etc., using our mappings in the setting of C^*-algebra valued fuzzy metric space. The works done in this paper thus open up a wide scope of investigation in C^*-algebra valued fuzzy metric space considering various emerging applications.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Kramosil, I., & Michálek, J. (1975). Fuzzy metrics and statistical metric spaces. *Kybernetika (Prague), 11*(5), 336–344. http://dml.cz/dmlcz/125556

2 Grabiec, M. (1988). Fixed points in fuzzy metric spaces. *Fuzzy Sets Syst., 27*(3), 385–389. https://doi.org/10.1016/0165-0114(88)90064-4

3 Roldán-López-de-Hierro, A.F., & Sintunavarat, W. (2016). Common fixed point theorems in fuzzy metric spaces using the CLRg property. *Fuzzy Sets Syst., 282*, 131–142. https://doi.org/10.1016/j.fss.2014.11.005

4 Javed, K., Uddin, F., Aydi, H., Mukheimer, A., & Arshad, M. (2021). Ordered-theoretic fixed point results in fuzzy b-metric spaces with an application. *J. Math., 2021*, Article ID 6663707, 7 pages. https://doi.org/10. 1155/2021/6663707

5 Patir, B., Goswami, N., & Mishra, L.N. (2018). Fixed point theorems in fuzzy metric spaces for mappings with some contractive type conditions. *Korean J. Math., 26*(2), 307–326. https://doi.org/10.11568/kjm.2018.26.2.307

6 Goswami, N., & Patir, B. (2021). Fixed point theorems in fuzzy metric spaces for mappings with $B_{\gamma,\mu}$ condition. *Proyecciones (Antofagasta), 40*(4), 837–857. http://dx.doi.org/10.22199/issn.0717-6279-3333

7 Huang, H., Carić, B., Došenović, T., Rakić, D., & Brdar, M. (2021). Fixed-point theorems in fuzzy metric spaces via fuzzy F-contraction. *Math., 9*(6), 641. https://doi.org/10.3390/math9060641

8 Patir, B., Goswami, N., & Mishra, V.N. (2018). Some results on fixed point theory for a class of generalized nonexpansive mappings. *Fixed Point Theory Appl., 2018*, 19. https://doi.org/10.1186/s13663-018-0644-1

9 Bouftouh, O., & Kabbaj, S. (2023). Fixed point theorems in C^*-algebra valued asymmetric metric spaces. *Palest. J. Math., 12*(1), 859–871.

10 Mani, G., Gnanaprakasam, A.J., Ege, O., Aloqaily, A., & Mlaiki, N. (2023). Fixed point results in C^*-algebra-valued partial b-metric spaces with related application. *Math., 11*(5), 1158. https://doi.org/10.3390/math11051158

11 Khan, M.S., Swaleh, M., & Sessa, S. (1984). Fixed point theorems by altering distances between the points. *Bull. Aust. Math. Soc., 30*(1), 1–9. https://doi.org/10.1017/S0004972700001659

12 Shen, Y., Qiu, D., & Chen, W. (2012). Fixed point theorems in fuzzy metric spaces. *Appl. Math. Lett., 25*(2), 138–141. https://doi.org/10.1016/j.aml.2011.08.002

13 Shoaib, A., Kumam, P., Shahzad, A., Phiangsungnoen, S., & Mahmood, Q. (2018). Fixed point results for fuzzy mappings in a b-metric space. *Fixed Point Theory Appl., 2018*, 2. https://doi.org/10.1186/s13663-017-0626-8

14 Ma, Z., Jiang, L., & Sun, H. (2014). C^*-algebra valued metric spaces and some related fixed point theorems. *Fixed Point Theory Appl., 2014*, 206. https://doi.org/10.1186/1687-1812-2014-206

15 Madadi, M., O' Regan, D., Park, C., de la Sen, M., & Saadati, R. (2020). On the topology induced by C^*-algebra valued fuzzy metric spaces. *Math., 8*(6), 905. https://doi.org/10.3390/math8060905

16 Khaofong, C., & Khammahawong, K. (2021). Fixed point theorems in C^*-algebra valued fuzzy metric spaces with application. *Thai J. Math., 19*(3), 964–970.

17 George, A., & Veeramani, P. (1994). On some results in fuzzy metric spaces. *Fuzzy Sets Syst., 64*(3), 395–399. https://doi.org/10.1016/0165-0114(94)90162-7

18 Goswami, N., & Patir, B. (2019). Fixed point theorems for asymptotically regular mappings in fuzzy metric spaces. *Korean J. Math., 27*(4), 861–877. https://doi.org/10.11568/kjm.2019.27.4.861

19 Qiaoling, X., Lining, J., & Zhenhua, M. (2016). Common fixed point theorems in C^*-algebra valued metric spaces. *J. Nonlinear Sci. Appl., 9*(6), 4617–4627. http://dx.doi.org/10.22436/jnsa.009.06.100

20 Douglas, R.G. (1998). *Banach Algebra Techniques in Operator Theory.* Springer, Berlin.

21 Murphy, J.G. (1990). *C*-Algebras and Operator Theory.* Academic Press, San Diego.

22 Ruddin, W. (1991). *Functional Analysis.* McGraw-Hill, Inc., New York.

23 Conway, J.B. (1989). *A Course in Functional Analysis.* Springer, New York.

24 Wairojjana, N., Došenović, T., Rakić, D., Gopal, D., & Kumam, P. (2015). An altering distance function in fuzzy metric fixed point theorems. *Fixed Point Theory Appl., 2015*, 69. https://doi.org/ 10.1186/s13663-015-0318-1

25 Pathak, H.K., & Shahzad, N. (2008). Fixed points for generalized contractions and applications to control theory. *Nonlinear Anal., 68*(8), 2181–2193. https://doi.org/10.1016/j.na.2007.01.041

26 Rhoades, B.E., Pathak, H.K., & Mishra, S.N. (2012). Some weakly contractive mapping theorems in partially ordered spaces and applications. *Demonstr. Math., 45*(3), 621–636. https://doi.org/10.1515/dema-2013-0388

27 Evans, L.C. (1997). *Partial Differential Equations.* Amer. Math. Soc.

28 Mani, G., Gnanaprakasam, A.J., Kumar, S., Ege, O., & De la Sen, M. (2023). Fixed point theorems for nonlinear contraction in fuzzy-controlled bipolar metric spaces. *Axioms, 12*(4), 1–12. https://doi.org/10.3390/axioms12040396

29 Ramalingam, B., Ege, O., Aloqaily, A., & Mlaiki, N. (2023). Fixed point theorems on fuzzy bipolar *b*-metric spaces. *Symmetry, 15*(10), 1–17. https://doi.org/10.3390/sym15101831

30 Gnanaprakasam, A.J., Ramalingam, B., Mani, G., Ege, O., & George, R. (2024). A numerical scheme and application to the fractional integro-differential equation using fixed-point techniques. *Fractal Fract., 8*(1), 34, 1–17. https://doi.org/10.3390/fractalfract8010034

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