Bulletin of the Karaganda University. Mathematics Series, No. 4(116), 2024, pp. 159–169

https://doi.org/10.31489/2024M4/159-169 *Research article*

Advances in the generalized Cesàro polynomials

N. Özmen[1], E. Erkuş-Duman[2,*]

[1] *Düzce University, Düzce, Türkiye;*
[2] *Gazi University, Ankara, Türkiye*
(E-mail: nejlaozmen@duzce.edu.tr, eerkusduman@gmail.com)

Cesàro polynomials have been extended in various ways and applied in diverse areas. In this paper, we aim to introduce a multivariable and multiparameter generalization of Cesàro polynomials. Then we explore several generating functions, an addition formula, a differential-recurrence relation, a multiple integral formula for this extended Cesàro polynomial, as well as a multiple integral formula whose kernel is this extended Cesàro polynomial. Also we present several bilinear and bilateral generating functions for this extended Cesàro polynomial, two of whose examples are demonstrated.

Keywords: Cesàro polynomials, generating function, recurrence relation, hypergeometric function, integral representation.

2020 Mathematics Subject Classification: 33C45, 33C70.

Introduction

The generalized Cesàro polynomials $g_n^{(s)}(\lambda, x)$ are defined by [1]

$$g_n^{(s)}(\lambda, x) = \binom{s+n}{n} \, {}_2F_1 \left[\begin{array}{cc} -n, & \lambda; \\ & -s-n; \end{array} x \right], \tag{1}$$

where

$$g_n^{(s)}(x) := g_n^{(s)}(1, x), \ \ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \tag{2}$$

are the Cesàro polynomials [2–7]. Here ${}_2F_1$ denotes the hypergeometric function (or Gaussian hypergeometric function) [8]:

$${}_2F_1\left(a, b; c; x\right) = F\left(a, b; c; x\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k,$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol.

The generalized Cesàro polynomials $g_n^{(s)}(\lambda, x)$ in (1) have the following generating function [9]:

$$\sum_{n=0}^{\infty} g_n^{(s)}(\lambda, x) t^n = (1-t)^{-s-1}(1-xt)^{-\lambda}. \tag{3}$$

Recall the following double series manipulations: Let f, $g : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$ be functions and $p \in \mathbb{N}$. Then

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} f(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} f(k, n+pk), \tag{4}$$

*Corresponding author. *E-mail: eerkusduman@gmail.com*
Received: 28 January 2024; *Accepted:* 28 May 2024.

$$\sum_{k=0}^{n} \sum_{j=0}^{[k/p]} g(k,j) = \sum_{j=0}^{[n/p]} \sum_{k=0}^{n-pj} g(k+pj,j), \tag{5}$$

where $[\lambda]$ denotes the integer part of $\lambda \in \mathbb{R}$.

Cesàro polynomials have been generalized in various ways and used in diverse areas [1–7], [10; 62]. For example, Malik [11] has introduced Cesàro polynomials in two and three variables and has given their generating functions. In this paper, we provide a multivariable and multiparameter generalization of Cesàro polynomials. Then we investigate several generating functions, an addition formula, a differential-recurrence relation, a multiple integral formula for this extended Cesàro polynomial, as well as a multiple integral formula whose kernel is this extended Cesàro polynomial. Also we explore several bilinear and bilateral generating functions for this extended Cesàro polynomial, two examples of which are considered.

1 *Multivariable and multiparameter Cesàro polynomials*

In this section, we define a multivariable and multiparameter extension of the generalized Cesàro polynomials $g_n^{(s)}(\lambda, x)$ in (1) and obtain their generating functions. Also, we derive several properties for these polynomials.

Definition 1. Let $m \in \mathbb{N}$; $n \in \mathbb{N}_0$; $s \in \mathbb{C} \setminus \mathbb{N}_0$; λ_j, $x_j \in \mathbb{C}$ $(j = 1, \ldots, m)$. Then an m variable and m parameter extension of the generalized Cesàro polynomials is defined by

$$g_n^{(s)}(\lambda_1, \ldots, \lambda_m; x_1, \ldots, x_m) := \sum_{r_1+\cdots+r_m=n} \binom{s+n}{n} \frac{(-n)_{\delta_m}}{(-s-n)_{\delta_m}} \prod_{j=1}^{m} \frac{(\lambda_j)_{r_j}\, x_j^{r_j}}{(r_j)!}, \tag{6}$$

where

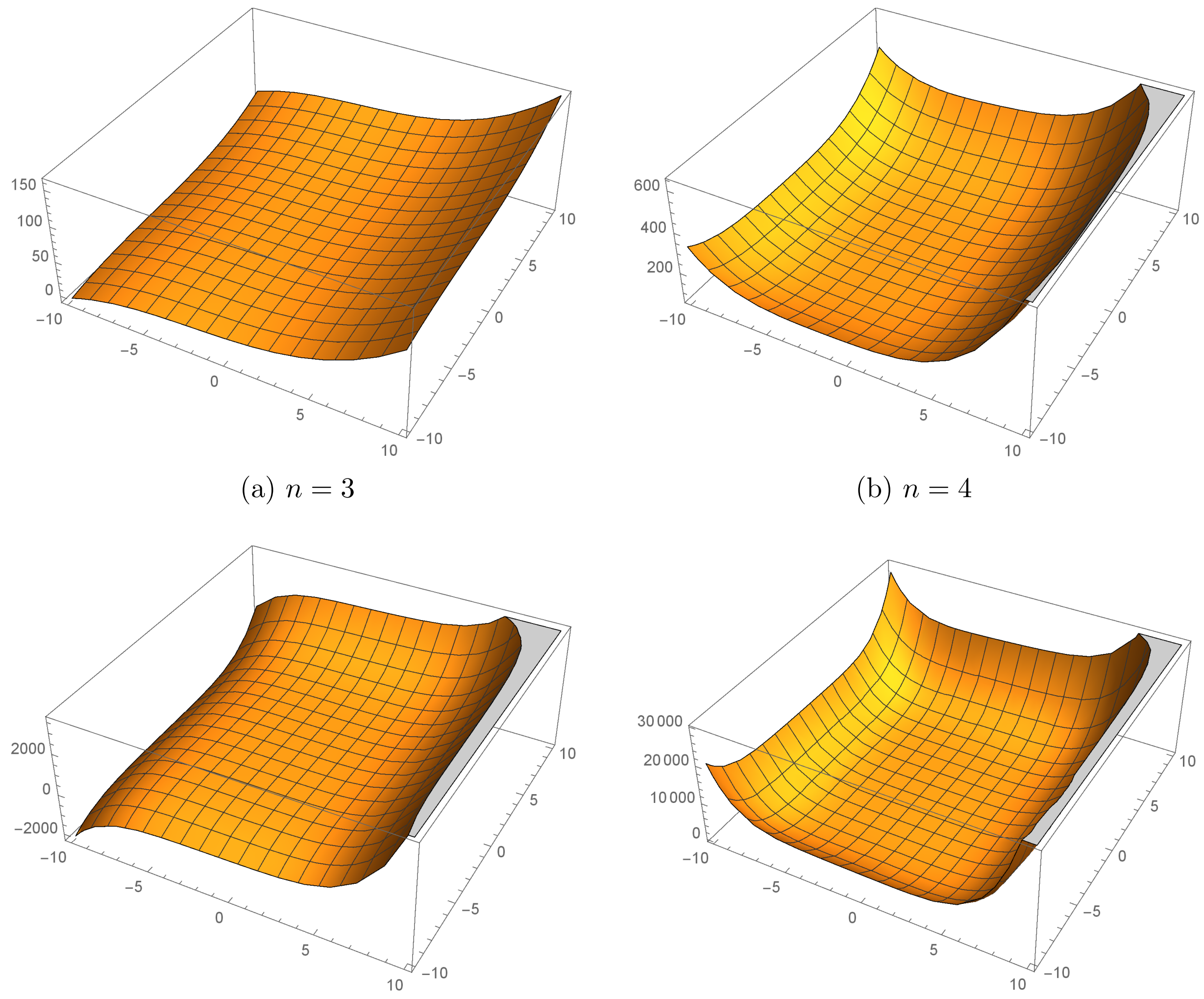
$$\delta_m := r_1 + \cdots + r_m. \tag{7}$$

The summation notation $\sum\limits_{r_1+\cdots+r_m=n}$ in (6) represents the following m-ple series:

$$\sum_{r_1+\cdots+r_m=n} = \sum_{r_1=0}^{n} \sum_{r_2=0}^{n-r_1} \cdots \sum_{r_m=0}^{n-r_1-\cdots-r_{m-1}}. \tag{8}$$

Figure demonstrates the surfaces of the generalized Cesàro polynomials $g_n^{(s)}(\lambda_1, \lambda_2, x_1, x_2)$ in two variables for some parameter values. We should remark that the special case of $m = 1$ in (6) immediately reduces to the generalized Cesàro polynomials $g_n^{(s)}(\lambda, x)$ in (1). Also if we take $\lambda_j = 1$ $(j = 1, \ldots, m)$ in (6), we get the following multivariable generalization of the Cesàro polynomials $g_n^{(s)}(x)$ in (2):

$$g_n^{(s)}(x_1, \ldots, x_m) := \sum_{r_1+\cdots+r_m=n} \binom{s+n}{n} \frac{(-n)_{\delta_m}}{(-s-n)_{\delta_m}} \prod_{j=1}^{m} x_j^{r_j}.$$



(a) $n = 3$ (b) $n = 4$

(c) $n = 5$ (d) $n = 6$

Figure. Surfaces of the generalized Cesàro polynomials $g_n^{(s)}(\lambda_1, \lambda_2, x_1, x_2)$ in two variables for the parameter values $s = 4$, $\lambda_1 = 1/10$, $\lambda_2 = 1/20$ and $n = 3, 4, 5, 6$

In the study of special functions, a theoretical relationship to the unification of generating functions is critical. Several researchers have made strides in this approach [12–14].

The following theorems present two generating function relations for the multivariable-multiparameter Cesàro polynomials in (6).

Theorem 1. The multivariable-multiparameter generalized Cesàro polynomials in (6) are generated by the following function:

$$\sum_{n=0}^{\infty} g_n^{(s)}(\lambda_1, \dots, \lambda_m; x_1, \dots, x_m)\, t^n = (1-t)^{-s-1} \prod_{j=1}^{m} (1 - x_j t)^{-\lambda_j}, \tag{9}$$

where $|t| < \min\left\{ |x_1|^{-1}, \dots, |x_m|^{-1}, 1 \right\}$ and $m \in \mathbb{N}$.

Proof. Let $\mathcal{L}_1$ be the left member of (9).

Replacing the $g_n^{(s)}(\lambda_1,\ldots,\lambda_m;x_1,\ldots,x_m)$ with (6) and (8), we get

$$\begin{aligned}\mathcal{L}_1 = & \sum_{n=0}^{\infty}\sum_{r_1=0}^{n}\sum_{r_2=0}^{n-r_1}\cdots\sum_{r_m=0}^{n-r_1-\cdots-r_{m-1}}\binom{s+n}{n} \\ & \times \frac{(-n)_{\delta_m}}{(-s-n)_{\delta_m}}\prod_{j=1}^{m}\frac{(\lambda_j)_{r_j}\;x_j^{r_j}}{(r_j)!}\,t^n.\end{aligned} \tag{10}$$

Employing the case $p=1$ of (4) in the first double sums in (10) gives

$$\begin{aligned}\mathcal{L}_1 = & \sum_{r_1=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r_2=0}^{n}\sum_{r_3=0}^{n-r_2}\cdots\sum_{r_m=0}^{n-r_2-\cdots-r_{m-1}}\binom{s+n+r_1}{n+r_1} \\ & \times \frac{(-n-r_1)_{\delta_m}}{(-s-n-r_1)_{\delta_m}}\prod_{j=1}^{m}\frac{(\lambda_j)_{r_j}\;x_j^{r_j}}{(r_j)!}\,t^{n+r_1}.\end{aligned} \tag{11}$$

Applying the same procedure as in getting (11) to the 2nd and 3rd double sums (11), and repeating the similar process, we find

$$\mathcal{L}_1 = \sum_{n=0}^{\infty}\sum_{r_1=0}^{\infty}\cdots\sum_{r_m=0}^{\infty}\binom{s+n+\delta_m}{n+\delta_m}\frac{(-n-\delta_m)_{\delta_m}}{(-s-n-\delta_m)_{\delta_m}}\prod_{j=1}^{m}\frac{(\lambda_j)_{r_j}\;x_j^{r_j}}{(r_j)!}\,t^{n+\delta_m}, \tag{12}$$

where δ_m is given in (7).

Consider the following easily-derivable identity:

$$(-m-p)_p = (-1)^p\,\frac{(m+p)!}{m!} \qquad (m,\,p\in\mathbb{N}_0)\,. \tag{13}$$

Employing (13) in (12) offers

$$\begin{aligned}\mathcal{L}_1 &= \sum_{n=0}^{\infty}\binom{s+n}{n}t^n\sum_{r_1=0}^{\infty}...\sum_{r_m=0}^{\infty}\prod_{j=1}^{m}\frac{(\lambda_j)_{r_j}\;x_j^{r_j}}{(r_j)!}\,t^{\delta_m} \\ &= \sum_{n=0}^{\infty}(s+1)_n\,\frac{t^n}{n!}\sum_{r_1=0}^{\infty}\frac{(\lambda_1)_{r_1}\,(x_1t)^{r_1}}{r_{1!}}\cdots\sum_{r_m=0}^{\infty}\frac{(\lambda_m)_{r_m}\,(x_mt)^{r_m}}{r_{m!}}.\end{aligned} \tag{14}$$

Using the generalized binomial theorem

$$(1-z)^{-\alpha} = \sum_{n=0}^{\infty}(\alpha)_n\,\frac{z^n}{n!} \qquad (|z|<1,\,\alpha\in\mathbb{C})$$

in each sum of the 2nd equality in (14), we arrive at the right member of (9).

Remark 1. The case $m=1$ of the generating function relation (9) reduces to the generating function relation (3).

Theorem 2. The multivariable-multiparameter generalized Cesàro polynomials in (6) are generated by the following generating function:

$$\begin{aligned}&\sum_{n=0}^{\infty}\binom{n+k}{n}g_{n+k}^{(s)}(\lambda_1,\ldots,\lambda_m;x_1,\ldots,x_m)\,t^n = (1-t)^{-s-k-1} \\ &\times\prod_{j=1}^{m}(1-x_jt)^{-\lambda_j}g_k^{(s)}\left(\lambda_1,\ldots,\lambda_m;\frac{x_1(1-t)}{1-x_1t},\cdots,\frac{x_m(1-t)}{1-x_mt}\right),\end{aligned} \tag{15}$$

where $m \in \mathbb{N}$, $k \in \mathbb{N}_0$, and $|t| < \min\left\{|x_1|^{-1}, \ldots, |x_m|^{-1}, 1\right\}$.

Proof. Replacing t by $t+u$ in (9) gives

$$\sum_{n=0}^{\infty} g_n^{(s)}(\lambda_1, \ldots, \lambda_m; x_1, \ldots, x_m)(t+u)^n = (1-t-u)^{-s-1} \prod_{j=1}^{m} (1 - x_j t - x_j u)^{-\lambda_j},$$

which, upon using binomial theorem, yields

$$\begin{aligned} \sum_{n=0}^{\infty} g_n^{(s)}(\lambda_1, \ldots, \lambda_m; x_1, \ldots, x_m) \sum_{k=0}^{n} \binom{n}{k} t^{n-k} u^k = (1-t)^{-s-1} \\ \times \left(1 - \frac{u}{1-t}\right)^{-s-1} \prod_{j=1}^{m} (1-x_j t)^{-\lambda_j} \left(1 - \frac{x_j u}{1 - x_j t}\right)^{-\lambda_j}. \end{aligned} \tag{16}$$

Using (9) on the right member of (16), with the aid of the case $p = 1$ of (4), offers

$$\begin{aligned} &\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{n+k}{k} g_{n+k}^{(s)}(\lambda_1, \ldots, \lambda_m; x_1, \ldots, x_m) t^n u^k \\ = \;& (1-t)^{-s-1} \prod_{j=1}^{m} (1-x_j t)^{-\lambda_j} \\ & \times \sum_{k=0}^{\infty} g_k^{(s)} \left(\lambda_1, \ldots, \lambda_m; \frac{x_1(1-t)}{1-x_1 t}, \ldots, \frac{x_m(1-t)}{1-x_m t}\right) \left(\frac{u}{1-t}\right)^k, \end{aligned}$$

which, upon equating the coefficients of u^k on both sides, yields the desired identity (15).

Theorem 3. The following identity holds true:

$$\begin{aligned} & g_n^{(s_1+s_2+1)}(\lambda_1 + \mu_1, \ldots, \lambda_m + \mu_m; x_1, \ldots, x_m) \\ & = \sum_{k=0}^{n} g_{n-k}^{(s_1)}(\lambda_1, \ldots, \lambda_m; x_1, \ldots, x_m) g_k^{(s_2)}(\mu_1, \ldots, \mu_m; x_1, \ldots, x_m). \end{aligned} \tag{17}$$

Proof. From (9), we find

$$\begin{aligned} & \sum_{n=0}^{\infty} g_n^{(s_1+s_2+1)}(\lambda_1 + \mu_1, \ldots, \lambda_m + \mu_m; x_1, \ldots, x_m) t^n \\ & = (1-t)^{-s_1-s_2-2} \prod_{j=1}^{m} (1-x_j t)^{-\lambda_j - \mu_j} \\ & = \sum_{n=0}^{\infty} g_n^{(s_1)}(\lambda_1, \ldots, \lambda_m; x_1, \ldots, x_m) t^n \sum_{k=0}^{\infty} g_k^{(s_2)}(\mu_1, \ldots, \mu_m, x_1, \ldots, x_m) t^k \\ & = \sum_{n=0}^{\infty} \sum_{k=0}^{n} g_{n-k}^{(s_1)}(\lambda_1, \ldots, \lambda_m; x_1, \ldots, x_m) g_k^{(s_2)}(\mu_1, \ldots, \mu_m; x_1, \ldots, x_m)\, t^n. \end{aligned}$$

Matching the coefficients of the first and last members yields the desired identity (17).

Theorem 4. The following differential-recurrence relation holds true:

$$\begin{aligned} &\frac{\partial}{\partial x_{j_0}} g_{n+1}^{(s)}(\lambda_1,\ldots,\lambda_m;x_1,\ldots,x_m) \\ &= \lambda_{j_0} g_n^{(s)}(\lambda_1,\ldots,\lambda_{j_0}+1,\lambda_{j_0+1},\lambda_m;x_1,\ldots,x_m), \end{aligned} \tag{18}$$

where $1 \le j_0 \le m$.

Proof. We will prove, when $j_0 = 1$. By symmetry, it will be easy to interpret the result into the general $1 \le j_0 \le m$.

Differentiating both sides of (9) with respect to x_1, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{\partial}{\partial x_1} g_n^{(s)}(\lambda_1,\ldots,\lambda_m;x_1,\ldots,x_m)\, t^{n-1} \\ &= \lambda_1 (1-t)^{-s-1} \left[(1-x_1t)^{-\lambda_1-1} \prod_{j=2}^{m} (1-x_jt)^{-\lambda_j} \right], \end{aligned}$$

which, upon using (9), yields

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{\partial}{\partial x_1} g_{n+1}^{(s)}(\lambda_1,\ldots,\lambda_m;x_1,\ldots,x_m)\, t^{n} \\ &= \lambda_1\, g_n^{(s)}(\lambda_1+1,\lambda_2,\ldots,\lambda_m;x_1,\ldots,x_m)\, t^n. \end{aligned} \tag{19}$$

Equating the coefficients of t^n on both sides of (19) leads to the identity (18) when $j_0 = 1$.

Integrating both sides of (6) with respect to each of the variables x_j $(j = 1, \ldots, m)$ from 0 to 1 gives the result in the following theorem.

Theorem 5. Let $m \in \mathbb{N}$; $n \in \mathbb{N}_0$; $s \in \mathbb{C} \setminus \mathbb{N}_0$; $\lambda_j \in \mathbb{C}$ $(j = 1, \ldots, m)$. Then

$$\begin{aligned} &\int_0^1 \cdots \int_0^1 g_n^{(s)}(\lambda_1,\ldots,\lambda_m;x_1,\ldots,x_m)\, dx_1 \cdots dx_m \\ &= \sum_{r_1+\cdots+r_m=n} \binom{s+n}{n} \frac{(-n)_{\delta_m}}{(-s-n)_{\delta_m}} \prod_{j=1}^{m} \frac{(\lambda_j)_{r_j}}{(r_j+1)!}, \end{aligned}$$

where δ_m is the same as in (7).

The following theorem provides an integral representation of the multivariable-multiparameter generalized Cesàro polynomials.

Theorem 6. Let $m \in \mathbb{N}$; $n \in \mathbb{N}_0$; $s \in \mathbb{C} \setminus \mathbb{N}_0$; $\lambda_j, x_j \in \mathbb{C}$ $(j = 1, \ldots, m)$. Also let $\Re(s+1) > 0$, $\Re(\lambda_j) > 0$ $(j = 1, \ldots, m)$. Then

$$\begin{aligned} &g_n^{(s)}(\lambda_1,\ldots,\lambda_m;x_1,\ldots,x_m) \\ &= \frac{1}{n!\,\Gamma(s+1) \prod_{j=1}^{m} \Gamma(\lambda_j)} \int_0^{\infty} \cdots \int_0^{\infty} e^{-(u+u_1+\cdots+u_m)} \\ &\times \left(u + \sum_{j=1}^{m} u_j x_j\right)^n u^s u_1^{\lambda_1-1} \cdots u_m^{\lambda_m-1} du du_1 \cdots du_m. \end{aligned} \tag{20}$$

Proof. Recall that the well-known identity as

$$c^{-v}=\frac{1}{\Gamma(v)}\int\limits_0^\infty e^{-ct}t^{v-1}dt \quad (\Re(c)>0,\,\Re(\nu)>0)\,. \tag{21}$$

Using (21) in each factor of the right member of (9), under the restrictions in Theorem 1, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} g_n^{(s)}(\lambda_1,\ldots,\lambda_m,x_1,\ldots,x_m)t^n\\
&=\frac{1}{\Gamma(s+1)}\int\limits_0^\infty e^{-(1-t)u}u^s du\frac{1}{\Gamma(\lambda_1)}\int\limits_0^\infty e^{-(1-x_1t)u_1}u_1^{\lambda_1-1}du_1\\
&\times\cdots\frac{1}{\Gamma(\lambda_m)}\int\limits_0^\infty e^{-(1-x_mt)u_m}u_m^{\lambda_m-1}du_m\\
&=\frac{1}{\Gamma(s+1)\Gamma(\lambda_1)\cdots\Gamma(\lambda_m)}\int\limits_0^\infty\cdots\int\limits_0^\infty e^{-(u+u_1+\cdots+u_m)}u^s u_1^{\lambda_1-1}\cdots u_m^{\lambda_m-1}\\
&\times\sum_{n=0}^{\infty}\frac{(u+u_1x_1+\cdots+u_mx_m)^n}{n!}du du_1\cdots du_m\,t^n.
\end{aligned}$$

Equating the coefficients of t^n on the first and last members of the last resulting identity yields the desired integral representation (20).

2 Miscellaneous generating function relations

Now, we obtain new substantial families of bilinear and bilateral generating function relations for the multivariable-multiparameter generalized Cesàro polynomials in (6).

Throughout this section, let $m,\,p,\,q,\,r\in\mathbb{N};\ l\in\mathbb{N}_0;\ \ \mu,\,\nu\in\mathbb{C};\ a_k\in\mathbb{C}\setminus\{0\}\ (k\in\mathbb{N}_0)$. Also let

$$\Omega_\mu:\mathbb{C}^r\longrightarrow\mathbb{C}\setminus\{0\}$$

be a bounded function.

Theorem 7. Let

$$\Lambda_{\mu,\nu}(y_1,\ldots,y_r;\eta):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\nu k}(y_1,\ldots,y_r)\,\eta^k$$

and

$$\begin{aligned}
&\Theta_{n,p}^{\mu,\nu}\left(\lambda_1,\ldots,\lambda_m,x_1,\ldots,x_m;y_1,\ldots,y_r;\xi\right)\\
&:=\sum_{k=0}^{[n/p]}a_k g_{n-pk}^{(s)}\left(\lambda_1,\ldots,\lambda_m,x_1,\ldots,x_m\right)\Omega_{\mu+\nu k}(y_1,\ldots,y_r)\,\xi^k.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{n=0}^{\infty}\Theta_{n,p}^{\mu,\nu}\left(\lambda_1,\ldots,\lambda_m,x_1,\ldots,x_m;y_1,\ldots,y_r;\tfrac{\eta}{t^p}\right)t^n\\
&=(1-t)^{-s-1}\prod_{j=1}^{m}(1-x_jt)^{-\lambda_j}\Lambda_{\mu,\nu}(y_1,\ldots,y_r;\eta).
\end{aligned} \tag{22}$$

Proof. Let $\mathcal{L}_2$ be the left member of (22). Then we have

$$\mathcal{L}_2 = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k g_{n-pk}^{(s)}\left(\lambda_1,\ldots,\lambda_m,x_1,\ldots,x_m\right)\Omega_{\mu+\nu k}(y_1,\ldots,y_r)\eta^k t^{n-pk}.$$

Using (4), we obtain

$$\begin{aligned}\mathcal{L}_2 &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k\, g_n^{(s)}\left(\lambda_1,\ldots,\lambda_m,x_1,\ldots,x_m\right)\Omega_{\mu+\nu k}(y_1,\ldots,y_r)\eta^k t^n \\ &= \sum_{n=0}^{\infty} g_n^{(s)}\left(\lambda_1,\ldots,\lambda_m,x_1,\ldots,x_m\right)t^n \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(y_1,\ldots,y_r)\eta^k \\ &= (1-t)^{-s-1}\prod_{j=1}^{m}(1-x_jt)^{-\lambda_j}\Lambda_{\mu,\nu}(y_1,\ldots,y_r;\eta),\end{aligned}$$

which is the right member of (22).

Theorem 8. Let

$$N_{n,l,q}^{\mu,p}(y_1,\ldots,y_r;z) := \sum_{k=0}^{[n/q]}\binom{l+n}{n-qk} a_k \Omega_{\mu+pk}(y_1,\ldots,y_r)z^k. \tag{23}$$

Also let

$$\begin{aligned}&\Lambda_{l,q}^{\mu,p}\left[\lambda_1,\ldots,\lambda_m,x_1,\ldots,x_m;y_1,\ldots,y_r;t\right] \\ &\quad := \sum_{n=0}^{\infty} a_n g_{l+qn}^{(s)}(\lambda_1,\ldots,\lambda_m,x_1,\ldots,x_m)\Omega_{\mu+pn}(y_1,\ldots,y_r)\,t^n.\end{aligned}$$

Then

$$\begin{aligned}&\sum_{n=0}^{\infty} g_{l+n}^{(s)}(\lambda_1,\ldots,\lambda_m,x_1,\ldots,x_m)N_{n,l,q}^{\mu,p}(y_1,\ldots,y_r;z)t^n \\ &= (1-t)^{-s-l-1}\prod_{j=1}^{m}(1-x_jt)^{-\lambda_j} \\ &\times \Lambda_{l,q}^{\mu,p}\left[\lambda_1,\ldots,\lambda_m,\frac{x_1(1-t)}{1-x_1t},\ldots,\frac{x_m(1-t)}{1-x_mt};y_1,\ldots,y_r;z\Big(\frac{t}{1-t}\Big)^q\right].\end{aligned} \tag{24}$$

Proof. Let $\mathcal{L}_3$ be the left member of (24). Using (23), we have

$$\mathcal{L}_3 = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/q]} g_{l+n}^{(s)}(\lambda_1,\ldots,\lambda_m,x_1,\ldots,x_m)\binom{l+n}{n-qk} a_k \Omega_{\mu+pk}(y_1,\ldots,y_r)z^k\, t^n.$$

Employing (4), in view of the result in Theorem 2, we may write

$$\begin{aligned}\mathcal{L}_3 &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} g_{l+n+qk}^{(s)}(\lambda_1,\ldots,\lambda_m,x_1,\ldots,x_m)\binom{l+n+qk}{n} \\ &\quad\times a_k \Omega_{\mu+pk}(y_1,\ldots,y_r)z^k t^{n+qk}\end{aligned}$$

$$
\begin{aligned}
&= (1-t)^{-s-l-1}\prod_{j=1}^{m}(1-x_jt)^{-\lambda_j}\\
&\quad\times\sum_{k=0}^{\infty}a_k g_{l+qk}^{(s)}\left(\lambda_1,\ldots,\lambda_m,\frac{x_1(1-t)}{1-x_1t},\ldots,\frac{x_m(1-t)}{1-x_mt}\right)\\
&\quad\times\Omega_{\mu+pk}(y_1,\ldots,y_r)\frac{z^k t^{qk}}{(1-t)^{qk}}\\
&= (1-t)^{-s-l-1}\prod_{j=1}^{m}(1-x_jt)^{-\lambda_j}\\
&\quad\times\Lambda_{l,q}^{\mu,p}\left[\lambda_1,\ldots,\lambda_m,\frac{x_1(1-t)}{1-x_1t},\ldots,\frac{x_m(1-t)}{1-x_mt};y_1,\ldots,y_r;z\left(\frac{t}{1-t}\right)^q\right],
\end{aligned}
$$

which is the right member of (24).

Theorem 9. Let

$$
\begin{aligned}
&\Lambda_{\mu,\nu}^{n,p}(\lambda_1+\beta_1,\ldots,\lambda_m+\beta_m,x_1,\ldots,x_m;y_1,\ldots,y_r;z)\\
&:=\sum_{k=0}^{[n/p]}a_k g_{n-pk}^{(s_1+s_2+1)}(\lambda_1+\beta_1,\ldots,\lambda_m+\beta_m,x_1,\ldots,x_m)\Omega_{\mu+\nu k}(y_1,\ldots,y_r)z^k.
\end{aligned}
$$

Then, for $n\in\mathbb{N}_0$, we have

$$
\begin{aligned}
&\sum_{k=0}^{n}\sum_{l=0}^{[k/p]}a_l g_{n-k}^{(s_1)}(\lambda_1,\ldots,\lambda_m,x_1,\ldots,x_m)g_{k-pl}^{(s_2)}(\beta_1,\ldots,\beta_m,x_1,\ldots,x_m)\\
&\qquad\times\Omega_{\mu+\nu l}(y_1,\ldots,y_r)z^l\\
&\qquad=\Lambda_{\mu,\nu}^{n,p}(\lambda_1+\beta_1,\ldots,\lambda_m+\beta_m,x_1,\ldots,x_m;y_1,\ldots,y_r;z).
\end{aligned}
\tag{25}
$$

Proof. Let $\mathcal{L}_4$ be the left member of (25). Using (5) and then using addition formula (17) for the multivariable-multiparameter generalized Cesàro polynomials, we get

$$
\begin{aligned}
\mathcal{L}_4 &= \sum_{l=0}^{[n/p]}\sum_{k=0}^{n-pl}a_l g_{n-k-pl}^{(s_1)}(\lambda_1,\ldots,\lambda_m,x_1,\ldots,x_m)g_k^{(s_2)}(\beta_1,\ldots,\beta_m,x_1,\ldots,x_m)\\
&\quad\times\Omega_{\mu+\nu l}(y_1,\ldots,y_r)z^l\\
&= \sum_{l=0}^{[n/p]}a_l g_{n-pl}^{(s_1+s_2+1)}(\lambda_1+\beta_1,\ldots,\lambda_m+\beta_m,x_1,\ldots,x_m)\Omega_{\mu+\nu l}(y_1,\ldots,y_r)z^l\\
&= \Lambda_{\mu,\nu}^{n,p}(\lambda_1+\beta_1,\ldots,\lambda_m+\beta_m,x_1,\ldots,x_m;y_1,\ldots,y_r;z),
\end{aligned}
$$

which is the right member of (25).

3 Concluding remarks and examples

We proposed an extension of Cesàro polynomials to several variables and parameters. Then we investigated several generating functions, an addition formula, a differential-recurrence relation, a multiple integral formula for this extended Cesàro polynomial, as well as a multiple integral formula kernel of which is this extended Cesàro polynomial. Also we explored several bilinear and bilateral generating functions for this extended Cesàro polynomial, two examples of which are demonstrated in Examples 1 and 2.

Since the multivariable function $\Omega_{\mu+\nu k}(y_1,\ldots,y_r)$ is very general, we may deduce a number of particular formulas from the results in Sections 1 and 2. We just use Theorem 7 to present the following two examples.

Example 1. The Bessel function $J_\mu(x)$ are generated by (see, e.g., [15; p. 141])

$$\left(1-\frac{2t}{x}\right)^{-\mu/2} J_\mu\left(\sqrt{x^2-2xt}\right)=\sum_{n=0}^{\infty} J_{\mu+n}(x)\frac{t^n}{n!}. \tag{26}$$

If we take $r=1$, $a_k=\frac{1}{k!}$, $\nu=1$ and substitute the Bessel function for $\Omega_{\mu+\nu k}$ in Theorem 7, using the relation (26), we can obtain the following result providing a class of bilateral generating function relation for the multivariable generalized Cesàro polynomials and the Bessel functions:

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} g_{n-pk}^{(s)}(\lambda_1,\ldots,\lambda_m,x_1,\ldots,x_m)\,J_{\mu+k}(y)\eta^k t^{n-pk}\\&=\left(1-\frac{2\eta}{y}\right)^{-\mu/2} J_\mu\left(\sqrt{y^2-2y\eta}\right)(1-t)^{-s-1}\prod_{j=1}^{m}(1-x_jt)^{-\lambda_j}.\end{aligned}$$

Example 2. Taking $r=m$, $a_k=1$, $\mu=0$, $\nu=1$ and substituting the multivariable-multiparameter generalized Cesàro polynomials for $\Omega_{\mu+\nu k}$ in Theorem 7, and using the generating relation (9), we may get the following class of bilinear generating functions for the multivariable-multiparameter generalized Cesàro polynomials:

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} g_{n-pk}^{(s)}(\lambda_1,\ldots,\lambda_m,x_1,\ldots,x_m)\\&\qquad\times g_k^{(s)}(\beta_1,\ldots,\beta_m,y_1,\ldots,y_m)\,\eta^k t^{n-pk}\\&=[(1-t)(1-\eta)]^{-s-1}\prod_{j=1}^{m}(1-x_jt)^{-\lambda_j}(1-y_j\eta)^{-\beta_j},\end{aligned}$$

where each variable, each parameter, and each index can be suitably restricted so that this formula is meaningful.

Obviously, many other particular cases of Theorem 7 can be provided. Further, the results in the other theorems in Sections 1 and 2 can reduce to yield a variety of identities about the extended Cesàro polynomials (6) and their simpler ones.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Lin, S.-D., Liu, S.-J., Lu, H.-C., & Srivastava, H.M. (2011). Integral representations for the generalized Bedient polynomials and the generalized Cesáro polynomials. *Appl. Math. Comput., 218*(4), 1330–1341. https://doi.org/10.1016/j.amc.2011.06.016

2 Agarwal, A.K., & Manocha, H.L. (1980). On some new generating functions. *Mat. Vesnik, 4(17)*(32), 395–402.

3 Basu, D.K. (1976). On sets generated by $A(t)\psi(xt)$. *C. R. Acad. Bulgare Sci., 29*(8), 1107–1110.

4 Chatterjea, S.K. (1977). An extension of a class of bilateral generating functions for certain special functions. *Bull. Inst. Math. Acad. Sinica, 5*(2), 323–331.

5 Chihara, T.S. (1978). *An Introduction to Orthogonal Polynomials*. Gordon and Breach Science Publishers. Inc. New York, London, Paris.

6 Dai, F., & Ge, Y. (2021). Sharp estimates of the Cesàro kernels for weighted orthogonal polynomial expansions in several variables. *J. Funct. Anal., 280*(4), ID 108865. https://doi.org/10.1016/j.jfa.2020.108865

7 Özmen, N. (2019). New generating function relations for the q-generalized Cesáro polynomials. *J. Funct. Spaces*, Art. ID 3829620, 1–7. https://doi.org/10.1155/2019/3829620

8 Rainville, E.D. (1960). *Special Functions*, Macmillan Company, New York.

9 Özmen N., & Erkus-Duman, E. (2018). Some families of generating functions for the generalized Cesàro polynomials. *J. Comput. Anal. Appl., 25*(4), 670–683.

10 Debnath, L., & Bhatta, D. (2015). *Integral Transforms and Their Applications*. Third Edi. CRC Press, Taylor & Francis Group, Boca Raton, London, New York.

11 Malik, M.A. (2013). A note on generating functions of Cesàro polynomials of several variables. *Mat. Vesnik, 65*(4), 425–430.

12 Korkmaz-Duzgun D., & Erkus-Duman, E. (2019). Generating functions for k-hypergeometric functions. *Int. J. Appl. Phys. Math., 9*(3), 119–126. https://doi.org/10.17706/ijapm.2019.9.3.119-126

13 Liu, S.-J. (2009). Bilateral generating functions for the Lagrange polynomials and the Lauricella functions. *Integral Transforms Spec. Funct., 20*(7), 519–527. https://doi.org/10.1080/10652460802568291

14 Liu, S.-J., Lin, S.-D., Srivastava, H.M., & Wong, M.-M. (2012). Bilateral generating functions for the Erkuş-Srivastava polynomials and the generalized Lauricella functions. *Appl. Math. Comput., 218*(15), 7685–7693. https://doi.org/10.1016/j.amc.2012.01.005

15 Watson, G.N. (1944). *A Treatise on the Theory of Bessel Functions*. 2nd Ed. Cambridge Univ. Press, Cambridge, London and New York.

*Author Information**

Nejla Özmen — Associate Professor, Düzce University, Faculty of Art and Science, Department of Mathematics, Konuralp TR-81620, Düzce, Türkiye; e-mail: *nejlaozmen@duzce.edu.tr*; https://orcid.org/0000-0001-7555-1964

Esra Erkuş-Duman — Full Professor, Gazi University, Faculty of Science, Department of Mathematics, Teknikokullar TR-06500, Ankara, Türkiye; e-mail: *eerkusduman@gmail.com*; https://orcid.org/0000-0003-1848-5248

*The author's name is presented in the order: First, Middle and Last Names.