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Conditions for maximal regularity of solutions to fourth-order differential equations

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This article investigates a fourth-order differential equation defined in a Hilbert space, with an unbounded intermediate coefficient and potential. The key distinction from previous research lies in the fact that the intermediate term of the equation does not obey to the differential operator formed by its extreme terms. The study establishes that the generalized solution to the equation is maximally regular, if the intermediate coefficient satisfies an additional condition of slow oscillation. A corresponding coercive estimate is obtained, with the constant explicitly expressed in terms of the coefficients' conditions. Fourth-order differential equations appear in various models describing transverse vibrations of homogeneous beams or plates, viscous flows, bending waves, and etc. Boundary value problems for such equations have been addressed in numerous works, and the results obtained have been extended to cases with smooth variable coefficients. The smoothness conditions imposed on the coefficients in this study are necessary for the existence of the adjoint operator. One notable feature of the results is that the constraints only apply to the coefficients themselves; no conditions are placed on their derivatives. Secondly, the coefficient of the lowest order in the equation may be zero, moreover, it may not be unbounded from below.

Keywords: fourth-order differential equation, unbounded coefficient, solution, existence, uniqueness, smoothness, operator, separability, regularity, coercive estimate.

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1 Introduction. Formulation of the problem

Fourth-order differential equations describe various physical phenomena, such as transversal oscillations of homogeneous beams or plates, viscoelastic and inelastic flows, bending waves, and other [1, 2]. The issues of existence and uniqueness of solutions to boundary value problems posed for linear and nonlinear fourth-order differential equations have been studied extensively in the literature [3–5]. In the case of an infinite domain, the Cauchy problem for a fourth-order waves equation is considered in [6]. However, in these works, the coefficients of the equations are either constant or assumed to be bounded functions. Additionally, when investigating nonlinear equations, excessively strict restrictions are imposed on the coefficients to ensure the uniqueness of solutions [3–5]. In light of both theoretical and practical needs, there is a growing relevance in studying the solvability of fourth-order differential equations with variable coefficients and relaxing constraints on these coefficients. This concern is particularly pertinent to differential equations with independently growing coefficients that are given in an infinite domain.

Consider the following fourth-order differential equation defined on the real line:

$$L_0 y = y^{(4)} + p(x)\, y^{(3)} + q(x)\, y = F(x), \tag{1}$$

where $x \in \mathbb{R} = (-\infty, \infty)$, $p(x) > 0$, $p(x) \in C^{(3)}_{loc}(\mathbb{R})$, $q(x)$ is a continuous function, and $F(x) \in L_2(\mathbb{R})$.

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Let L denote the closure in the $L_2(\mathbb{R})$ norm of the operator

$$L_0 y = y^{(4)} + p(x)\, y^{(3)} + q(x)\, y$$

defined on the set $C_0^{(4)}(\mathbb{R})$ of continuously differentiable up to the fourth order functions with compact support. A solution to equation (1) is an element $y \in D(L)$ satisfying the equality $Ly = F$.

Our goal is to establish conditions sufficient for the fulfillment of the inequality

$$\left\| y^{(4)} \right\|_2 + \left\| p y^{(3)} \right\|_2 + \|(1+|q|)y\|_2 \leq C\left(\|F\|_2 + \|y\|_2\right), \tag{2}$$

for a solution y, where $\|\cdot\|_2$ denotes the norm of the $L_2(\mathbb{R})$ space. Inequality (2) is referred to as a coercive estimate or an estimate of maximal regularity of the solution.

The equation (1) has been primarily studied in the case of $p = 0$ [7]. In addition, if $q \geq \delta > 0$, then (1) is a unique solvable. And if the oscillation of q satisfies certain additional conditions, then the inequality (2) is satisfied for a solution of (1). However, when $p(x)$ is a non-zero, rapidly growing function, the method of [7] is inapplicable. This is because the operator $p\frac{d^3}{dx^3}$ may not obey $\frac{d^4}{dx^4} + q(x)\, E$ (E is the identity operator). For the sake of completeness, we provide statements about the existence and uniqueness of solutions with proofs.

The aforementioned problem of unique and coercive solvability has been addressed in [8, 9] for second-order differential equations with rapidly growing intermediate coefficients, and in [10] for third-order differential equations. In [11], the authors developed an effective method for investigating the spectrum of a degenerate symmetric fourth-order differential operator. We build upon the ideas of the last four works. Unique and coercive solvability of various types of singular differential equations with intermediate coefficients is studied in [12–15].

In what follows, by C we will denote positive constants, which may have, in general, different values in the different places.

2 On an auxiliary binomial differential equation

Let us consider the operator $l_0 y = y^{(4)} + p(x)\, y^{(3)}$, $D(l_0) = C_0^{(4)}(\mathbb{R})$. We denote its closure in $L_2(\mathbb{R})$ by l.

Lemma 1. Suppose the function $p(x) \in C_{loc}^{(3)}(\mathbb{R})$ such that

$$p(x) \geq \varepsilon > 0. \tag{3}$$

Then, for any $y \in C_0^{(4)}(\mathbb{R})$, the following estimate holds

$$\left\| \sqrt{p}\, y^{(3)} \right\|_2 \leq \left\| \frac{l_0 y}{\sqrt{p}} \right\|_2. \tag{4}$$

Proof. Let $y \in C_0^{(4)}(\mathbb{R})$. We concider the scalar product $A = \left(l_0 y, y^{(3)}\right)$. Since y is a function with compact support, the following equalities hold:

$$A = \int_{-\infty}^{\infty} y^{(4)}(x) y^{(3)}(x)\, dx + \int_{-\infty}^{\infty} p(x) \left[y^{(3)}(x)\right]^2 dx = \int_{-\infty}^{\infty} p(x) \left[y^{(3)}(x)\right]^2 dx. \tag{5}$$

On the other hand, using condition (3) and the Holder inequality, we obtain:

$$A \leq \left(\int_{-\infty}^{\infty} |l_0 y|^2 \frac{1}{p(x)}\, dx\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} p(x) \left|y^{(3)}\right|^2 dx\right)^{\frac{1}{2}}.$$

From this and (5), inequality (4) follows. The right-hand side of (4) is bounded under the condition (3).

Let $\rho(t)$ and $v(t) \neq 0$ be given continuous functions, and k is a natural number. We introduce the following notations:

$$\alpha_{\rho,v,k} = \sup_{x>0} \left(\int_0^x \rho^2(t)dt \right)^{\frac{1}{2}} \left(\int_x^{\infty} t^{2(k-1)2} v^{-2}(t)\, dt \right)^{\frac{1}{2}},$$

$$\beta_{\rho,v,k} = \sup_{x<0} \left(\int_x^0 \rho^2(s)ds \right)^{\frac{1}{2}} \left(\int_{-\infty}^{x} s^{2(k-1)} v^{-2}(s)\, ds \right)^{\frac{1}{2}},$$

$$\gamma_{\rho,v,k} = \max\left(\alpha_{\rho,v,k}, \beta_{\rho,v,k}\right).$$

Lemma 2. [11] If functions $\rho(t)$ and $v(t)$ satisfy the relation

$$\gamma_{\rho,v,k} < \infty \, (k \in \mathbb{N}),$$

then for each $f \in C_0^{(k)}(\mathbb{R})$ the following inequality holds:

$$\|\rho f\|_2 \leq \frac{2}{(k-1)!} \gamma_{\rho,v,k} \left\| v f^{(k)} \right\|_2 .$$

Lemma 3. Suppose the function $p(x)$ satisfies condition (3) and $\gamma_{1,\sqrt{p},3} < \infty$. Then the operator l is invertible, and for each $y \in D(l)$, the inequality holds

$$\|y\|_2 + \left\| \sqrt{p} y^{(3)} \right\|_2 \leq C \, \|ly\|_2 . \tag{6}$$

Proof. Let $y \in C_0^{(3)}(\mathbb{R})$. According to the condition $\gamma_{1,\sqrt{p},3} < \infty$, Lemma 2, and estimate (4), we obtain the following inequalities:

$$\|y\|_2 \leq C \left\| \sqrt{p} y^{(3)} \right\|_2 \leq C \left\| \frac{l_0 y}{\sqrt{p}} \right\|_2 .$$

By (3), we have

$$\left\| \sqrt{p} y^{(3)} \right\|_2 \leq \sqrt{\varepsilon} \, \|l_0 y\|_2 \tag{7}$$

and

$$\|y\|_2 \leq C \sqrt{\varepsilon} \, \|l_0 y\|_2 . \tag{8}$$

Since $D(l_0) = C_0^{(3)}(\mathbb{R})$ and l is the closure of the operator l_0, from (7) and (8), the inequalities

$$\left\| \sqrt{p} y^{(3)} \right\|_2 \leq \sqrt{\varepsilon} \, \|ly\|_2$$

and

$$\|y\|_2 \leq C \sqrt{\varepsilon} \, \|ly\|_2$$

follow for each $y \in D(l)$, respectively. Combining them yields (6).

Consider the equation

$$ly = y^{(4)} + p(x)\, y^{(3)} = f(x). \tag{9}$$

An element $y \in D(l)$ satisfying $ly = f$ is called a solution to (9).

Lemma 4. Suppose that the conditions of Lemma 3 hold for $p(x)$. Then the solution to equation (9) is unique.

Proof. If y and z are two solutions to equation (9), then by definition, $y, z \in D(l)$ and $ly = f$, $lz = f$. For $v = y - z$, we have $lv = 0$. Then, by inequality (6), $\|v\| = 0$, i.e., $y = z$.

Lemma 5. Suppose that the conditions of Lemma 3 hold for $p(x)$. Then, for any $f(x) \in L_2(\mathbb{R})$, a solution to equation (9) exists.

Proof. According to Lemma 3, the operator l is invertible. It suffices to show that its range $R(l)$ coincides with the entire space $L_2(\mathbb{R})$. By Lemma 3, $y^{(3)} \in L_2(\mathbb{R})$, if $y \in D(l)$. Let $y^{(3)} = z$ and $\Theta z = z' + p(x) z$. Then $z \in L_2(\mathbb{R})$, and equation (9) takes the form:

$$\Theta z = z' + pz = f \in L_2(\mathbb{R}).$$

The equality $R(l) = R(\Theta)$ holds. Indeed,

$$R(\Theta) = \{v \in L_2(\mathbb{R}) : \exists z \in D(\Theta),\ \Theta z = v\} =$$

$$= \{v \in L_2(\mathbb{R}) : \exists y \in D(l),\ ly = v\} = R(l).$$

According to (6), $R(\Theta)$ is a closed set. It suffices to demonstrate that $R(\Theta) = L_2(\mathbb{R})$. Let us assume the opposite. Suppose that $R(\Theta) \neq L_2(\mathbb{R})$. Then there exists a non-zero element $w \in L_2(\mathbb{R})$, which is orthogonal to the set $R(\Theta)$: $(w, \Theta z) = 0, z \in D(\Theta)$. Since $(w, \Theta z) = (\Theta^* w, z)$, and the set $D(\Theta)$ is dense in $L_2(\mathbb{R})$, the function $w \in D(\Theta^*)$ satisfies the following homogeneous equation:

$$\Theta^* w = w - w' = 0.$$

Therefore, as $p(x)$ is continuous, it follows that $w' \in L_{2,loc}(\mathbb{R})$, then $w \in W^1_{2,loc}(\mathbb{R})$. Consequently, the function $w(x)$ is continuous, and

$$|w(x)| = |c|\, e^{\int_a^x p(t)\,dt},\ \forall x \in \mathbb{R}.$$

Hence, $|w(x)| \geq |c|$ for $x \geq a$, we obtain $w \notin L_2(\mathbb{R})$. This leads to a contradiction, demonstrating that $R(\Theta) = L_2(\mathbb{R})$.

3 Conditions for the separability of a binomial operator

Let $\lambda \in \mathbb{R}_+ = [0, +\infty)$. Consider the following differential operator $\Theta_{0\lambda} z = z' + (p + \lambda) z$, $D(\Theta_{0\lambda}) = C_0^{(1)}(\mathbb{R})$. Its closure in the space $L_2(\mathbb{R})$ we denote by Θ_λ.

Definition 1. It is said that the operator Θ_λ is separable in the space $L_2(\mathbb{R})$, if for any $z \in D(\Theta_\lambda)$, the following inequality holds:

$$\left\|z'\right\|_2 + \|pz\|_2 + \lambda \|z\|_2 \leq C(\|\Theta_\lambda z\|_2 + \|z\|_2). \tag{10}$$

It is evident that the operator Θ_λ is separable in the space $L_2(\mathbb{R})$, if and only if there exists $\mu \in \mathbb{R}$ such that the operator $\Theta_{\lambda+\mu} = \Theta_\lambda + \mu E$ is separable in this space.

Lemma 6. Let the coefficient p satisfy the conditions of Lemma 3 and the following relation:

$$\sup_{x,\eta \in \mathbb{R}, |x-\eta| \leq 1} \frac{p(x)}{p(\eta)} < \infty. \tag{11}$$

Then, the operator Θ_λ is separable in $L_2(\mathbb{R})$.

Proof. Let us observe that the conditions of Lemma 3 remain valid for the function p, and $\lambda \geq 0$. According to Lemma 4 and Lemma 5, the inverse operator $\Theta_\lambda^{-1}(\lambda \geq 0)$ exists and is continuous. We will now demonstrate that the operator Θ_λ is separable for at least one $\lambda \geq 0$.

Let $\Delta_j = [j, j+1)$, $\Omega_j = \left(j - \frac{1}{2}, j + \frac{3}{2}\right)(j \in Z)$. We choose the functions $\varphi_j(x)$ $(j \in Z)$ from $C_0^\infty(\Omega_j)\,(j \in Z)$, satisfying the following conditions:

a) $0 \leq \varphi_j(x) \leq 1$, $\varphi_j(x) = 1\ \forall x \in \Delta_j$, $\sup\limits_{x \in \Omega_j} \max\limits_{j \in Z} \left|\varphi_j'(x)\right| \leq M$.

Then

$$\acute{\Delta}_j \subset \Omega_j \subset \Delta_{j-1} \cup \Delta_j \cup \Delta_{j+1}, \Delta_j \cap \Delta_k = \emptyset\,(j \neq k),$$

$$\Omega_j \cap \Omega_m = \emptyset\ \ (|j - m| \geq 2), \sum_{j=-\infty}^{\infty} \varphi_j(x)\,\chi_{\Delta_j}(x) = 1.$$

Here χ_{Δ_j} is a characteristic function of Δ_j. Recall that the sequence $\{\varphi_j(x)\}_{j=-\infty}^{\infty}$, satisfying conditions a), exists [7].

Let $p_j(x)\,(j \in Z)$ be the extension to the entire $\mathbb{R}$ of the restriction in $\Omega_j\,(j \in Z)$ of the function $p(x)$ such that

$$\frac{1}{2} \inf_{z \in \Omega_j} p(z) \leq p_j(x) \leq 2 \sup_{z \in \Omega_j} p(z),\ \ x \in \mathbb{R}. \tag{12}$$

According to condition (11), such an extension exists [7]. Let

$$\widetilde{\theta}_{j,\lambda} z = z' + (p_j + \lambda)\, z,\ \ z \in C_0^{(1)}(\mathbb{R}).$$

Denote the closure of the operator $\widetilde{\theta}_{j,\lambda}$ in the space $L_2(\mathbb{R})$ as $\theta_{j,\lambda}$. By Lemma 3, for any $z \in D(\theta_{j,\lambda})$, we have

$$\left\|\sqrt{p_j + \lambda} z\right\|_2 \leq \left\|\sqrt{\frac{1}{p_j + \lambda}} \theta_{j,\lambda} z\right\|_2.$$

Then,

$$\|z\|_2 \leq \frac{1}{\mathop{inf}\limits_{x \in \mathbb{R}} (p_j(x) + \lambda)} \|\theta_{j,\lambda} z\|_2\ \ (j \in Z). \tag{13}$$

In particular, based on (3) and (12), we obtain

$$\|z\|_2 \leq \frac{2}{\varepsilon + 2\lambda} \|\theta_{j,\lambda} z\|_2\ \ \ (j \in Z). \tag{14}$$

Therefore, the operator $\theta_{j,\lambda}$ is invertible. Due to Lemma 5, the operator $\theta_{j,\lambda}^{-1}\,(j \in Z)$ is continuous. Let $f \in C_0^{(1)}(\mathbb{R})$. Consider the following operators M_λ and B_λ:

$$M_\lambda f = \sum_j \varphi_j \theta_{j,\lambda}^{-1}\left(\chi_{\Delta_j} f\right),\ B_\lambda f = \sum_j \varphi_j' \theta_{j,\lambda}^{-1}\left(\chi_{\Delta_j} f\right).$$

Since f is a function with compact support, the number of terms in the sums on the right-hand side of the last equalities is finite. By our choice, for $z \in \Omega_j$, the equality $\Theta_\lambda z = \theta_{j,\lambda} z$, $z \in D(\Theta_\lambda)$, holds. Considering this and the properties of the function $\varphi_j \in C_0^\infty(\Omega_j)$, we can easily demonstrate the equality

$$\Theta_\lambda(M_\lambda f) = (B_\lambda + E)\, f. \tag{15}$$

Note that the multiplicity of the intersection of intervals $\Omega_j\,(j\in Z)$ is at most two. Therefore, the following inequalities hold:

$$\|B_\lambda f\|_2^2=\sum_{j=-\infty}^{\infty}\int_{\Delta_j}|B_\lambda f|^2\,dx\le\sum_{j=-\infty}^{\infty}\int_{\Delta_j}\left[\sum_{k=j-1}^{j+1}\left|\varphi_k'(x)\right|\left|\theta_{k,\lambda}^{-1}\left(\chi_{\Delta_k}f\right)\right|\right]^2dx\le$$

$$\le 3\sum_{j=-\infty}^{\infty}\int_{\Delta_j}\sum_{k=j-1}^{j+1}\left|\varphi_k'\left(x\right)\right|^2\left|\theta_{k,\lambda}^{-1}\left(\chi_{\Delta_k}f\right)\right|^2dx\le 3M^2\sum_{j=-\infty}^{\infty}\int_{\Delta_j}\sum_{k=j-1}^{j+1}\left|\theta_{k,\lambda}^{-1}\left(\chi_{\Delta_k}f\right)\right|^2dx=$$

$$=3M^2\sum_{j=-\infty}^{\infty}\left\|\theta_{j,\lambda}^{-1}\left(\chi_{\Delta_j}f\right)\right\|_2^2.$$

According to inequality (14), we have

$$\|B_\lambda f\|_2^2\le 3M^2\left(\frac{2}{\varepsilon+2\lambda}\right)^2\|f\|_2^2.$$

Therefore, if we denote $\lambda_0=\sqrt{3}M\theta^{-1}-0,5\varepsilon$, then for $\lambda\ge\lambda_0$, we have

$$\|B_\lambda\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})}\le\mu\;\left(0<\mu<1\right),$$

where $\|\cdot\|=\|\cdot\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})}$ is the operator norm. By the well-known Banach theorem on small perturbations of a linear operator, for $\lambda\ge\lambda_0$, the operator $E+B_\lambda$ is invertible, and its inverse $(E+B_\lambda)^{-1}$ is bounded. The following inequalities are easily proven:

$$\frac{1}{1+\mu}\le\left\|(E+B_\lambda)^{-1}\right\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})}\le\frac{1}{1-\mu}\;\;(\lambda\ge\lambda_0).\tag{16}$$

By (15), we obtain the following operator equality

$$\Theta_\lambda^{-1}=M_\lambda\left(E+B_\lambda\right)^{-1},\;\lambda\ge\lambda_0.\tag{17}$$

Let us estimate the norm $\left\|(p+\lambda)\Theta_\lambda^{-1}\right\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})}$. By (16) and (17),

$$\left\|(p+\lambda)\,\Theta_\lambda^{-1}\right\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})}\le\frac{1}{1-\mu}\left\|(p+\lambda)\,M_\lambda\right\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})}.$$

But

$$\|(p+\lambda)\,M_\lambda f\|_2^2=\sum_{j=-\infty}^{\infty}\int_{\Delta_j}\left(p\left(x\right)+\lambda\right)^2\left|\sum_{k=j-1}^{j+1}\varphi_k\left(x\right)\theta_{k,\lambda}^{-1}\left(\chi_{\Delta_k}f\right)\right|^2dx\le$$

$$\le 3\sum_{j=-\infty}^{\infty}\int_{\Delta_j}(p(x)+\lambda)^2\left[\left|\varphi_{j-1}\theta_{j-1,\lambda}^{-1}\chi_{\Delta_{j-1}}f(x)\right|^2+\left|\varphi_j\theta_{j,\lambda}^{-1}\chi_{\Delta_j}f(x)\right|^2\right]dx+$$

$$+3\sum_{j=-\infty}^{\infty}\int_{\Delta_j}(p(x)+\lambda)^2\left|\varphi_{j+1}\theta_{j+1,\lambda}^{-1}\left(\chi_{\Delta_{j+1}}f\right)\right|^2dx\le$$

$$\le 3\left(\sup_{x\in\Omega_j}p\left(x\right)+\lambda\right)^2\sum_{j=-\infty}^{\infty}\int_{\mathbb{R}}\left|\varphi_j\left(x\right)\theta_{j,\lambda}^{-1}\left(\chi_{\Delta_j}f\right)\right|^2dx.$$

According to inequality (13), property a) of the sequence $\{\varphi_j(x)\}_{j=-\infty}^{\infty}$ and condition (11), we obtain

$$\|(p+\lambda)\,M_\lambda f\|_2^2 \le 3\left(\sup_{x\in\Omega_j} p\,(x)+\lambda\right)^2 \sum_{j=-\infty}^{\infty}\int_R \left|\theta_{j,\lambda}^{-1}\left(\chi_{\Delta_j} f\right)\right|^2 dx \le$$

$$\le 3\left(\sup_{x\in\Omega_j} p\,(x)+\lambda\right)^2 \frac{1}{\left(\inf\limits_{x\in R} p_j\,(x)+\lambda\right)^2}\sum_{j=-\infty}^{\infty}\int_R \left|\left(\chi_{\Delta_j} f\right)\right|^2 dx \le$$

$$\le 12\left(\frac{\sup\limits_{t\in\Omega_j} p\,(t)+\lambda}{\sup\limits_{t\in\Omega_j} p\,(t)+\lambda}\right)^2 \int_{\mathbb{R}}\left(\sum_{j=-\infty}^{\infty}\chi_{\Delta_j}^2\right) f^2\,(x)\,dx \le 12\left(\sup_{x\in\Omega_j}\frac{p\,(x)}{p\,(t)}+1\right)^2 \|f\|_2^2 .$$

So

$$\|(p+\lambda)M_\lambda f\|_2^2 \le 12(K+1)^2\,\|f\|_2^2\ \ (\lambda\ge\lambda_0)\,,\ K=\sup_{x\in\Omega_j}\frac{p\,(x)}{p\,(t)}. \tag{18}$$

For $z\in D\,(\Theta_\lambda)\,,\ \Theta_\lambda z=f,\ \lambda\ge\lambda_0$, we have $z=\Theta_\lambda^{-1}f$. Therefore, according to (17), (18) and (16),

$$\|(p+\lambda)\,z\|_2=\left\|(p+\lambda)\,M_\lambda\,(E+B_\lambda)^{-1}\,f\right\|_2\le$$

$$\le 2\sqrt{3}\,(K+1)\left\|(E+B_\lambda)^{-1}\,f\right\|_2 \le 2\sqrt{3}\,(K+1)\,\frac{1}{1-\mu}\,\|f\|_2 .$$

Furthermore

$$\left\|z'\right\|_2=\|f-(p+\lambda)\,z\|_2\le\left[2\sqrt{3}\,(K+1)\,\frac{1}{1-\mu}+1\right]\|f\|_2 .$$

Consequently,

$$\left\|z'\right\|_2+\|pz\|_2+\|\lambda z\|_2\le(6\sqrt{3}(K+1)\frac{1}{1-\mu}+1)\,\|f\|_2 .$$

So, we have proven the inequality (10), and lemma.

From this lemma, taking into account the notation $(l+\lambda E)\,y=y^{(4)}+(p+\lambda)\,y^{(3)},\ y^{(3)}=z$, and Lemma 3, we come to the following conclusion.

Lemma 7. Let the function p satisfy the conditions of Lemma 3 and the relation (11). Then, the operator $l+\lambda E(\ \lambda\ge 0)$ is boundedly invertible in $L_2\,(\mathbb{R})$. Moreover, for any $y\in D(l+\lambda E)$, the following inequality holds:

$$\left\|y^{(4)}\right\|_2+\left\|(p+\lambda)y^{(3)}\right\|_2+\|y\|_2\le C\,\|(l+\lambda E)\,y\|_2 .$$

Remark 1. The condition (3), which was used in the proofs of Lemmas 3, 6, and 7, can be replaced with the condition $p\,(x)\ge 1$. Indeed, if we denote $x=\varepsilon^{-1}t\ (\ t>0)\ ,\ \widehat{y}\,(t)=y\left(\varepsilon^{-1}t\right)$ and $\widehat{p}\,(t)=p\left(\varepsilon^{-1}t\right)$. The operator $ly=y^{(4)}+p\,(x)\,y^{(3)}$ is transformed into

$$\varepsilon^4\widehat{l}\widehat{y}\,(t)=\widehat{y}^{(4)}\,(t)+\varepsilon^{-1}\widehat{p}\,(t)\,\widehat{y}^{(3)}\,(t)\,,$$

where $\varepsilon^{-1}\widehat{p}\,(t)\ge 1$.

4 Main result and its proof

Theorem 1. Assume that $p(x)$ satisfies conditions (3), $\gamma_{1,\sqrt{p},3} < \infty$ and $\gamma_{q,p,3} < \infty$. Then for any $f \in L_2(\mathbb{R})$ there exists a solution to equation (1) and it is unique. If, in addition, the relation (11) holds, then the solution y satisfies the following maximal regularity estimate

$$\left\|y^{(4)}\right\|_2 + \left\|py^{(3)}\right\|_2 + \|(1+|q|)y\|_2 \le C\,\|f\|_2 . \tag{19}$$

Proof. In equation (1), we introduce a new variable t using the formula $x = \frac{t}{a}$. Let us denote:

$$\widetilde{y}(t) = y\left(a^{-1}t\right),\ \widetilde{p}(t) = p\left(a^{-1}t\right),\ \widetilde{q}(t) = q\left(a^{-1}t\right),\ \widetilde{F}(t) = a^{-4}F\left(a^{-1}t\right) \quad (t \in \mathbb{R}).$$

Then, equation (1) takes the form:

$$\widetilde{L}_{0a}\widetilde{y} = \widetilde{y}^{(4)}(t) + a^{-1}\widetilde{p}(t)\,\widetilde{y}^{(3)}(t) + a^{-4}\widetilde{q}(t)\,\widetilde{y}(t) = \widetilde{F}(t). \tag{20}$$

Let l_a be the closure of the differential operator

$$l_{0a}\widetilde{y} = \widetilde{y}^{(4)}(t) + a^{-1}\widetilde{p}(t)\,\widetilde{y}^{(3)}(t),\ \widetilde{y} \in C_0^{(4)}(\mathbb{R}),$$

in the space $L_2(\mathbb{R})$. It can be easily verified that $\gamma_{1,\sqrt{a^{-1}\widetilde{p}},3} = a^3\gamma_{1,\sqrt{p},3} < \infty$. By Lemma 3, the operator l_a is continuously invertible. Moreover, by Lemma 6, for each $\widetilde{y} \in D(l_a)$, we have

$$\left\|\widetilde{y}^{(4)}(t)\right\|_2 + \left\|a^{-1}\left(\widetilde{p}(t)+\lambda\right)\widetilde{y}^{(3)}(t)\right\|_2 + \|\widetilde{y}\|_2 \le C_a\,\|l_a\widetilde{y}\|_2 . \tag{21}$$

Further, $\gamma_{a^{-4}\widetilde{q},a^{-1}\widetilde{p},3} = \frac{1}{\sqrt{a}}\gamma_{q,p,3}$. Consequently, by Lemma 1, we obtain $\left\|a^{-4}\widetilde{q}\widetilde{y}\right\|_2 \le \frac{2}{\sqrt{a}}\gamma_{q,p,3}C_a\,\|l_a\widetilde{y}\|_2$. If we choose the parameter a such that $a \ge \max\left(\frac{4C_a^2}{\nu^2}\gamma_{q,p,3}^2,\ 1\right)$ $(0 < \nu < 1)$, then the following inequality holds:

$$\left\|a^{-4}\widetilde{q}\widetilde{y}\right\|_2 \le \nu\,\|l_a\widetilde{y}\|_2,\ 0 < \nu < 1. \tag{22}$$

Then, by the theorem on small perturbations, the closure $\widetilde{L}_a$ in $L_2(\mathbb{R})$ of the operator $\widetilde{L}_{0a}\widetilde{y} = l_a\widetilde{y} + a^{-4}\widetilde{q}(t)\,\widetilde{y}(t)$ is invertible, and its inverse $\widetilde{L}_a^{-1}$ is continuous. So, for each right-hand side $\widetilde{F}(t) \in L_2(\mathbb{R})$, the solution $\widetilde{y}$ of the equation (20) exists and is unique. Furthermore, by (22),

$$\left\|\widetilde{l}_a\widetilde{y}\right\|_2 \le \frac{1}{(1-\nu)}\left\|\widetilde{L}_a\widetilde{y}\right\|_2 .$$

In accordance with (21), we have

$$\left\|\widetilde{y}^{(4)}(t)\right\|_2 + \left\|a^{-1}\widetilde{p}(t)\,\widetilde{y}^{(3)}(t)\right\|_2 + \left\|a^{-4}\widetilde{q}\widetilde{y}\right\|_2 \le \left[C_a + \frac{1}{1-\nu}\right]\left\|\widetilde{L}_a\widetilde{y}\right\|_2 .$$

Returning by the substitution $x = \frac{1}{a}t$ to the variable x in this inequality, we obtain the estimate

$$\left\|y^{(4)}\right\|_2 + \left\|py^{(3)}\right\|_2 + \|qy\|_2 \le C\,\|F\|_2 .$$

From here, the inequality (19) easily follows.

Conclusion

The qualitative properties of a fourth-order differential equation with unlimited intermediate and minor coefficients are studied in the work. For a wide class of coefficients the correctness of equation is proved and a maximal regularity estimate of the solution in the norm of the Hilbert space is obtained.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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