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Solution of a boundary value problem for a third-order inhomogeneous equation with multiple characteristics with the construction of the Green's function

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In the paper the second boundary value problem in a rectangular domain for an inhomogeneous third-order partial differential equation with multiple characteristics with constant coefficients was considered. The uniqueness of the solution to the problem posed is proven by the method of energy integrals. A counterexample is constructed in case when the uniqueness theorem's conditions are violated. Using the method of separation of variables, the solution to the problem is sought in the form of a product of two functions $X(x)$ and $Y(y)$. To determine $Y(y)$, we obtain a second-order ordinary differential equation with two boundary conditions at the boundaries of the segment $[0, q]$. For this problem, the eigenvalues and the corresponding eigenfunctions are found for $n = 0$ and $n \in N$. To determine $X(x)$, we obtain a third-order ordinary differential equation with three boundary conditions at the boundaries of the segment $[0, p]$. Using the Green's function method, we constructed solution of the specified problem. A separate Green's function for $n = 0$ and a separate Green's function for the case when n is natural were constructed. The solution for $X(x)$ is written in terms of the constructed Green's function. After some transformations, an integral Fredholm equation of the second kind is obtained, the solution of which is written through the resolvent. Estimates for the resolvent and Green's function are obtained. The uniform convergence of the solution and the possibility of its term-by-term differentiation under certain conditions on given functions are proven. When justifying the uniform convergence of the solution, the absence of a "small denominator" is proven.

Keywords: differential equation, the third order, multiple characteristics, the second boundary value problem, regular solution, uniqueness, existence, Green's function.

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Introduction

Third-order partial differential equations are considered in solving problems in the theory of nonlinear acoustics and in the hydrodynamic theory of space plasma, fluid filtration in porous media [1].

In the aggregate, all third-order equations occupy a special place in terms of their specific character, equations with multiple characteristics.

The first results on a third-order equation with multiple characteristics were obtained by H. Block [2], E. Del Vecchio [3].

L. Cattabriga in [4] for equation $D_x^{2n+1}u - D_y^2 u = 0$ constructed a fundamental solution in the form of a double improper integral.

In [5], a fundamental solution of a third-order equation with multiple characteristics containing the second derivative with respect to time was constructed, their properties were studied, and estimates were found for $|t| \to \infty$.

In works [6–9], boundary value problems for third-order equations with multiple characteristics are considered using the construction of the Green's function. Also, we note the works [10–21], in which

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the boundary value problems for third-order equations are considered. Boundary value problems close to the topic of this work were studied in [22, 23]. In [24, 25], a solution to the problem posed for a third-order equation was found with other boundary conditions.

1 *Formulation of the problem*

In the domain $D = \{(x, y): \ 0 < x < p, \ 0 < y < q\}$, we consider the following third-order equation of the form:

$$L(u) = U_{xxx} - U_{yy} + A_1 U_{xx} + A_2 U_x + A_3 U_y + A_4 U = g_1(x, y), \tag{1}$$

where $A_i, \ p, \, q \in R, \ i = \overline{1,4}$, are given sufficiently smooth functions.

By the replacement

$$U\left(x, y\right) = u\left(x, y\right) e^{-\frac{A_1}{3}x + \frac{A_3}{2}y},$$

equation (1) can be reduced to the form

$$u_{xxx} - u_{yy} + a_1 u_x + a_2 u = g(x, y), \tag{2}$$

where $a_1 = -\frac{{A_1}^2}{3} + A_2, \ a_2 = \frac{2A_1^3}{27} + \frac{A_3^2}{2} - \frac{A_1 A_2}{3} + A_4, \ g(x, y) = g_1(x, y) \cdot e^{\frac{A_1}{3}x - \frac{A_3}{2}y}$.

Problem A_2. Find function $u\left(x, y\right)$ from class $C_{x,y}^{3,2}\left(D\right) \cap C_{x,y}^{2,1}\left(\overline{D}\right)$, that satisfies equation (2) and the following boundary conditions:

$$u_y\left(x, 0\right) = 0, \quad u_y\left(x, q\right) = 0, \quad 0 \leq x \leq p, \tag{3}$$

$$u\left(p, y\right) = \psi_2\left(y\right), \quad u_x\left(p, y\right) = \psi_3\left(y\right), \quad u_{xx}\left(0, y\right) = \psi_1\left(y\right), \quad 0 \leq y \leq q, \tag{4}$$

where $\psi_i\left(y\right), \ i = \overline{1,3}, \, g\left(x, y\right)$ are given functions. Note that in works [9–12] the case $a_1 = a_2 = 0$ was considered.

2 *The uniqueness of solution*

Theorem 1. If problem A_2 has a solution, then if conditions $a_1 \leq 0, \ a_2 \geq 0$ are met, it is unique.

Proof. Let's assume the opposite. Let problem A_2 have two solutions $u_1\left(x, y\right)$ and $u_2\left(x, y\right)$. Then function $u\left(x, y\right) = u_1\left(x, y\right) - u_2\left(x, y\right)$ satisfies the homogeneous equation (2) with homogeneous boundary conditions. Let's prove that $u\left(x, y\right) \equiv 0$ is in $\overline{D}$.

In the domain D the identity

$$uL\left[u\right] = uu_{xxx} - uu_{yy} + a_1 uu_x + a_2 u^2 = 0$$

or

$$\frac{\partial}{\partial x}\left(uu_{xx} - \frac{1}{2}u_x^2 + \frac{1}{2}a_1 u^2\right) - \frac{\partial}{\partial y}\left(uu_y\right) + u_y^2 + a_2 u^2 = 0 \tag{5}$$

holds. Integrating identity (5) over the domain D and taking into account homogeneous boundary conditions, we obtain

$$-\frac{1}{2}a_1 \int\limits_0^q u^2\left(0, y\right) dy + \frac{1}{2}\int\limits_0^q u_x^2\left(0, y\right) dy + \int\limits_0^p \int\limits_0^q u_y^2 dx dy + a_2 \int\limits_0^p \int\limits_0^q u^2 dx dy = 0.$$

If $a_1, \ a_2 \neq 0$, from the fourth term, we get $u\left(x, y\right) \equiv 0, \, (x, y) \in \overline{D}$. If $a_2 = 0$, then from the third term $u_y\left(x, y\right) = 0$. From the equation and taking into account the homogeneous boundary conditions (4) we obtain $u\left(x, y\right) \equiv 0$ is in $\overline{D}$. The theorem has been proven.

Remark 1. Note that if the conditions of Theorem 1 are violated, the homogeneous problem A_2 for the homogeneous equation (2) may have a nontrivial solution. For example, problem

$$\begin{cases} u_{xxx}(x,y) + \left(\frac{(2k+1)\pi}{2p}\right)^2 u_x(x,y) - \left(\frac{\pi n}{q}\right)^2 u(x,y) - u_{yy}(x,y) = 0, \\ u_y(x,0) = 0, \quad u_y(x,q) = 0, \quad 0 \le x \le p, \\ u(p,y) = 0, \quad u_x(p,y) = 0, \quad u_{xx}(0,y) = 0, \quad 0 \le y \le q \end{cases}$$

has a nontrivial solution in the form:

$$u(x,y) = \left(1 + (-1)^{k+1} \sin\left(\frac{(2k+1)\pi}{2p} x\right)\right) \cos\left(\frac{\pi n}{q} y\right), \quad n,k \in Z.$$

3 *Existence of a solution*

Theorem 2. If the following conditions are met:

1) $\psi_i(y) \in C^3[0,q], \quad {\psi_i}'(0) = {\psi_i}'(q) = 0, \; i = \overline{1,3};$

2) $\dfrac{\partial^3 g(x,y)}{\partial x \partial y^2} \in C[0,q], \quad g_y(x,0) = g_y(x,q) = 0, \quad 0 \le x \le p;$

3) $0 \le C < \min\left\{\dfrac{1}{p^2 + \frac{1}{2}p^3}, \dfrac{\lambda_1^2}{Kp(\lambda_1+1)}\right\},$

then a solution to the problem exists.
Here $C = \max\{|a_1|, |a_2|\}$, $\lambda_1 = \sqrt[3]{\left(\frac{\pi}{q}\right)^2}$, $K = \frac{16}{3}\left(1 - \exp\left(-\frac{2\sqrt{3}\pi}{3}\right)\right)^{-1}$.

In works [9–12] $C = 0$. The 3rd condition is satisfied at $C = 0$.

Proof. Consider the following Sturm-Liouville problem taking into account the boundary conditions (3):

$$\begin{cases} Y''(y) + \lambda^3 Y(y) = 0, \\ Y_y(0) = Y_y(q) = 0, \end{cases} \tag{6}$$

eigenvalues and eigenfunctions of problem (6) have the form:

$$Y_n(y) = \begin{cases} \frac{1}{\sqrt{q}}, \quad {\lambda_0}^3 = 0, \quad n = 0, \\ \sqrt{\frac{2}{q}} \cos\left(\frac{\pi n}{q} y\right), \quad {\lambda_n}^3 = \left(\frac{\pi n}{q}\right)^2, \quad n \in N. \end{cases}$$

Let's expand $g(x,y)$ into a Fourier series of $\{Y_n(y)\}$:

$$g(x,y) = \sum_{n=1}^{\infty} g_n(x) Y_n(y),$$

here $g_n(x) = \sqrt{\frac{2}{q}} \int\limits_0^q g(x,\eta) \cos\left(\frac{\pi n}{q}\eta\right) d\eta$. We integrate function $g_n(x)$ by parts twice and taking into account condition 2, Theorem 2, we obtain the estimate $|g_n(x)| \le \frac{M}{n^2} |F_n(x)|$. Here $F_n(x) = \sqrt{\frac{2}{q}} \int\limits_0^q g_{\eta\eta}(x,\eta) \cos\frac{\pi n}{q}\eta d\eta$.

Further on we will denote all arbitrary positive constants by M.

We look for a solution to problem A_2 in the form

$$u(x,y)=\sum_{n=0}^{\infty}X_n(x)\,Y_n(y). \tag{7}$$

Substituting (7) into equation (2) and taking into account condition (4) we have the following problem:

$$\begin{cases} X'''+a_1X'+a_2X+\lambda_n^3X=g(x), \\ X''(0)=\psi_{1n},\ X(p)=\psi_{2n},\ X'(p)=\psi_{3n}, \end{cases} \tag{8}$$

where $\psi_{in}=\sqrt{\frac{2}{q}}\int\limits_0^q\psi_{in}(\eta)\cos\left(\frac{\pi n}{q}\eta\right)d\eta,\ i=\overline{1,3}.$

Using the function

$$V(x)=X(x)-\rho(x), \tag{9}$$

boundary conditions (8) are transformed into homogeneous ones. Function $\rho(x)$ looks like:

$$\rho_n(x)=\psi_{2n}-\psi_{3n}p+\frac{\psi_{1n}}{2}p^2+(\psi_{3n}-\psi_{1n}p)\,x+\frac{\psi_{1n}}{2}x^2.$$

Substituting (9) into (8) we obtain the problem

$$\begin{cases} V'''+\lambda_n^3V=\lambda_n^3f_n(x)-a_1V'-a_2V, \\ \qquad V''(0)=V(p)=V'(p)=0, \end{cases} \tag{10}$$

here

$$f_n(x)=\left(\frac{a_1p-a_1x+a_2px}{\lambda_n^3}-\frac{a_2p^2+a_2x^2}{2\lambda_n^3}-\frac{p^2+x^2}{2}+px\right)\psi_{1n}-$$
$$-\left(\frac{a_2}{\lambda_n^3}+1\right)\psi_{2n}+\left(\frac{a_2p-a_1-a_2x}{\lambda_n^3}+p-x\right)\psi_{3n}+\frac{g(x)}{\lambda_n^3}.$$

Then we have estimates

$$\begin{gathered} |f_n(x)|\le\frac{M}{n^3}\left(|\Psi_{1n}|+|\Psi_{2n}|+|\Psi_{3n}|+\frac{1}{n}|F_n(x)|\right), \\ |f'_n(x)|\le\frac{M}{n^3}\left(|\Psi_{1n}|+|\Psi_{3n}|+\frac{1}{n}|F'_n(x)|\right). \end{gathered} \tag{11}$$

Let's consider cases $n=0$ and $n\in N$ separately. Problem (10) for $\lambda_0=0$ has the form:

$$\begin{cases} V_0'''=f_0(x)-a_1V_0'-a_2V_0, \\ V_0''(0)=V_0(p)=V_0'(p)=0, \end{cases} \tag{12}$$

here

$$f_0(x)=g_0(x)+\left(a_1(x-p)+a_2\left(px-\frac{p^2}{2}-\frac{x^2}{2}\right)\right)\psi_{1\,0}-a_2\psi_{2\,0}+(a_2(p-x)-a_1)\,\psi_{3\,0}.$$

Problem (12) is equivalent to the integro-differential equation

$$V_0(x)=\int\limits_0^pG_n(x,\xi)f_n(\xi)\,d\xi+a_1\int\limits_0^pG_0(x,\xi)\,V'_0(\xi)\,d\xi-a_2\int\limits_0^pG_0(x,\xi)\,V_0(\xi)\,d\xi, \tag{13}$$

here $G_0(x,\xi)$ is the Green's function of problem (12), it has the following properties:

$$\frac{\partial^3G_0(x,\xi)}{\partial x^3}=0,$$

$$G_{10xx}(0,\xi) = G_{20}(p,\xi) = G_{20x}(p,\xi) = 0,$$

$$\begin{gathered} G_{20}(\xi,\xi) - G_{10}(\xi,\xi) = 0, \\ G_{20x}(\xi,\xi) - G_{10x}(\xi,\xi) = 0, \\ G_{20xx}(\xi,\xi) - G_{10xx}(\xi,\xi) = 1. \end{gathered}$$

Function $G_0(x,\xi)$ has the form

$$G_0(x,\xi) = \frac{1}{2}\begin{cases} (p-\xi)(p+\xi-2x), & 0 \le x < \xi \le p, \\ (x-p)^2, & 0 \le \xi < x \le p. \end{cases} \tag{14}$$

It is easy to verify that the function defined by formula (14) has all the properties formulated in the definition of the Green's function.

Integrating by parts the second integral in (13) and introducing the notation

$$\begin{gathered} \alpha_0(x) = \int_0^p G_0(x,\xi) f_0(\xi)\, d\xi, \\ \bar{G}_0(x,\xi) = a_1 G_{0\xi}(x,\xi) - a_2 G_0(x,\xi), \end{gathered}$$

we get

$$V_0(x) = \alpha_0(x) + \int_0^p \bar{G}_0(x,\xi) V_0(\xi)\, d\xi. \tag{15}$$

Equation (15) is the Fredholm integral equation of the second kind. We solve (15) using the iteration method.

Taking the zero approximation $V_0(x) = \alpha_0(x)$, we write (15) as follows:

$$V_m(x) = \alpha_0(x) + \int_0^p \bar{G}_0(x,\xi) V_{m-1}(\xi)\, d\xi, \quad m = 1, 2, \ldots$$

The first approximation is

$$V_1(x) = \alpha_0(x) + \int_0^p \bar{G}_0(x,\xi)\, \alpha_0(\xi)\, d\xi,$$

the second approximation is

$$\begin{gathered} V_2(x) = \alpha_0(x) + \int_0^p \bar{G}_0(x,s) V_1(s)\, ds = \alpha_0(x) + \\ + \int_0^p \bar{G}_0(s,\xi)\left(\alpha_0(s) + \int_0^p \bar{G}_0(s,\xi)\,\alpha_0(\xi)\, d\xi\right) ds = \\ = \alpha_0(x) + \int_0^p \bar{G}_0(x,\xi)\,\alpha_0(\xi)\, d\xi + \int_0^p \bar{G}_0(x,s)\, ds \int_0^p \bar{G}_0(s,\xi)\,\alpha_0(\xi)\, d\xi, \end{gathered}$$

by changing the order of integration in the iterated integral and making the replacement

$$\bar{G}_1(x,\xi) = \int_0^p \bar{G}_0(x,s)\,\bar{G}_0(s,\xi)\, ds,$$

then we get

$$V_2(x) = \alpha_0(x) + \int_0^p \left(\bar{G}_0(x,\xi) + \bar{G}_1(x,\xi)\right)\alpha_0(\xi)\,d\xi.$$

If we continue the process indefinitely, we get

$$V_0(x) = \alpha_0(x) + \int_0^p \left(\bar{G}_0(x,\xi) + \sum_{m=1}^{\infty} \bar{G}_m(x,\xi)\right)\alpha_0(\xi)\,d\xi.$$

Here

$$\bar{G}_m(x,\xi) = \int_0^p \bar{G}_0(x,s)\,\bar{G}_{m-1}(s,\xi)\,ds, \quad m = 1, 2, 3, \ldots$$

If we denote

$$R_0(x,\xi) = \bar{G}_0(x,\xi) + \sum_{m=1}^{\infty} \bar{G}_m(x,\xi),$$

then we have a solution in the form

$$V_0(x) = \alpha_0(x) + \int_0^p R_0(x,\xi)\,\alpha_0(\xi)\,d\xi.$$

Then we get a solution for $\lambda_0 = 0$ in the form

$$u_0(x) = \frac{1}{\sqrt{q}}\left(V_0(x) + \rho_0(x)\right).$$

Let's evaluate this solution. First let's find the estimate $G_0(x,\xi)$:

$$|G_0(x,\xi)| \le \frac{1}{2}p^2, \quad |G_{0\xi}(x,\xi)| \le p.$$

For the resolvent $|R_0(x,\xi)| \le \left|\bar{G}_0(x,\xi)\right| + \left|\bar{G}_1(x,\xi)\right| + \ldots + \left|\bar{G}_m(x,\xi)\right| + \ldots$ we find an estimate using the majorant series:

$$\left|\bar{G}_0(x,\xi)\right| \le C\left(p + \frac{1}{2}p^2\right) \le \frac{1}{p}(J_0 p),$$

$$\left|\bar{G}_1(x,\xi)\right| \le \int_0^p \left|\bar{G}_0(x,s)\right|\left|\bar{G}_0(s,\xi)\right| ds \le \frac{1}{p}(J_0 p)^2,$$

$$\left|\bar{G}_2(x,\xi)\right| \le \int_0^p \left|\bar{G}_0(x,s)\right|\left|\bar{G}_1(s,\xi)\right| ds \le \frac{1}{p}(J_0 p)^3,$$

. . . .

$$\left|\bar{G}_m(x,\xi)\right| \le \int_0^p \left|\bar{G}_0(x,s)\right|\left|\bar{G}_{m-1}(s,\xi)\right| ds \le \frac{1}{p}(J_0 p)^{m+1},$$

. . . .

Here $C = \max\left\{|a_1|, |a_2|\right\}$, $J_0 = C\left(p + \frac{1}{2}p^2\right)$. Hence the majorant series looks

$$\frac{1}{p}\sum_{m=1}^{\infty}(J_0 p)^m.$$

Condition 3, Theorem 2 can be written as

$$C < \frac{2}{p^3 + 2p^2} \Rightarrow C\left|\frac{1}{2}p^2 + p\right| < \frac{1}{p},$$

hence

$$J_0 p < 1,$$

then the majorizing series is the sum of the terms of an infinite decreasing geometric progression. In this case, the resolvent converges uniformly, and its estimate has the form

$$|R_0(x,\xi)| \le \frac{J_0}{1 - J_0 p} \le M.$$

For $\alpha_0(x)$ the estimate is

$$|\alpha_0(x)| \le \int_0^p |G_0(x,\xi)|\,|g_0(\xi)|\,d\xi \le M.$$

Then

$$|u_0(x)| \le M, \quad \left|u_0'''(x)\right| \le M.$$

The solution to problem (10), at $n \in N$, is sought as follows:

$$V_n(x) = \lambda_n^3 \int_0^p G_n(x,\xi) f_n(\xi)\,d\xi - a_1 \int_0^p G_n(x,\xi)\,V_n'(\xi)\,d\xi - a_2 \int_0^p G_n(x,\xi)\,V_n(\xi)\,d\xi, \tag{16}$$

where $G_n(x,\xi)$ is the Green's function of problem (10), which has the following properties:

$$\frac{\partial^3 G_n(x,\xi)}{\partial x^3} + \lambda_n^3 G_n(x,\xi) = 0,$$

$$G_{1nxx}(0,\xi) = G_{2n}(p,\xi) = G_{2nx}(p,\xi) = 0, \tag{17}$$

$$\begin{gathered} G_{2n}(\xi,\xi) - G_{1n}(\xi,\xi) = 0, \\ G_{2nx}(\xi,\xi) - G_{1nx}(\xi,\xi) = 0, \\ G_{2nxx}(\xi,\xi) - G_{1nxx}(\xi,\xi) = 1. \end{gathered} \tag{18}$$

Let's construct the Green's function. Since linearly independent solutions to Equation $X'''_n + \lambda_n^3 X_n = 0$ have the form:

$$X_1(x) = e^{-\lambda_n x}, \quad X_2(x) = e^{\frac{\lambda_n}{2}x}\cos\beta_n x, \quad X_3(x) = e^{\frac{\lambda_n}{2}x}\sin\beta_n x, \quad \beta_n = \frac{\sqrt{3}}{2}\lambda_n,$$

let us represent the required Green's function in the form

$$G_n(x,\xi) = \begin{cases} a_1 e^{-\lambda_n x} + a_2 e^{\frac{\lambda_n}{2}x}\cos\beta_n x + a_3 e^{\frac{\lambda_n}{2}x}\sin\beta_n x\,, & 0 \le x \le \xi, \\ b_1 e^{-\lambda_n x} + b_2 e^{\frac{\lambda_n}{2}x}\cos\beta_n x + b_3 e^{\frac{\lambda_n}{2}x}\sin\beta_n x\,, & \xi \le x \le p, \end{cases} \tag{19}$$

where $a_1, a_2, a_3, b_1, b_2, b_3$ are currently unknown functions from ξ.

From properties (18) of the Green's function and setting $c_n(\xi) = b_n(\xi) - a_n(\xi)$, $n = 1, 2, 3$, we obtain a system of linear equations for finding the functions $c_n(\xi)$:

$$\begin{cases} c_1 e^{-\lambda_n \xi} + c_2 e^{\frac{\lambda_n}{2}\xi} cos\beta_n\xi + c_3 e^{\frac{\lambda_n}{2}\xi} \sin\beta_n\xi = 0, \\ -c_1 e^{-\lambda_n \xi} + c_2 e^{\frac{\lambda_n}{2}\xi} cos\left(\beta_n\xi + \frac{\pi}{3}\right) + c_3 e^{\frac{\lambda_n}{2}\xi} \sin\left(\beta_n\xi + \frac{\pi}{3}\right) = 0, \\ c_1 e^{-\lambda_n \xi} + c_2 e^{\frac{\lambda_n}{2}\xi} cos\left(\beta_n\xi + \frac{2\pi}{3}\right) + c_3 e^{\frac{\lambda_n}{2}\xi} \sin\left(\beta_n\xi + \frac{2\pi}{3}\right) = \frac{1}{\lambda_n^2}. \end{cases}$$

The determinant of this system is equal to the value of the Wronski determinant $W(X_1, X_2, X_3)$ at point $x = \xi$, and therefore is nonzero and equal to $W(X_1, X_2, X_3) = \frac{3\sqrt{3}}{2}$. Having calculated Δc_i, $i = 1, 2, 3$, we get:

$$c_1(\xi) = \frac{e^{\lambda_n \xi}}{3\lambda_n^2}, \quad c_2(\xi) = -\frac{2e^{-\frac{\lambda_n}{2}\xi} \sin\left(\beta_n\xi + \frac{\pi}{6}\right)}{3\lambda_n^2}, \quad c_3(\xi) = \frac{2e^{-\frac{\lambda_n}{2}\xi} \cos\left(\beta_n\xi + \frac{\pi}{6}\right)}{3\lambda_n^2}.$$

Next, we will use property (17) of the Green's function; in our case, these relations take the form:

$$\begin{cases} 2b_1 - b_2 + \sqrt{3}b_3 = \frac{2}{3{\lambda_n}^2}\left(e^{\lambda_n \xi} + 2e^{-\frac{\lambda_n}{2}\xi} \cos\left(\frac{\sqrt{3}}{2}\lambda_n\xi\right)\right), \\ b_1 e^{-\lambda_n p} + b_2 e^{\frac{\lambda_n}{2}p} \cos\frac{\sqrt{3}}{2}\lambda_n p + b_3 e^{\frac{\lambda_n}{2}p} \sin\frac{\sqrt{3}}{2}\lambda_n p = 0, \\ -b_1 e^{-\lambda_n p} + b_2 e^{\frac{\lambda_n}{2}p} \cos\left(\frac{\sqrt{3}}{2}\lambda_n p + \frac{\pi}{3}\right) + b_3 e^{\frac{\lambda_n}{2}p} \sin\left(\frac{\sqrt{3}}{2}\lambda_n p + \frac{\pi}{3}\right) = 0. \end{cases}$$

Due to the linear independence of ${X_1}''(0)$, ${X_2}'(p)$, ${X_3}'(p)$, the determinant of this system is:

$$\Delta = \sqrt{3}e^{\lambda_n p}\left(1 + 2e^{\frac{-3\lambda_n}{2}p} \cos\left(\frac{\sqrt{3}}{2}\lambda_n p\right)\right) = \sqrt{3}e^{\lambda_n p}\overline{\Delta},$$

here $\overline{\Delta} = 1 + 2e^{\frac{-3\lambda_n}{2}p} \cos\left(\frac{\sqrt{3}}{2}\lambda_n p\right)$.

Consider the following function

$$\overline{\Delta} = 1 + 2e^{-\sqrt{3}t} \cos t, \quad t = \frac{\sqrt{3}}{2}\lambda_n t.$$

The critical points of this function are

$$t_k = \frac{2\pi}{3} + \pi k, \quad k = 0, 1, 2, 3, \ldots$$

$\overline{\Delta}(t)$ takes minimum value only at $k = 0$. Then

$$\overline{\Delta} \geq 1 - \exp\left(-\frac{2\sqrt{3}\pi}{3}\right) > 0.$$

This proves the absence of a "small denominator", hence $\Delta \neq 0$.

Having calculated $\Delta b_i,\ \ i=1,2,3,$ we obtain:

$$b_1=\frac{e^{\lambda_n p}}{\sqrt{3}\lambda_n^2\Delta}\left(e^{\lambda_n\xi}+2e^{-\frac{\lambda_n}{2}\xi}\cos\left(\frac{\sqrt{3}}{2}\lambda_n\xi\right)\right),$$

$$b_2=-\frac{2e^{-\frac{\lambda_n}{2}p}}{\sqrt{3}\lambda_n^2\Delta}\left(e^{\lambda_n\xi}+2e^{-\frac{\lambda_n}{2}\xi}\cos\left(\frac{\sqrt{3}}{2}\lambda_n\xi\right)\right)\sin\left(\frac{\sqrt{3}}{2}\lambda_n p+\frac{\pi}{6}\right),$$

$$b_3=\frac{e^{-\frac{\lambda_n}{2}p}}{\sqrt{3}\lambda_n^2\Delta}\left(e^{\lambda_n\xi}+2e^{-\frac{\lambda_n}{2}\xi}\cos\left(\frac{\sqrt{3}}{2}\lambda_n\xi\right)\right)\cos\left(\frac{\sqrt{3}}{2}\lambda_n p+\frac{\pi}{6}\right).$$

Considering $a_k\left(\xi\right)=b_k\left(\xi\right)-c_k\left(\xi\right)\ ,\ \ k=1,2,3$ we have $a_k,\ k=1,2,3$:

$$a_1=\frac{2}{\sqrt{3}\lambda_n^2\Delta}\left(e^{\lambda_n\left(p-\frac{\xi}{2}\right)}\cos\left(\frac{\sqrt{3}}{2}\lambda_n\xi\right)-e^{\lambda_n\left(\xi-\frac{p}{2}\right)}\cos\left(\frac{\sqrt{3}}{2}\lambda_n p\right)\right),$$

$$a_2=\tfrac{2}{\sqrt{3}\lambda_n^2\Delta}\left(e^{-\lambda_n\left(\frac{\xi}{2}-p\right)}\left(1+2e^{\frac{-3\lambda_n}{2}p}\cos\left(\tfrac{\sqrt{3}}{2}\lambda_n p\right)\right)\sin\left(\tfrac{\sqrt{3}}{2}\lambda_n\xi+\tfrac{\pi}{6}\right)-\right.$$
$$\left.-e^{-\lambda_n\left(\frac{p}{2}-\xi\right)}\left(1+2e^{-\frac{3\lambda_n}{2}\xi}\cos\left(\tfrac{\sqrt{3}}{2}\lambda_n\xi\right)\right)\sin\left(\tfrac{\sqrt{3}}{2}\lambda_n p+\tfrac{\pi}{6}\right)\right),$$

$$a_3=\tfrac{2}{\sqrt{3}\lambda_n^2\Delta}\left(e^{-\lambda_n\left(\frac{p}{2}-\xi\right)}\left(1+2e^{-\frac{3\lambda_n}{2}\xi}\cos\left(\tfrac{\sqrt{3}}{2}\lambda_n\xi\right)\right)\cos\left(\tfrac{\sqrt{3}}{2}\lambda_n p+\tfrac{\pi}{6}\right)-\right.$$
$$\left.-e^{-\lambda_n\left(\frac{\xi}{2}-p\right)}\left(1+2e^{\frac{-3\lambda_n}{2}p}\cos\left(\tfrac{\sqrt{3}}{2}\lambda_n p\right)\right)\cos\left(\tfrac{\sqrt{3}}{2}\lambda_n\xi+\tfrac{\pi}{6}\right)\right).$$

Putting the found values into (19), we obtain function $G_n\left(x,\xi\right)$ in the form:

$$G_n\left(x,\xi\right)=\begin{cases}G_{1n}\left(x,\xi\right),\ 0\le x<\xi,\\ G_{2n}\left(x,\xi\right),\ \xi<x\le p,\end{cases}$$

here

$$G_{1n}\left(x,\xi\right)=\tfrac{1}{\sqrt{3}\lambda_n^2\Delta}\left(e^{-\lambda_n x}\left(2e^{\lambda_n\left(p-\frac{\xi}{2}\right)}\cos\left(\tfrac{\sqrt{3}}{2}\lambda_n\xi\right)-2e^{\lambda_n\left(\xi-\frac{p}{2}\right)}\cos\left(\tfrac{\sqrt{3}}{2}\lambda_n p\right)\right)-\right.$$
$$-2e^{-\frac{\lambda_n}{2}(p-x)}\left(e^{\lambda_n\xi}+2e^{-\frac{\lambda_n}{2}\xi}\cos\left(\tfrac{\sqrt{3}}{2}\lambda_n\xi\right)\right)\sin\left(\tfrac{\sqrt{3}}{2}\lambda_n\left(p-x\right)+\tfrac{\pi}{6}\right)+$$
$$\left.+2e^{\lambda_n\left(\frac{x}{2}-\frac{\xi}{2}\right)}\left(e^{\lambda_n p}+2e^{\frac{-\lambda_n}{2}p}\cos\left(\tfrac{\sqrt{3}}{2}\lambda_n p\right)\right)\sin\left(\tfrac{\sqrt{3}}{2}\lambda_n\left(\xi-x\right)+\tfrac{\pi}{6}\right)\right),$$

$$G_{2n}(x,\xi)=\frac{1}{\sqrt{3}\lambda_n^2\Delta}\left(e^{\lambda_n\xi}+2e^{-\frac{\lambda_n}{2}\xi}\cos\left(\frac{\sqrt{3}}{2}\lambda_n\xi\right)\right)\left(e^{\lambda_n(p-x)}-2e^{-\frac{\lambda_n}{2}(p-x)}\sin\left(\frac{\sqrt{3}}{2}\lambda_n(p-x)+\frac{\pi}{6}\right)\right).$$

The estimate for $G_n\left(x,\xi\right)$ has the form

$$\left|G_n\left(x,\xi\right)\right|\le\frac{K}{\lambda_n^2},\quad\left|G_{n\xi}\left(x,\xi\right)\right|\le\frac{K}{\lambda_n}.\tag{20}$$

Integrating by parts the second integral in (16) and introducing the notation

$$V_{0n}\left(x\right)=\lambda_n^3\int\limits_0^p G_n\left(x,\xi\right)f_n\left(\xi\right)d\xi,$$

$$\bar{G}_n\left(x,\xi\right)=a_1G_{n\xi}\left(x,\xi\right)-a_2G_n\left(x,\xi\right),$$

then (16) has the form

$$V_n(x) = V_{0n}(x) + \int\limits_0^p \bar{G}_n(x,\xi)\, V_n(\xi)\, d\xi. \tag{21}$$

Equation (21) is the Fredholm integral equation of the second kind. Let us write the solution (21) using the resolvent in the form

$$V_n(x) = V_{0n}(x) + \int\limits_0^p R_n(x,\xi)\, V_{0n}(\xi)\, d\xi,$$

where

$$R_n(x,\xi) = \bar{G}_n(x,\xi) + \sum_{m=1}^{\infty} \bar{G}_{mn}(x,\xi), \tag{22}$$

here

$$\bar{G}_{mn}(x,\xi) = \int\limits_0^p \bar{G}_n(x,s)\, \bar{G}_{(m-1)n}(s,\xi)\, ds, \ \ m = 1, 2, \ldots, \quad \bar{G}_{0n}(x,\xi) = \bar{G}_n(x,s).$$

The following relations are valid for functions $G_n(x,\xi)$, $\bar{G}_n(x,\xi)$

$$\begin{gathered} G_{nxx}(x, x-0) - G_{nxx}(x, x+0) = 1, \\ G_{n\xi\xi}(x, x-0) - G_{n\xi\xi}(x, x+0) = 1, \\ G_{nx\xi}(x, x-0) - G_{nx\xi}(x, x+0) = -1, \end{gathered} \tag{23}$$

$$\begin{gathered} \bar{G}_n(x, x-0) - \bar{G}_n(x, x+0) = 0, \\ \bar{G}_{nx}(x, x-0) - \bar{G}_{nx}(x, x+0) = -a_1, \\ \bar{G}_{nxx}(x, x-0) - \bar{G}_{nxx}(x, x+0) = -a_2, \\ \bar{G}_{nxxx}(x, x-0) - \bar{G}_{nxxx}(x, x+0) = 0. \end{gathered} \tag{24}$$

Let us evaluate solution (22). From

$$R_n(x,\xi) = \bar{G}_{1n}(x,\xi) + \bar{G}_{2n}(x,\xi) + \ldots + \bar{G}_{mn}(x,\xi) + \ldots,$$

let's find the estimate

$$|R_n(x,\xi)| \le \left|\bar{G}_{1n}(x,\xi)\right| + \left|\bar{G}_{2n}(x,\xi)\right| + \ldots + \left|\bar{G}_{mn}(x,\xi)\right| + \ldots, \tag{25}$$

using equality $\bar{G}_n(x,\xi) = a_1 G_{n\xi}(x,\xi) - a_2 G_n(x,\xi)$ taking into account (20), we have an estimate for $\bar{G}_n(x,\xi)$ in the form

$$\left|\bar{G}_n\right| \le |a_1|\,|G_{n\xi}| + |a_2|\,|G_n| \le \left(\frac{1}{\lambda_n} + \frac{1}{\lambda_n^2}\right) M.$$

For the right side of inequality (25), we construct a majorizing series. By entering the designation

$$J = \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_1^2}\right) M,$$

we have

$$\left|\bar G_{1n}\left(x,\xi\right)\right| \le \left|\bar G_{n}\left(x,\xi\right)\right| \le MN\left(\frac{1}{\lambda_n}+\frac{1}{\lambda_n^2}\right) \le \frac{1}{p}Jp,$$

$$\left|\bar G_{2n}\left(x,\xi\right)\right| \le \int\limits_0^p \left|\bar G_{1n}\left(x,s\right)\right| \left|\bar G_{1n}\left(s,\xi\right)\right| ds \le \frac{1}{p}J^2p^2,$$

$$\left|\bar G_{3n}\left(x,\xi\right)\right| \le \int\limits_0^p \left|\bar G_{1n}\left(x,s\right)\right| \left|\bar G_{2n}\left(s,\xi\right)\right| ds \le \frac{1}{p}J^3p^3,$$

. . . .

$$\left|\bar G_{mn}\left(x,\xi\right)\right| \le \int\limits_0^p \left|\bar G_{1n}\left(x,s\right)\right| \left|\bar G_{(m-1)n}\left(s,\xi\right)\right| ds \le \frac{1}{p}J^mp^m,$$

. . . .

Then the majorizing series has the form

$$\frac{1}{p}\sum_{m=1}^{\infty}\left(Jp\right)^m.$$

Condition 3, Theorem 2 can be written as

$$C < \frac{\lambda_1^2}{Kp\left(\lambda_1+1\right)} \quad \Rightarrow \quad \left(\frac{1}{\lambda_1}+\frac{1}{\lambda_1^2}\right)KC < \frac{1}{p},$$

from here

$$Jp < 1,$$

then the majorizing series is the sum of the terms of an infinite decreasing geometric progression. In this case, the resolvent converges uniformly, and its estimate has the form

$$\left|R\left(x,\xi\right)\right| \le \frac{J}{1-Jp} \le M. \tag{26}$$

Substituting $G_n\left(x,\xi\right) = -\frac{1}{\lambda_n^3}G_{n\xi\xi\xi}\left(x,\xi\right)$ into $V_{0n}\left(x\right)$ and integrating, we have

$$V_{0n}\left(x\right) = -f_n\left(x\right) + f_n\left(0\right)G_{2n\xi\xi}\left(x,0\right) - f_n\left(p\right)G_{1n\xi\xi}\left(x,p\right) + \int\limits_0^p G_{n\xi\xi}\left(x,\xi\right)f_n{}'\left(\xi\right)d\xi.$$

Taking into account estimates (11) and

$$\left|G_{2n\xi\xi}\left(x,0\right)\right| \le K, \quad \left|G_{1n\xi\xi}\left(x,p\right)\right| \le K,$$

we get

$$\left|V_{0n}\left(x\right)\right| \le \frac{M}{n^4}\left(1+\left|F_n\left(x\right)\right|+\left|F_n\left(0\right)\right|+\left|F_n\left(p\right)\right|\right) + \frac{M}{n^3}\left(\left|\Psi_{1n}\right|+\left|\Psi_{2n}\right|+\left|\Psi_{3n}\right|\right). \tag{27}$$

From (26) and (27) we obtain the estimate

$$\begin{gathered}\left|V_n\left(x\right)\right| \le \left|V_{0n}\left(x\right)\right| + \int\limits_0^p \left|R\left(x,\xi\right)\right|\left|V_{0n}\left(\xi\right)\right|d\xi \le \\ \le \tfrac{M}{n^4}\left(1+\left|F_n\left(x\right)\right|+\left|F_n\left(0\right)\right|+\left|F_n\left(p\right)\right|\right) + \tfrac{M}{n^3}\left(\left|\Psi_{1n}\right|+\left|\Psi_{2n}\right|+\left|\Psi_{3n}\right|\right).\end{gathered}$$

Due to (7) and (9), the solution to the problem has the form

$$u(x,y)=\sum_{n=1}^{\infty}\left(V_n(x)+\rho_n(x)\right)\cos\left(\frac{\pi n}{q}y\right).$$

Let's check this solution for convergence. Considering the assessment

$$|\rho_n(x)|\le\frac{M}{n^3}\left(|\Psi_{1n}|+|\Psi_{2n}|+|\Psi_{3n}|\right),$$

we have

$$|u(x,y)|\le M\sum_{n=1}^{\infty}\frac{1}{n^4}\left(1+|F_n(x)|+|F_n(0)|+|F_n(p)|\right)+M\sum_{n=1}^{\infty}\frac{1}{n^3}\left(|\Psi_{1n}|+|\Psi_{2n}|+|\Psi_{3n}|\right).$$

Let's show the convergence $u_{xxx}(x,y)$. Taking into account (23) and (24), we find the derivatives of $V_n(x)$ with respect to x of the third order.

$$V'''_n(x)=\lambda_n^3 f_n(x)-a_1\left(V'_{0n}(x)+\int_0^p R_{nx}(x,\xi)V_{0n}(\xi)\,d\xi\right)-$$
$$-a_2\left(V_{0n}(x)+\int_0^p R_n(x,\xi)V_{0n}(\xi)\,d\xi\right)-\lambda_n^3\left(V_{0n}(x)+\int_0^p R_n(x,\xi)V_{0n}(\xi)\,d\xi\right).$$

Using estimate (23) and the properties of the Green's function, we get

$$\left|V'_{0n}(x)\right|\le\frac{M}{n^{\frac{7}{3}}}\left(|\Psi_{1n}|+|\Psi_{2n}|+|\Psi_{3n}|+\frac{|F_n(0)|}{n}+1\right),$$

$$|R_{nx}(x,\xi)|\le n^{\frac{2}{3}}M,$$

next we have

$$\left|V'''_n(x)\right|\le\frac{M^2}{n}\sum_{i=1}^{3}|\Psi_{in}|+\frac{M}{n^2}\left(|F_n(x)|+|F_n(0)|+|F_n(p)|+1\right).$$

From here

$$|u_{xxx}(x,y)|\le\sum_{n=1}^{\infty}\frac{M}{n}\left(|\Psi_{1n}|+|\Psi_{2n}|+|\Psi_{3n}|\right)+\sum_{n=1}^{\infty}O\left(n^{-2}\right).$$

Using the Cauchy-Bunyakovsky and Bessel inequalities, we obtain:

$$|u_{xxx}(x,y)|\le M\left(\sqrt{\sum_{n=1}^{\infty}|\Psi_{1n}|^2}+\sqrt{\sum_{n=1}^{\infty}|\Psi_{2n}|^2}+\sqrt{\sum_{n=1}^{\infty}|\Psi_{3n}|^2}\right)\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^2}}+\sum_{n=1}^{\infty}O\left(n^{-2}\right)\le$$
$$\le M\sqrt{\frac{\pi^2}{6}}\left(\|\psi'''_1(y)\|+\|\psi'''_2(y)\|+\|\psi'''_3(y)\|\right)+\sum_{n=1}^{\infty}O\left(n^{-2}\right)<\infty.$$

Here

$$\sum_{n=1}^{\infty}|\Psi_{in}|^2\le\left\|\psi_i'''\right\|^2_{L_2[0,q]},\ i=\overline{1,3},\quad \sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

Given the inequality

$$|u_{yy}(x,y)|\le|u_{xxx}(x,y)|+|a_1|\,|u_x(x,y)|+|a_2|\,|u(x,y)|,$$

we can conclude that u_{yy} also converge.

From the solution of problems (11) and (13) we obtain a solution to problem A_2 in explicit form:

$$u(x,y)=\frac{1}{\sqrt{q}}\left(\alpha_0(x)+\int\limits_0^p R_0(x,\xi)\,\alpha_0(\xi)\,d\xi.+\rho_0(x)\right)+$$
$$+\sqrt{\frac{2}{q}}\sum_{n=1}^{\infty}\cos\left(\frac{\pi n}{q}y\right)\int\limits_0^p G_n(x,\xi)\,\lambda_n^3 f_n(\xi)\,d\xi+$$
$$+\sqrt{\frac{2}{q}}\sum_{n=1}^{\infty}\cos\left(\frac{\pi n}{q}y\right)\left(\int\limits_0^p R_n(x,\xi)\int\limits_0^p G_n(x,s)\,\lambda_n^3 f_n(s)\,ds d\xi\right)+\sqrt{\frac{2}{q}}\sum_{n=1}^{\infty}\rho_n(x)\cos\left(\frac{\pi n}{q}y\right).$$

Thus, Theorem 2 is proved.

Conclusion

In this paper, we consider a boundary value problem for a third-order inhomogeneous equation with multiple characteristics, containing low-order terms with constant coefficients. The uniqueness and existence of a solution to the problem posed are investigated. Sufficient conditions are found for the coefficients under which the problem posed is uniquely solvable, and in the case of violating these conditions, an example of a nontrivial solution to a homogeneous problem is constructed. The solution to the problem is constructed in the form of a eigenfunctions' series for a one-dimensional spectral problem.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Грин функциясын құра отырып, еселі сипаттамалары бар үшінші ретті біртекті емес теңдеу үшін шеттік есептің шешімі

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Жұмыста тұрақты коэффициенттерімен еселі сипаттамалары бар дербес туындылы үшінші ретті біртекті емес дифференциалдық теңдеу үшін тікбұрышты облыста екінші шеттік есеп қарастырылған. Қойылған есептің шешімінің жалғыздығы энергия интегралдары әдісімен дәлелденді. Жалғыздық теоремасының шарттары бұзылған жағдайға қарсы мысал құрастырылды. Айнымалыларды бөліктеу әдісін қолданып, есептің шешімі $X(x)$ және $Y(y)$ екі функцияның көбейтіндісі ретінде ізделеді. $Y(y)$ анықтау үшін $[0,q]$ сегментінің шекараларында екі шекаралық шарттары бар екінші ретті қарапайым дифференциалдық теңдеуді аламыз. Бұл есеп үшін меншікті мәндері және оған сәйкес $n=0$ және $n \in N$ үшін меншікті функциялары табылды. $X(x)$ анықтау үшін $[0,p]$ сегментінің шекараларында үш шекаралық шарты бар үшінші ретті қарапайым дифференциалдық теңдеуді аламыз. Көрсетілген есептің шешімі Грин функциясы әдісі көмегімен шығарылған. $n=0$ үшін бөлек Грин функциясы және n натурал сан болған жағдай үшін бөлек Грин функциясы құрылды. $X(x)$ үшін шешім құрылған Грин функциясы арқылы жазылған. Кейбір түрлендірулерден кейін шешімі резольвента арқылы жазылған екінші текті интегралды Фредгольм теңдеуі алынды. Резольвента мен Грин функциясы үшін бағалаулар табылды. Шешімнің бірқалыпты жинақтылығы және берілген функцияларда кейбір шарттар үшін мүшелеп дифференциалдану мүмкіндігі дәлелденді. Шешімнің бірқалыпты жинақтылығын негіздеу кезінде «кіші бөлімнің» жоқтығы дәлелденген.

Кілт сөздер: дифференциалдық теңдеу, үшінші рет, еселі сипаттамалар, екінші шеттік есеп, тұрақты шешім, жалғыздық, бар болу, Грин функциясы.

Решение краевой задачи для неоднородного уравнения третьего порядка с кратными характеристиками с построением функции Грина

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В работе рассмотрена вторая краевая задача в прямоугольной области для неоднородного дифференциального уравнения в частных производных третьего порядка с постоянными коэффициентами с кратными характеристиками. Единственность решения поставленной задачи доказана методом интегралов энергии. Построен контрпример в случае нарушения условий теоремы единственности. Используя метод разделения переменных, решение задачи ищется в виде произведения двух функций $X(x)$ и $Y(y)$. Для определения $Y(y)$ получаем обыкновенное дифференциальное уравнение второго порядка с двумя граничными условиями на границах сегмента $[0, q]$. Для этой задачи найдены собственные значения и соответствующие им собственные функции при $n = 0$ и $n \in N$. Для определения $X(x)$ получаем обыкновенное дифференциальное уравнение третьего порядка с тремя граничными условиями на границах сегмента $[0, p]$. Методом функции Грина получено решение указанной задачи. Были построены отдельная функция Грина для $n = 0$ и отдельная функция Грина для случая, когда n – натуральное. Решение для $X(x)$ выписано через построенную функцию Грина. После некоторых преобразований получено интегральное уравнение Фредгольма второго рода, решение которой выписано через резольвенту. Получены оценки резольвенты и функции Грина. Доказаны равномерная сходимость решения и возможность его почленного дифференцирования при некоторых условиях на заданные функции. При обосновании равномерной сходимости решения доказано отсутствие «малого знаменателя».

Ключевые слова: дифференциальное уравнение, третий порядок, кратные характеристики, вторая краевая задача, регулярное решение, единственность, существование, функция Грина.

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