On a method for constructing the Green function of the Dirichlet problem for the Laplace equation

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The study of boundary value problems for elliptic equations is of both theoretical and applied interest. A thorough study of model physical and spectral problems requires an explicit and effective representation of the problem solution. Integral representations of solutions of problems of differential equations are one of the main tools of mathematical physics. Currently, the integral representation of the Green function of classical problems for the Laplace equation for an arbitrary domain is obtained only in a two-dimensional domain by the Riemann conformal mapping method. Starting from the three-dimensional case, these classical problems are solved only for spherical sectors and for the regions lying between the faces of the hyperplane. The problem of constructing integral representations of general boundary value problems and studying their spectral problems remains relevant. In this work, using the boundary condition of the Newtonian (volume) potential and the spectral property of the potential of a simple layer, the Green function of the Dirichlet problem for the Laplace equation was constructed.

Keywords: Laplace equation, Green function, Dirichlet problem, simple layer potential.

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Introduction

Let $\Omega \subset R^n$ is a bounded domain with a smooth boundary $\partial\Omega$.

The Dirichlet problem. Find in Ω the solution $u(x)$ of the Laplace equation

$$-\Delta_x u = f(x), \quad x \in \Omega,$$

satisfying the boundary condition

$$u|_{x \in \partial\Omega} = 0.$$

The function $G(x, y), x, y \in \Omega$ is called the Green function of the Dirichlet problem if

$$-\Delta_x G(x, y) = 0, \quad x \in \Omega, \quad G(x, y)|_{x \in \partial\Omega, y \in \Omega} = 0.$$

The solution of the Dirichlet problem using the Green's function $G(x, y)$ is representable in the following integral form

$$u(x) = \int_\Omega G(x, y) f(y) dy.$$

In the two-dimensional case, the method of conformal mapping of the analytical function is used to construct the Green's function. Starting from the three-dimensional case, the construction of the Green function is carried out by the method of Fredholm integral equations of the second kind, or by

the method of maps, which are ineffective. Therefore, in multidimensional cases, $G(x,y)$ is constructed only for spherical sectors and for half-spaces.

In this paper, we present a method for constructing the Green function, which essentially uses the boundary properties of the Newtonian potential (volume potential) and the spectral properties of the trace of the potential of a simple layer.

By $u = L_N^{-1} f$ we shall call the Newtonian potential (volumetric potential)

$$u(x) = L_N^{-1} f = \int\limits_{\Omega} \varepsilon(x,y) f(y) dy, \tag{1}$$

where $\varepsilon(x,y)$ is the fundamental solution of the Laplace equation

$$-\Delta_x \varepsilon(x,y) = \delta(x,y), \tag{2}$$

the function $\varepsilon(x,y)$ in (2) has the following form

$$\varepsilon(x,y) = \begin{cases} -\frac{1}{2\pi} \ln |x-y|, & n = 2, \\ \frac{1}{\omega_n (n-2)|x-y|^{n-2}}, & n > 2. \end{cases} \tag{3}$$

Next, we will use the following statement from the work of T.Sh. Kal'menov, D. Suragan [1].

Theorem 1. The Newtonian potential $u(x) \in W_2^2(\Omega)$ at $x \in \Omega$ satisfies the Laplace equation

$$-\Delta_x u = f(x) \tag{4}$$

and the boundary condition

$$-\frac{u(x)}{2} + \int\limits_{\partial\Omega} \left(\frac{\partial \varepsilon}{\partial n_y}(x-y) u(y) - \varepsilon(x-y) \frac{\partial u(y)}{\partial n_y} \right) dS_y = 0, \quad x \in \partial\Omega. \tag{5}$$

Inversely, if $u \in W_2^2(\Omega)$ satisfies equation (3) and boundary condition (4), then $u(x)$ coincides with the Newtonian potential (1).

Note that in the work of the Saito [2] it is also established that $u(x) = L_N^{-1} f(x)$ satisfies the boundary condition (4). In contrast to the work of the Saito, in our work it was found that if the solution satisfies equation (3) and boundary condition (4), it coincides with the Newton potential $u(x) = L_N^{-1} f(x)$.

It follows from Theorem 1 that the Green function of problem (3)-(4) in an arbitrary domain is the fundamental solution $\varepsilon(x,y)$.

Similarly, the lateral boundary conditions of the wave and heat potentials are found in [3–6].

Let $-\Delta_0$ be the closure in $L_2(\Omega)$ of the differential operator $-\Delta$ on subset of functions $u \in C^{2+\alpha}(\overline{\Omega})$, $u|_{\partial\Omega} = \frac{\partial u}{\partial n}|_{\partial\Omega} = 0$, and $-\Delta_0^*$ is its adjoint operator in $L_2(\Omega)$.

The operator L is called a correct restriction if

$$L \subset -(\Delta_0)^*, \ L^{-1} \text{ is invertible on all } L_2(\Omega).$$

Correct restriction L of the operator $-(\Delta_0)^*$ we call a regular boundary extension if

$$-\Delta_0 \subset L, \ \|L^{-1}\|_{L_2(\Omega) \to L_2(\Omega)} < \infty.$$

The description of correct boundary value problems for general elliptic operators by the method of regular extensions of operators in Hilbert space is given by M.M. Vishik [7], and the description of correct restrictions for maximal operators is given by M.O. Otelbaev [8].

Next, we look for regular solutions of equation (3) in the form

$$u(x) = \int\limits_{\Omega} \varepsilon(x,y)f(y)dy + \int\limits_{\partial\Omega} \nu(\xi)\varepsilon(x,\xi)dS_\xi, \tag{6}$$

where

$$u_\nu(x) = \int\limits_{\partial\Omega} \nu(\xi)\varepsilon(x,\xi)dS_\xi \tag{7}$$

is the potential of a simple layer, and $\nu(\xi)$ is the density of the potential of the simple layer (6).

Suppose first that $\nu(x) \in C(\partial\Omega)$ and for each $x \in \partial\Omega$, and $\nu(x)$ is a linear continuous functional of $f(x) \in L_2(\Omega)$, i.e. $\nu(x) = \nu(x, f)$.

According to Riesz's theorem, $\nu(\xi, f)$ is representable as

$$\nu(\xi) = \nu(\xi, f) = \int\limits_{\Omega} \tilde{q}(\xi, y)f(y)dy, \tag{8}$$

where $\tilde{q}$ is continuous over $\xi \in \partial\Omega$ and $\tilde{q}(\xi, y) \in L_2(\Omega)$ over variable $y \in \Omega$, i.e.,

$$\|\tilde{q}(\xi, y)\|_{L_2(\Omega) \cap C(\partial\Omega)} = \|\nu(\xi)\|_{C(\partial\Omega)}.$$

Substituting the right part (7) into the formula (6), we get

$$u_\nu(x) = \int\limits_{\partial\Omega} \varepsilon(x,\xi) \int\limits_{\Omega} \tilde{q}(\xi, y)f(y)dydS_\xi =$$

$$= \int\limits_{\Omega} f(y) \int\limits_{\partial\Omega} \varepsilon(x,\xi)\tilde{q}(\xi, y)dS_\xi dy = \int\limits_{\Omega} \tilde{q}(x,y)f(y)dy,$$

$$q(x,y) = \int\limits_{\partial\Omega} \varepsilon(x,\xi)\tilde{q}(\xi, y)dS_\xi. \tag{9}$$

Thus, the operator

$$u_\nu = \mathcal{L}^{-1}f = \int\limits_{\Omega} q(x,y)f(y)dy, \quad x \in \Omega$$

converts an arbitrary function $f \in L_2(\Omega)$ to $\ker\triangle_0^*$, i.e. $-\triangle_y\mathcal{L}^{-1}f \equiv 0$.

Now we will rewrite the integral operator (5) in the form

$$u(x) = L_R^{-1}f = \int\limits_{\Omega} (\varepsilon(x,y) + q(x,y))f(y)dy.$$

By construction $-\Delta u = f(x)$. Therefore, the operator $u = L_R^{-1}f$ is a correct restriction of the maximal operator $-\Delta_0^*$, i.e. a invertible generalized solution of equation (3).

Remark. It is not difficult to establish that in the representation (8) we can consider $\tilde{q} \in L_2(\partial\Omega) \cap L_2(\Omega)$.

According to the theory of correct restrictions generated by integral operators (T.Sh. Kal'menov, M. Otelbaev [9]), a correct restriction of L_R^{-1} generates regular boundary operators if and only if adjoint to (L_R^{-1}) operator $(L_R^{-1})^*$ is a correct restriction, i.e. the operator

$$(L_R^{-1})^* g = \int\limits_{\Omega} \varepsilon(y,x)g(x)dx + \int\limits_{\Omega} q(x,y)g(x)dx$$

is a correct restriction.

According to [8], this can only be the case when

$$-\Delta_y q(x, y) = 0,$$

i.e.

$$-\Delta_y \left[\int_{\partial\Omega} \varepsilon(x, \xi) q_0(\xi, y) dS_\xi \right] = -\int_{\partial\Omega} \varepsilon(x, \xi) \Delta_y \widetilde{q}(\xi, y) dS_\xi = 0. \tag{10}$$

The following statement takes place

Lemma 1. The trace of the potential operator of a simple layer on $\partial\Omega$, given by the integral

$$(D_S^{-1}\nu)(x) = \int_{\partial\Omega} \varepsilon(x, \xi)\nu(\xi) dS_\xi, \quad x \in \partial\Omega$$

is a completely continuous self-adjoint operator in $L_2(\Omega)$ and its kernel $\varepsilon(x, \xi), x, \xi \in \partial\Omega$ is represented as

$$\varepsilon(x, \xi) = \sum_{|m|=1}^{\infty} \frac{e_m(x)e_m(\xi)}{\lambda_m},$$

where $e_m(x)$ is a complete orthonormal system of eigenfunctions of the operator D_S^{-1} corresponding to the eigenvalues of $\frac{1}{\lambda_m}$.

Indeed, from $\varepsilon(x, \xi) = \varepsilon(\xi, x)$ and its weak divergence on $\partial\Omega$ follows the validity of Lemma 1.

It is easy to check that

$$D_S^{-1}e_m(x) = \frac{e_m(x)}{\lambda_m}, \quad D_S e_m(x) = \lambda_m e_m(x), \tag{11}$$

where D_S is inverse operator to D_S^{-1}.

Using Fourier series expansions

$$\varepsilon(x, \xi) = \sum_{|m|=1}^{\infty} \frac{e_m(x)e_m(\xi)}{\lambda_m}, \quad x \in \partial\Omega, \quad \xi \in \partial\Omega$$

and

$$-\Delta_y \tilde{g}(\xi, y) = -\sum_{|m|=1}^{\infty} \frac{(-\Delta_y \tilde{g})_m(y)e_m(\xi)}{\lambda_m}, \quad y \in \partial\Omega, \quad \xi \in \partial\Omega,$$

$$(-\Delta_y g(y))_m = \int_{\partial\Omega} (-\Delta_y g(\xi, y))e_m(\xi) dS_\xi.$$

From the equality (9) at $x \in \partial\Omega$ it follows that $(-\Delta_y \tilde{g}(y))_m = 0$, which is equivalent to $-\Delta_y \tilde{g}(\xi, y) \equiv 0$.

In particular,

$$\int_{\partial\Omega} \varepsilon(x, \xi)\Delta_y \tilde{g}(\xi, y) dS_\xi = 0, \quad x \in \Omega.$$

Now we are looking for the Green function $G(x, y)$ in the form

$$G(x, y) = \varepsilon(x, y) - \int_{\partial\Omega} \varepsilon(x, \xi)D_S \tilde{q}(\xi, y) dS_\xi. \tag{12}$$

Since $G(x,y) \equiv G(y,x)$, it follows from (11) that

$$G(x,y) = \varepsilon(x,y) - \int\limits_{\partial\Omega} \varepsilon(x,\xi)D_S\tilde{q}(\xi,y)dS_\xi =$$

$$= \varepsilon(y,x) - \int\limits_{\partial\Omega} \tilde{q}(\xi,x)D_S\varepsilon(y,\xi)dS_\xi =$$

$$= \varepsilon(x,y) - \int\limits_{\partial\Omega} \tilde{q}(\xi,x)D_S\varepsilon(y,\xi)dS_\xi.$$

It follows that

$$\tilde{q}(\xi,x) = \varepsilon(x,\xi).$$

Therefore,

$$G(x,y) = \varepsilon(x,y) - \int\limits_{\partial\Omega} \varepsilon(x,\xi)D_S\varepsilon(y,\xi)dS_\xi. \tag{13}$$

From (12) it is easy to verify that

$$-\Delta_x G = \delta(x-y),$$
$$-\Delta_x \int\limits_{\partial\Omega} \varepsilon(x,\xi)D_S\varepsilon(y,\xi)dS_\xi = -\Delta_y \int\limits_{\partial\Omega} \varepsilon(x,\xi)D_S\varepsilon(y,\xi)dS_\xi = 0, \ x \in \Omega, \ y \in \Omega. \tag{14}$$

It takes place
Lemma 2. The following equality is true

$$- \int\limits_{x\in\partial\Omega, y\in\Omega} \varepsilon(x,\xi)D_S\varepsilon(y,\xi)dS_\xi = -\varepsilon(y,x) = -\varepsilon(x,y). \tag{15}$$

Proof. Let us set

$$\widetilde{e_m}(y) = \int\limits_{\partial\Omega} \varepsilon(y,\xi)e_m(\xi)dS_\xi, \quad y \in \Omega,$$

it is obvious that

$$-\Delta_y \widetilde{e_m}(y) = 0, \quad y \in \Omega,$$

$$\varepsilon(y,\xi) = \sum_{|m|=1}^{\infty} \widetilde{e_m}(y)e_m(\xi). \tag{16}$$

By construction

$$\varepsilon(y,\xi)|_{y\in\partial\Omega} = \sum_{|m|=1}^{\infty} \frac{e_m(y)e_m(\xi)}{\lambda_m}.$$

Taking into account the formula (10), from (15) we obtain

$$D_S\varepsilon(y,\xi) = \sum_{|m|=1}^{\infty} \lambda_m \widetilde{e_m}(y)e_m(\xi), \quad y \in \Omega.$$

Based on (16) and the ratio

$$\varepsilon(x,\xi) = \sum_{|m|=1}^{\infty} \frac{e_m(x)e_m(\xi)}{\lambda_m}, \quad x \in \partial\Omega,$$

from (14) at $y \in \Omega$ we have

$$-\int_{\partial\Omega} \varepsilon(x,\xi)D_S\varepsilon(y,\xi)dS_\xi \bigg|_{x\in\partial\Omega,\ y\in\Omega} =$$

$$= -\left(\sum_{|m|=1}^{\infty} \frac{e_m(x)e_m(\xi)}{\lambda_m}, \sum_{|\bar{m}|=1}^{\infty} \lambda_m\widetilde{e}_{\bar{m}}(y)e_{\bar{m}}(\xi)\right)_{L_2(\partial\Omega)} =$$

$$= -\sum_{|m|=1}^{\infty} e_m(x)\widetilde{e_m}(y) = -\varepsilon(y,x) = -\varepsilon(x,y), \quad y \in \Omega, \ \ x \in \partial\Omega.$$

Using this, from (11) we will make sure that

$$G(x,y)|_{x\in\partial\Omega} = \varepsilon(x,y) - \int_{\partial\Omega} \varepsilon(x,y)D_S\varepsilon(y,\xi)d\xi_y =$$

$$= \varepsilon(x,y) - \varepsilon(x,y)|_{y\in\Omega, x\in\partial\Omega} = 0.$$

Lemma 2 is proved.

Equality (13) and Lemma 2 follow

Theorem 2. The Green function $G(x,y)$ of the Dirichlet problem is given by the formula

$$G(x,y) = \varepsilon(x,y) - \int_{\partial\Omega} \varepsilon(x,\xi)D_S\varepsilon(y,\xi)dS_\xi,$$

where $\varepsilon(x,y)$ is the fundamental solution of the Laplace equation, and D_S is the operator defined by the formula (10).

Conflict of Interest

The author declares no conflict of interest.

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Лаплас теңдеуі үшін Дирихле есебінің Грин функциясының интегралдық көрсетілімі туралы

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Эллиптикалық теңдеулерге арналған шеткі есептерді зерттеу теориялық және қолданбалы қызығушылық тудырады. Модельдік физикалық және спектрлік есептерді мұқият зерттеу үшін есептің шешімін нақты және тиімді ұсыну қажет. Дифференциалдық теңдеулер есептерінің шешімдерінің интегралдық көрсетілімі математикалық-физиканың негізгі құралдарының бірі. Қазіргі уақытта еркін аймақ үшін Лаплас теңдеуі үшін классикалық есептердің Грин функциясының интегралды көрсетілімі Риманның конформды бейнелеу әдісімен тек екі өлшемді аймақта алынды. Үш өлшемді жағдайдан бастап, бұл классикалық есептер тек шар секторлары үшін және гиперсызықтықтың беттері арасында орналасқан аймақтар үшін шешіледі. Жалпы шеткі есептердің интегралды көрсетілімін құру және олардың спектрлік мәселелерін зерттеу мәселесі өзекті болып қала береді. Жұмыста ньютондық (көлемдік) потенциалдың шекаралық шартын және қарапайым қабат потенциалының спектрлік қасиеттерін пайдалана отырып, Лаплас теңдеуі үшін Дирихле есебінің Грин функциясы құрастырылған.

Кілт сөздер: Лаплас теңдеуі, Грин функциясы, Дирихле есебі, қарапайым қабат потенциалы.

Об интегральном представлении функции Грина задачи Дирихле для уравнения Лапласа

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Изучение краевых задач для эллиптических уравнений представляет и теоретический, и прикладной интерес. Для тщательного изучения модельных физических и спектральных задач требуется явное и эффективное представление решения задачи. Интегральные представления решений задач дифференциальных уравнений являются одними из основных инструментов математической физики. В настоящее время интегральное представление функции Грина классических задач для уравнения Лапласа для произвольной области получено только в двумерной области методом конформного отображения Римана. Начиная с трехмерного случая, эти классические задачи решены только для шаровых секторов и для областей, лежащих между гранями гиперплоскости. Вопрос построения интегральных представлений общих краевых задач и изучения их спектральных проблем остается актуальным. В работе, пользуясь граничным условием ньютонового (объемного) потенциала и спектральным свойством потенциала простого слоя, построена функция Грина задачи Дирихле для уравнения Лапласа.

Ключевые слова: уравнение Лапласа, функция Грина, задача Дирихле, потенциал простого слоя.

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