https://doi.org/10.31489/2024M4/170-184 *Research article*

Solving Volterra-Fredholm integral equations by non-polynomial spline functions

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It depends on our information, non-polynomial spline functions have not been applied for solving Volterra-Fredholm integral equations of the second kind yet. In this paper, we want to use such functions for finding approximation solutions of Volterra-Fredholm integral equations. In our approach, the coefficients of the non-polynomial spline were found by solving a system of linear equations. Then, these functions were utilized to reduce the fredholm integral equations to the solution of algebraic equations. Analysis of convergences investigated. Finally, three examples were presented to show the effectiveness of the method. This was done with the help of a computer program that used the Python code program version 3.9.

Keywords: Volterra integral equation, Fredholm integral equation, non-polynomial spline function.

2020 Mathematics Subject Classification: 45B05, 45B05, 41A15.

Introduction

Integral equations are a fundamental class of equations in mathematical analysis that involve an unknown function under an integral sign. They arise naturally in various fields such as physics, engineering, and applied mathematics, where they model phenomena ranging from heat conduction to quantum mechanics [1–3].

Volterra-Fredholm integral equations are vital in numerous scientific and engineering disciplines, such as mechanics, electrical engineering, and physics. These equations are instrumental in modeling intricate phenomena involving integration, which is essential for comprehending and addressing issues in these areas. To approximate solutions for these equations, numerical methods utilizing non-polynomial spline functions have been developed. This technique serves as an effective means for resolving integral equations lacking analytical solutions, delivering precise numerical answers for a range of significant problems. In recent years, there has been a growing interest in using non-polynomial splines to find numerical solutions for integral equations and other types of equations, as evidenced by the increasing number of published articles on the topic. Non-polynomial spline functions are a type of interpolation function that can be used to approximate the solution of integral equations. Non-polynomial spline functions are particularly useful because they can provide good approximation properties and can often handle irregularities in the solution or the kernel function better than polynomial-based methods. Numerous researchers utilize non-polynomial splines to solve Volterra and Fredholm integral equations [4–16].

Recently, the interplay of Volterra-Fredholm integral equation using many numerical techniques has been investigated [17–24]. For the first time, Salim, et al [25–27] used linear, quadratic and cubic

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Received: 7 January 2024; *Accepted:* 20 August 2024.
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spline function for solving the following linear Volterra-Fredholm integral equation of the second kind,

$$u(x) = f(x) + \lambda_1 \int_a^x K(x,t)u(t)dt + \lambda_2 \int_a^b L(x,t)u(t)dt, \tag{1}$$

where the functions $f(x)$, and the kernels $K(x,t)$ and $L(x,t)$ are known L^2 analytic functions and λ_1, λ_2 are arbitrary constants, x is variable and $u(x)$ is the unknown continuous function to be determined.

In this paper, we introduce a novel non-polynomial spline function to obtain the numerical solution of equation (1) for the first time.

The structure of this paper is as follows: Section 2 introduces our method for solving equation (1). Section 3 details our methodology, while Section 4 focuses on the convergence analysis. Section 5 presents several numerical examples to demonstrate the effectiveness of our technique. Finally, Section 6 offers some tentative conclusions.

1 *Non-polynomial spline function*

We describe the non-polynomial spline for solving equation (1) in similar manner of [11] The numerical scheme has been developed on the domain of integration $\omega = [a,b]$ with partitions

$$a = x_0 < x_1 < \cdots < x_n = b,$$

where $x_i = x_0 + ih$, $i = 0, \cdots, n$ and $h = \dfrac{b-a}{n}$. Let $S_i(x)$ be the interpolating non-polynomial spline function which interpolate y at x_i defined by [11–13]

$$S_i(x) = a_i + b_i(x - x_i) + c_i \sin \tau(x - x_i) + d_i \cos \tau(x - x_i), \tag{2}$$

where a_i, b_i, c_i and d_i are real numbers and τ is an arbitrary parameter. We denote the following relations

$$S_i(x_i,\tau) = y_i, \quad S_i(x_{i+1},\tau) = y_{i+1}, \quad S_i''(x_k,\tau) = M_i, \quad S_i''(x_{i+1},\tau) = M_{i+1}. \tag{3}$$

Using equation (2) and equation (3) we have the following expressions

$$a_i = y_i + \frac{M_i}{\tau^2}, \quad b_i = \frac{y_{i+1} - y_i}{h} + \frac{M_{i+1} - M_i}{\tau\theta}, \quad c_i = \frac{M_i \cos\theta - M_{i+1}}{\tau^2 \sin\theta},$$

$$d_i = \frac{-M_i}{\tau^2}. \tag{4}$$

With the continuity of first derivatives of $S_{i-1}(x)$ and $S_i(x)$ at $x = x_i$, $i = 1, 2, \cdots, n-1$, we obtain the following consistency relation,

$$\alpha M_{i-1} + 2\beta M_i + \alpha M_{i+1} = \frac{1}{h}(y_{i+1} - 2y_i + y_{i-1}), \tag{5}$$

where $\alpha = (\frac{1}{\theta^2})(\theta \csc\theta - 1)$, $\quad \beta = (\frac{1}{\theta^2})(1 - \theta \cot\theta)$.

Using the finite difference operator $E = e^{hD}$ in the above consistency relation where D is differential operator and by expanding in powers of hD, the error for relation equation (5) can be expressed as follows:

$$Error = (2\alpha + 2\beta - 1)(M_i - y_i'') + D^2h^2 \left(\alpha - \frac{1}{12}\right)(M_i - y_i'') +$$

$$+ D^4h^4 \left(\frac{\alpha}{12} - \frac{1}{360}\right)(M_i - y_i'') + O(h^6). \tag{6}$$

The consistency relation equation (5) for the above equation leads to the equation $2\alpha + 2\beta = 1$, which may also be expressed as $\tan\left(\frac{\theta}{2}\right) = \left(\dfrac{\theta}{2}\right)$. This equation has a zero root and an infinitely many of non-zero roots, the smallest positive root being $\theta = 8.98881$.

We use this θ as an optimal value in the convergence analysis and numerical computation. In this case, we have

$$\left|M_i - y_i''\right| \le k_2 h^2, \quad k_2 = 0.22 \max\left|y_i^4\right|,$$

provided that $2\alpha + 2\beta = 1$.

If we let $\alpha = \dfrac{1}{12}$ and $\beta = \dfrac{5}{12}$, the second term in the error equation (6) is also zero and the method can be modified and has a precision of the order $O(h^4)$ to calculate vector M:

$$M_{i+1} + 10M_i + M_{i-1} = \frac{12}{h}(y_{i+1} - 2y_i + y_{i-1}),$$

$$\left|M_i - y_i''\right| \le k_2 h^4, \quad k_2 = \frac{1}{240}\max\left|y_i^6\right|. \tag{7}$$

Provided that $2\alpha + 2\beta = 1$. If we let, $\alpha = \dfrac{1}{12}$ and $\beta = \dfrac{5}{12}$ the second term in the error equation (6) is also zero and the system (5) with natural cubic Spline initial condition $M_0 = M_n = 0$ is strictly diagonally dominant and has a unique solution to obtain $M_1, M_2, \cdots, M_{n-1}$.

From equation (7), we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 \\ 1 & 10 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 10 & 1 & \cdots & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \cdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 1 & 10 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ \vdots \\ M_{n-2} \\ M_{n-1} \\ M_n \end{bmatrix} = \frac{12}{h^2} \begin{bmatrix} 0 \\ y_0 - 2y_1 + y_2 \\ y_1 - 2y_2 + y_3 \\ \vdots \\ \vdots \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n \\ 0 \end{bmatrix}. \tag{8}$$

The above matrix form can be expressed as follows:

$$WM = \frac{12}{h^2}JY \quad \Rightarrow \quad M = \frac{12}{h^2}W^{-1}JY, \tag{9}$$

where $Y = (y_0, y_1, y_2, \cdots, y_{n-1}, y_n)^T$ and $M = (M_0, M_1, M_2, \cdots, M_{n-1}, M_n)^T$.

2 Methodology

In this section, we present numerical scheme to approximate equation (1). From equation (2) and equation (4) we have

$$U_i = y_i + \frac{M_i}{\tau^2} + \left(\frac{y_{i+1}}{h} + \frac{M_{i+1}}{h}\right)(x - x_i) - \left(\frac{y_i}{h} + \frac{M_i}{h}\right)(x - x_i) +$$

$$+ \frac{M_i \cos\theta}{\tau^2 \sin\theta}\sin\tau(x - x_i) - \frac{M_{i+1}\cos\theta}{\tau^2 \sin\theta}\sin\tau(x - x_i) - \frac{M_i}{\tau^2}\cos\tau(x - x_i). \tag{10}$$

By replacing equation (10) in equation (1) and using the collocation method, we have

$$
\begin{aligned}
U_i &= f(x_i) + \int_a^x K(x_i,t)u(t)dt + \int_a^b L(x_i,t)u(t)dt \\
&= f(x_i) + \sum_{j=0}^{i} \int_{x_j}^{x_{j+1}} K(x_i,t)u_j(t)dt + \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} L(x_i,t)u_j(t)dt \\
&= f(x_i) + \sum_{j=0}^{i} u_{j(t)} \int_{x_j}^{x_{j+1}} K(x_i,t)dt + \sum_{j=0}^{n-1} u_{j(t)} \int_{x_j}^{x_{j+1}} L(x_i,t)dt \\
&= f(x_i) + \sum_{j=0}^{i} (y_j + \frac{M_j}{\tau^2}) \int_{x_j}^{x_{j+1}} K(x_i,t)dt + \sum_{j=0}^{n-1} (y_j + \frac{M_j}{\tau^2}) \int_{x_j}^{x_{j+1}} L(x_i,t)dt \\
&+ \sum_{j=0}^{i} \left(\frac{y_{j+1}}{h} + \frac{M_{j+1}}{\tau\theta}\right) \int_{x_j}^{x_{j+1}} K(x_i,t)(t-t_j)dt + \sum_{j=0}^{n-1} \left(\frac{y_{j+1}}{h} + \frac{M_{j+1}}{\tau\theta}\right) \int_{x_j}^{x_{j+1}} L(x_i,t)(t-t_j)dt \\
&- \sum_{j=0}^{i} \left(\frac{y_j}{h} + \frac{M_{j+1}}{\tau\theta}\right) \int_{x_j}^{x_j} K(x_i,t)(t-t_j)dt - \sum_{j=0}^{n-1} \left(\frac{y_j}{h} + \frac{M_{j+1}}{\tau\theta}\right) \int_{x_j}^{x_{j+1}} L(x_i,t)(t-t_j)dt \\
&+ \sum_{j=0}^{i} \frac{M_j \cos\theta}{\tau^2 \sin\theta} \int_{x_j}^{x_{j+1}} K(x_i,t)\sin\tau(t-t_j)dt + \sum_{j=0}^{n-1} \frac{M_j \cos\theta}{\tau^2 \sin\theta} \int_{x_j}^{x_{j+1}} L(x_i,t)\sin\tau(t-t_j)dt \\
&+ \sum_{j=0}^{i} \frac{M_{j+1} \cos\theta}{\tau^2 \sin\theta} \int_{x_j}^{x_{j+1}} K(x_i,t)\sin\tau(t-t_j)dt + \sum_{j=0}^{n-1} \frac{M_{j+1} \cos\theta}{\tau^2 \sin\theta} \int_{x_j}^{x_{j+1}} L(x_i,t)\sin\tau(t-t_j)dt \\
&+ \sum_{j=0}^{i} \frac{M_j}{\tau^2} \int_{x_j}^{x_{j+1}} K(x_i,t)\cos\tau(t-t_j)dt + \sum_{j=0}^{n-1} \frac{M_j}{\tau^2} \int_{x_j}^{x_{j+1}} L(x_i,t)\cos\tau(t-t_j)dt \\
&= f(x_i) + \sum_{j=0}^{i} (y_j + \frac{M_j}{\tau^2})a_{ij} + \sum_{j=0}^{n-1} (y_j + \frac{M_j}{\tau^2})a^*_{ij} + \sum_{j=0}^{i} \left(\frac{y_{j+1}}{h} + \frac{M_{j+1}}{\tau\theta}\right) b_{ij+1} \\
&+ \sum_{j=0}^{n-1} \left(\frac{y_{j+1}}{h} + \frac{M_{j+1}}{\tau\theta}\right) b^*_{ij+1} - \sum_{j=0}^{i} \left(\frac{y_j}{h} + \frac{M_j}{\tau\theta}\right) c_{ij} - \sum_{j=0}^{n-1} (\frac{y_j}{h} + \frac{M_j}{\tau\theta})c^*_{ij} + \sum_{j=0}^{i} \frac{M_j \cos\theta}{\tau^2 \sin\theta} d_{ij} \\
&+ \sum_{j=0}^{n-1} \frac{M_j \cos\theta}{\tau^2 \sin\theta} d^*_{ij} + \sum_{j=0}^{i} \frac{M_{j+1}}{\tau^2 sin\theta} e_{ij+1} + \sum_{j=0}^{n-1} \frac{M_{j+1}}{\tau^2 \sin\theta} e^*_{ij+1} + \sum_{j=0}^{i} \frac{M_j}{\tau^2} p_{ij} + \sum_{j=0}^{n-1} \frac{M_j}{\tau^2} p^*_{ij} \\
&= f(i) + \sum_{j=0}^{i} (y_j + \frac{M_j}{\tau^2})a_{ij} + \sum_{j=0}^{n} (y_j + \frac{M_j}{\tau^2})a^*_{ij} + (-y_n - \frac{M_n}{\tau^2} a^*_{in}) \\
&- (\frac{y_0}{h} + \frac{M_0}{\tau\theta})b_{i0} + \sum_{j=0}^{i} (\frac{y_j}{h} + \frac{M_j}{\tau\theta})b_{ij} + \left(\frac{y_{i+1}}{h} + \frac{M_{i+1}}{\tau\theta}\right) b_{i,i+1} - (\frac{y_0}{h} + \frac{M_0}{\tau\theta})b^*_{i0} \\
&+ \sum_{j=0}^{n} \left(\frac{y_j}{h} + \frac{M_j}{\tau\theta}\right) b^*_{ij} - \sum_{j=0}^{i} (\frac{y_j}{h} + \frac{M_j}{\tau\theta})c_{ij} - \sum_{j=0}^{n} \left(\frac{y_j}{h} + \frac{M_j}{\tau\theta}\right) c^*_{ij} + \left(-\left(\frac{y_n}{h} + \frac{M_n}{\tau\theta}\right)\right) c^*_{in} \\
&+ \sum_{j=0}^{i} \frac{M_j \cos\theta}{\tau^2 \sin\theta} d_{ij} + \sum_{j=0}^{n-1} \frac{M_j \cos\theta}{\tau^2 \sin\theta} d^*_{ij} - \frac{M_n \cos\theta}{\tau^2 \sin\theta} d^*_{in} - M_o e_{io} + \sum_{j=0}^{i} \frac{M_j}{\tau^2 \sin\theta} e_{ij} \\
&+ M_{i+1} e_{i,i+1} - M_o e^*_{io} + \sum_{j=0}^{n} \frac{M_j}{\tau^2 \sin\theta} e^*_{ij} + \sum_{j=0}^{i} \frac{M_j}{\tau^2} p_{ij} + \sum_{j=0}^{n} \frac{M_j}{\tau^2} p^*_{ij} - \frac{M_n}{\tau^2} p_{in}
\end{aligned}
$$

$$
\begin{aligned}
&= f(i) + \sum_{j=0}^{i}(y_j + \frac{M_j}{\tau^2})a_{ij} + \sum_{j=0}^{n}(y_j + \frac{M_j}{\tau^2})a^*_{ij} + \sum_{j=0}^{i}\left(\frac{y_j}{h} + \frac{M_j}{\tau\theta}\right)b_{ij} \\
&+ \sum_{j=0}^{n}(\frac{y_j}{h} + \frac{M_j}{\tau\theta})b^*_{ij} - \sum_{j=0}^{i}(\frac{y_j}{h} + \frac{M_j}{\tau\theta})c_{ij} - \sum_{j=0}^{n}\left(\frac{y_j}{h} + \frac{M_j}{\tau\theta}\right)c^*_{ij} + \sum_{j=0}^{i}\frac{M_j\cos\theta}{\tau^2\sin\theta}d_{ij} \\
&+ \sum_{j=0}^{n-1}\frac{M_j\cos\theta}{\tau^2\sin\theta}d^*_{ij} + \sum_{j=0}^{i}\frac{M_j}{\tau^2\sin\theta}e_{ij} + \sum_{j=0}^{n}\frac{M_j}{\tau^2\sin\theta}e^*_{ij} + \sum_{j=0}^{i}\frac{M_j}{\tau^2}p_{ij} \\
&+ \sum_{j=0}^{n}\frac{M_j}{\tau^2}p^*_{ij} + O(h^4), \quad i = 0, 1, 2, \cdots, n. \qquad (11)
\end{aligned}
$$

After finding the above integration by quadrature rules, we assuming $a^*_{in} = b_{i0} = b^*_{i0} = b_{i,i+1} = c^*_{in} = d^*_{in} = e_{i0} = e_{i,i+1} = e^*_{i0} = p^*_{in} = 0$. Now, if we suppose $a_{ij} = A$, $b_{ij+1} = B$, $c_{ij} = C$, $d_{ij} = D$, $e_{ij+1} = E$ and $p_{ij+1} = P$. Also, $a^*_{ij} = A^*$, $b^*_{ij+1} = B^*$, $c^*_{ij} = C^*$, $d^*_{ij} = D^*$, $e^*_{ij+1} = E^*$, $p^*_{ij+1} = P^*$,

$$
\begin{aligned}
a_{ij} &= \int_{x_j}^{x_{j+1}} K(x_i,t)dt, & a^*_{ij} &= \int_{x_j}^{x_{j+1}} L(x_i,t)dt, \\
b_{ij+1} &= \int_{x_j}^{x_{j+1}} K(x_i,t)(t-t_j)dt, & b^*_{ij+1} &= \int_{x_j}^{x_{j+1}} L(x_i,t)(t-t_j)dt, \\
c_{ij} &= \int_{x_j}^{x_{j+1}} K(x_i,t)(t-t_j)dt, & c^*_{ij} &= \int_{x_j}^{x_{j+1}} L(x_i,t)(t-t_j)dt, \\
d_{ij} &= \int_{x_j}^{x_{j+1}} K(x_i,t)sin\tau(t-t_j)dt, & d^*_{ij} &= \int_{x_j}^{x_{j+1}} L(x_i,t)\sin\tau(t-t_j)dt, \\
e_{ij+1} &= \int_{x_j}^{x_{j+1}} K(x_i,t)\sin\tau(t-t_j)dt, & e^*_{ij+1} &= \int_{x_j}^{x_{j+1}} L(x_i,t)\sin\tau(t-t_j)dt, \\
p_{ij} &= \int_{x_j}^{x_{j+1}} K(x_i,t)\cos\tau(t-t_j)dt, & p^*_{ij} &= \int_{x_j}^{x_{j+1}} L(x_i,t)\cos\tau(t-t_j)dt,
\end{aligned}
$$

$$
\hat{M} \approx M = (M_0, M_1, M_2, \cdots, M_{n-1}, M_n)^T, \hat{u} \approx U = (U_0, U_1, U_2, \cdots, U_{n-1}, U_n)^T,
$$
$$
F = (f_0, f_1, f_2, \cdots, f_{n-1}, f_n)^T,
$$

we have

$$
\begin{aligned}
\hat{U} = F + [(A+\hat{A}) + \frac{1}{h}(B+\hat{B}) - \frac{1}{h}(C+\hat{C})]\hat{U} + \frac{1}{\tau^2}[(A+\hat{A}) + \frac{1}{h}(B+\hat{B}) \\
-\frac{1}{h}(C+\hat{C}) + \cot\theta D - \csc\theta E - P]\hat{M}.
\end{aligned}
$$

Using equation (9) we have

$$
\begin{aligned}
[I - [(A+\hat{A}) + \frac{1}{h}(B+\hat{B}) - \frac{1}{h}(C+\hat{C})] - \frac{12}{\theta^2}[(A+\hat{A}) + \frac{1}{h}(B+\hat{B}) \\
- \frac{1}{h}(C+\hat{C}) + \cot\theta D - csc\theta E - P]Z]\hat{U} = F. \qquad (12)
\end{aligned}
$$

Let

$$
H_1 = [(A+\hat{A}) + \frac{1}{h}(B+\hat{B}) - \frac{1}{h}(C+\hat{C})],
$$

and

$$H_2 = [(A+\hat{A}) + \frac{1}{h}(B+\hat{B}) - \frac{1}{h}(C+\hat{C}) + \cot\theta D - \csc\theta E - P]Z],$$

where $Z = W^{-1}J$. This implies that

$$[I - (H_1 + H_2 Z)]\hat{U} = F.$$

If we suppose $W^{-1} = (u_{ij}),\ 1 \le i,j \le n+1$, then

$$Z = \begin{bmatrix} u_{1,2} & u_{1,3} - 2u_{1,2} & z_{1,3} & \cdots & z_{1,n-1} & u_{1,n-1} - 2u_{1,n} & u_{1,n} \\ u_{2,2} & u_{2,3} - 2u_{2,2} & z_{1,3} & \cdots & z_{2,n-1} & u_{2,n-1} - 2u_{2,n} & u_{2,n} \\ u_{3,2} & u_{3,3} - 2u_{3,2} & z_{1,3} & \cdots & z_{3,n-1} & u_{3,n-1} - 2u_{3,n} & u_{3,n} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{n+1,2} & u_{n+1,3} - 2u_{n+1,2} & z_{n+1,3} & \cdots & z_{n+1,n-1} & u_{1,n-1} - 2u_{n+1,n} & u_{n+1,n} \end{bmatrix},$$

where $z_{i,j} = u_{i,j-1} - 2u_{i,j} + u_{i,j+1}$ for $3 \le j \le n-1$ and $1 \le i \le n+1$.

Finally we can approximate the exact solution y by the non-polynomial Spline function $\hat{U}$ such that $\hat{U} = \hat{U}_i$ on $[x_i, x_{i+1}]$, $i = 0, 1, 2, \cdots, n-1$, where

$$\hat{U}(x) = \hat{y}_i + \frac{\hat{M}_i}{\tau^2} + \left(\frac{\hat{y}_{i+1}}{h} + \frac{\hat{M}_{i+1}}{\tau\theta}\right)(x - x_i) - \left(\frac{\hat{y}_i}{h} + \frac{\hat{M}_i}{\tau\theta}\right)(x - x_i)$$

$$+ \left(\frac{\hat{M}_i \cos\theta}{\tau^2 \sin\theta}\right)\sin\tau(x - x_i) - \left(\frac{\hat{M}_{i+1}}{\tau^2 \sin\theta}\right)\sin\tau(x - x_i) - \left(\frac{\hat{M}_i}{\tau^2}\right)\cos\tau(x - x_i). \tag{13}$$

3 Analysis of convergence

Lemma 1. [28] Let A be a $n \times n$ matrix with $\|A\|_\infty < 1$, then the matrix $(I - A)$ is invertable. Moreover $\left\|(I - A)^{-1}\right\|_\infty < \frac{1}{1 - \|A\|_\infty}$.

Theorem 1. Let $f \in C^4(I)$ and $k \in C^4(I \times I)$ such that. $\frac{3}{2}\|K\|_\infty \|L\|_\infty (b - a) < 1$, then equation (13) defines a unique approximate and the resulting error $\hat{e} = y - \hat{U}$ satisfies

$$\|\hat{e}\|_\infty < \gamma h^4, \quad \forall\ r \in I,$$

where γ is a constant.

Proof. It is essay to show that $\|A\|_\infty$, $\|A^*\|_\infty$, $\|D\|_\infty$, $\|D^*\|_\infty$, $\|E\|_\infty$, $\|E^*\|_\infty$ and $\|P\|_\infty$, $\|P^*\|_\infty \le (\|K\|_\infty + \|L\|_\infty)(b - a)$ also $\|B\|_\infty$, $\|B^*\|_\infty$, $\|C\|_\infty$, $\|C^*\|_\infty \le (\|K\|_\infty + \|L\|_\infty)\dfrac{(b-a)h}{2}$.
Hence

$$\|H_1\|_\infty \le 2(\|K\|_\infty + \|L\|_\infty)(b - a)$$

and

$$\|H_2\|_\infty \le \frac{48}{100}(\|K\|_\infty + \|L\|_\infty)(b - a),$$

then we have

$$(\|H_1 + H_2 Z\|_\infty) < \frac{3}{2}(\|K\|_\infty + \|L\|_\infty) < 1.$$

Now by Lemma 1, the system (12) has a unique solution $\hat{y}$. It follows that the equation (13) defines a unique solution $\hat{U}$.

Now, let $\hat{e} = y - \hat{y} = (y_0 - \hat{y}_0, y_1 - \hat{y}_1, \cdots, y_n - \hat{y}_n)^T$. Then from equation (11), we get

$$(I - (H_1 + H_2 Z))\hat{e} = O(h^4).$$

Therefore,

$$\hat{e} = (I - (H_1 + H_2 Z))^{-1} = O(h^4),$$

for which implies by Lemma 1, that there exists α_0 such that

$$\|\hat{e}\|_\infty \leq \underbrace{\frac{\alpha_1 h^4}{1 - \frac{3}{2}(\|K\|_\infty + \|L\|_\infty)(b-a)}}_{\alpha_2}.$$

On the other hand, from equation (8), we have $(M - \hat{M}) = (\frac{12}{h^2})Z\hat{e}$. Therefore,

$$\left\|Z - \hat{Z}\right\|_\infty \leq 12\alpha_2 h^4.$$

In consequence, for all $i = 0, 1, \cdots, n-1$ and $x \in [x_i, x_{i+1}]$, we have

$$\left|U_i(X) - \hat{U}_i(X)\right| \leq 12\alpha_2 h^4.$$

It follows that

$$\left\|Y - \hat{U}\right\|_\infty \leq \|Y - U\|_\infty + \left\|U - \hat{U}\right\|_\infty \leq \alpha_1 h^4 + 12\alpha_2 h^4.$$

Thus, the proof is completed by taking $\gamma = \alpha_1 + 12\alpha_2$.

4 Numerical results

In this section, we present three examples to illustrate the efficiency and accuracy of the proposed method. The computed errors e_i are defined by $e_i = |u_i - S_i|$, where u_i is the exact solution of equation (1) and S_i is an approximate solution of the same equation. Also we compute Least square error(LSE)= $\sum_{i=0}^{n}(u_i - S_i)^2$ and all computations are performed using the Python program.

Example 1. Consider the linear Volterra-Fredholm integral equation

$$u(x) = -\frac{x^2}{2} - \frac{7x}{2} + 2 + \int_0^x u(t)dt + \int_0^1 xu(t)dt.$$

The exact solution to this equation is given by $u(x) = x + 2$.

Example 2. Consider the linear Volterra-Fredholm integral equation

$$u(x) = 2\cos(x) - 1 + \int_0^x (x-t)u(t)dt + \int_0^\pi u(t)dt.$$

The exact solution to this equation is given by $u(x) = \cos(x)$.

Example 3. Consider the linear Volterra-Fredholm integral equation

$$u(x) = -\frac{9x^5}{10} + 2x^3 - \frac{3x^2}{2} - \frac{3x}{2} + \frac{19}{10} + \int_0^x (x+t)u(t)dt + \int_0^1 (x-t)u(t)dt.$$

The exact solution to this equation is given by $u(x) = 2x^3 + 1$.

Table 1

The Numerical Results for Example 1 with $n = 5$ and $\tau = 1$

x_i	u_i	S_i	$\lvert u_i - S_i\rvert$	$\lvert u_i - S_i\rvert^2$
0	2	2	0	0
0.2	2.2	2.2	$8.8817842 \times 10^{-16}$	$7.88860905 \times 10^{-31}$
0.4	2.4	2.4	0	0
0.6	2.6	2.6	$1.33226763 \times 10^{-15}$	$1.77493704 \times 10^{-30}$
0.8	2.8	2.8	$1.33226763 \times 10^{-15}$	$1.77493704 \times 10^{-30}$
1	3.	3.	$1.33226763 \times 10^{-15}$	$1.77493704 \times 10^{-30}$
LSE				$6.113672015 \times 10^{-30}$

Table 2

The Numerical Results for Example 1 with $n = 5$ and $\tau = 5$

x_i	u_i	S_i	$\lvert u_i - S_i\rvert$	$\lvert u_i - S_i\rvert^2$
0	2	2	0	0
0.2	2.2	2.2	$8.88178420 \times 10^{-16}$	$7.88860905 \times 10^{-31}$
0.4	2.4	2.4	$4.44089210 \times 10^{-16}$	$1.97215226 \times 10^{-31}$
0.6	2.6	2.6	$4.44089210 \times 10^{-16}$	$1.97215226 \times 10^{-31}$
0.8	2.8	2.8	$8.88178420 \times 10^{-16}$	$7.88860905 \times 10^{-31}$
1	3.	3.	$2.22044605 \times 10^{-15}$	$4.93038066 \times 10^{-30}$
LSE				$6.902532920 \times 10^{-30}$

Table 3

The Numerical Results for Example 1 with $n = 5$ and $\tau = 179.7764$

x_i	u_i	S_i	$\lvert u_i - S_i\rvert$	$\lvert u_i - S_i\rvert^2$
0	2	2	0	0
0.2	2.2	2.2	0	0
0.4	2.4	2.4	$4.4408921 \times 10^{-16}$	$1.97215226 \times 10^{-31}$
0.6	2.6	2.6	$8.8817842 \times 10^{-16}$	$7.88860905 \times 10^{-31}$
0.8	2.8	2.8	$8.8817842 \times 10^{-16}$	$7.88860905 \times 10^{-31}$
1	3.	3.	0	0
LSE				$1.77493703674 \times 10^{-30}$

Table 4

The Numerical Results for Example 1 with $n = 10$ and $\tau = 1$

x_i	u_i	S_i	$\lvert u_i - S_i\rvert$	$\lvert u_i - S_i\rvert^2$
0	2	2	0	0
0.1	2.2	2.2	$4.4408921 \times 10^{-16}$	$1.97215226 \times 10^{-31}$
0.2	2.4	2.4	0	0
0.3	2.6	2.6	$4.4408921 \times 10^{-16}$	$1.97215226 \times 10^{-30}$
0.4	2.8	2.8	$4.4408921 \times 10^{-16}$	$1.97215226 \times 10^{-31}$
0.5	3.	3.	$8.88178420 \times 10^{-16}$	$7.88860905 \times 10^{-31}$
0.6	2	2	$1.77635684 \times 10^{-15}$	$3.15544362 \times 10^{-30}$
0.7	2.2	2.2	$1.33226763 \times 10^{-15}$	$1.77493704 \times 10^{-30}$
0.8	2.4	2.4	$1.33226763 \times 10^{-15}$	$1.77493704 \times 10^{-30}$
0.9	2.6	2.6	$1.33226763 \times 10^{-15}$	$1.77493704 \times 10^{-30}$
1	2.8	2.8	$1.33226763 \times 10^{-15}$	$1.77493704 \times 10^{-30}$
LSE				$1.16356983520 \times 10^{-29}$

Table 5

The Numerical Results for Example 1 with $n = 10$ and $\tau = 5$

x_i	u_i	S_i	$\lvert u_i - S_i \rvert$	$\lvert u_i - S_i \rvert^2$
0	2	2	0	0
0.1	2.2	2.2	0	$1.97215226 \times 10^{-31}$
0.2	2.4	2.4	0	0
0.3	2.6	2.6	0	0
0.4	2.8	2.8	$4.44089210 \times 10^{-16}$	$1.97215226 \times 10^{-31}$
0.5	3.	3.	$8.88178420 \times 10^{-16}$	$7.88860905 \times 10^{-31}$
0.6	2	2	$4.44089210 \times 10^{-16}$	$1.97215226 \times 10^{-31}$
0.7	2.2	2.2	$8.88178420 \times 10^{-16}$	$7.88860905 \times 10^{-31}$
0.8	2.4	2.4	$1.77635684 \times 10^{-15}$	$3.15544362 \times 10^{-30}$
0.9	2.6	2.6	$8.88178420 \times 10^{-16}$	$7.88860905 \times 10^{-31}$
1	2.8	2.8	0	0
LSE				$6.113672015462 \times 10^{-30}$

Table 6

The Numerical Results for Example 1 with $n = 10$ and $\tau = 179.7764$

0	2	2	0	0
0.1	2.2	2.2	0	$1.97215226 \times 10^{-31}$
0.2	2.4	2.4	0	0
0.3	2.6	2.6	$1.77635684 \times 10^{-15}$	$3.15544362 \times 10^{-30}$
0.4	2.8	2.8	$4.44089210 \times 10^{-15}$	$1.97215226 \times 10^{-31}$
0.5	3.	3.	$8.88178420 \times 10^{-16}$	$7.88860905 \times 10^{-31}$
0.6	2	2	$1.33226763 \times 10^{-15}$	$1.97215226 \times 10^{-31}$
0.7	2.2	2.2	$8.88178420 \times 10^{-16}$	$7.88860905 \times 10^{-31}$
0.8	2.4	2.4	$1.77635684 \times 10^{-15}$	$3.15544362 \times 10^{-30}$
0.9	2.6	2.6	$2.22044605 \times 10^{-15}$	$7.88860905 \times 10^{-31}$
1	2.8	2.8	$1.77635684 \times 10^{-15}$	$3.15544362 \times 10^{-30}$
LSE				$6.113672015462 \times 10^{-29}$

Table 7

The Numerical Results for Example 2 with $n = 5$ and $\tau = 1$

x_i	u_i	S_i	$\lvert u_i - S_i \rvert$	$\lvert u_i - S_i \rvert^2$
0	1	1.00692878	0.00692878	$4.80080263 \times 10^{-5}$
$\frac{\pi}{5}$	0.80901699	0.81390488	0.00488789	$2.38914209 \times 10^{-5}$
$\frac{2\pi}{5}$	0.30901699	0.31128372	0.00226672	$5.13803977 \times 10^{-5}$
$\frac{3\pi}{5}$	−0.3090169	−0.30814317	0.00087382	$7.63563793 \times 10^{-7}$
$\frac{4\pi}{5}$	−0.8090169	−0.80948866	0.00047166	$2.22464285 \times 10^{-7}$
π	−1.	−0.9488828	0.00057887	$3.35087778 \times 10^{-7}$
LSE				$7.835860281 \times 10^{-5}$

Table 8

The Numerical Results for Example 2 with $n = 5$ and $\tau = 5$

x_i	u_i	S_i	$\lvert u_i - S_i\rvert$	$\lvert u_i - S_i\rvert^2$
0	1	-1.93012882	2.93012882	8.5856549
$\frac{\pi}{5}$	-2.08618179	0.81390488	2.89519878	8.3821759
$\frac{2\pi}{5}$	-2.24223476	0.31128372	2.55125175	6.5088855
$\frac{3\pi}{5}$	-2.39828773	-0.30814317	2.08927073	4.36505219
$\frac{4\pi}{5}$	-2.55434069	-0.80948866	1.7453237	3.04615481
π	$-1.$	-2.71039366	1.71039366	2.92544648
LSE				33.813369868

Table 9

The Numerical Results for Example 2 with $n = 5$ and $\tau = 179.7764$

x_i	u_i	S_i	$\lvert u_i - S_i\rvert$	$\lvert u_i - S_i\rvert^2$
0	1	1.0178317	$1.78316984 \times 10^{-2}$	$3.17969468 \times 10^{-4}$
$\frac{\pi}{5}$	0.80901699	0.82402291	$1.50059187 \times 10^{-2}$	$2.25177596 \times 10^{-4}$
$\frac{2\pi}{5}$	0.30901699	0.31686699	$7.84999480 \times 10^{-3}$	$6.16224183 \times 10^{-5}$
$\frac{3\pi}{5}$	-0.30901699	-0.30908288	$6.58812345 \times 10^{-5}$	$4.34033706 \times 10^{-9}$
$\frac{4\pi}{5}$	-0.80901699	-0.8129492	$3.93220113 \times 10^{-3}$	$1.54622057 \times 10^{-5}$
π	$-1.$	-2.71039366	$1.21137812 \times 10^{-3}$	$1.46743694 \times 10^{-6}$
LSE				0.0006217034654

Table 10

The Numerical Results for Example 2 with $n = 10$ and $\tau = 1$

x_i	u_i	S_i	$\lvert u_i - S_i\rvert$	$\lvert u_i - S_i\rvert^2$
0	1	1.00094592	$9.45917105 \times 10^{-4}$	8.9475917×10^{-7}
$\frac{\pi}{10}$	$9.51056516 \times 10^{-1}$	$9.51847558 \times 10^{-1}$	$7.91041543 \times 10^{-4}$	$6.25746722 \times 10^{-7}$
$\frac{2\pi}{10}$	$5.87785252 \times 10^{-1}$	$8.09581066 \times 10^{-1}$	$5.64072056 \times 10^{-4}$	$3.18177284 \times 10^{-7}$
$\frac{3\pi}{10}$	$3.09016994 \times 10^{-1}$	$3.09309388 \times 10^{-1}$	$4.08708268 \times 10^{-4}$	$1.67042448 \times 10^{-7}$
$\frac{4\pi}{10}$	$6.12323400 \times 10^{-17}$	$2.05322609 \times 10^{-4}$	$2.92393614 \times 10^{-4}$	$8.54940254 \times 10^{-8}$
$\frac{5\pi}{10}$	$-3.09016994 \times 10^{-1}$	$-3.08878314 \times 10^{-1}$	$2.05322609 \times 10^{-4}$	$4.21573737 \times 10^{-8}$
$\frac{6\pi}{5}$	$-5.87785252 \times 10^{-1}$	$-5.87699570 \times 10^{-1}$	$7.84999480 \times 10^{-4}$	$1.92321364 \times 10^{-8}$
$\frac{7\pi}{5}$	$-0.30901699 \times 10^{-1}$	$-8.08974167 \times 10^{-1}$	$1.38679978 \times 10^{-4}$	$7.34153214 \times 10^{-9}$
$\frac{8\pi}{5}$	$-8.09016994 \times 10^{-1}$	$-9.51067784 \times 10^{-1}$	$8.56827412 \times 10^{-5}$	$1.83414589 \times 10^{-10}$
9π	$9.51056516 \times 10^{-1}$	$-9.51915530 \times 10^{-1}$	$4.28269295 \times 10^{-5}$	$1.26963313 \times 10^{-9}$
π	$-1.$	$-9.99915530 \times 10^{-1}$	$8.44696327 \times 10^{-5}$	$7.13511885 \times 10^{-9}$
LSE				$2.169046919839 \times 10^{-6}$

Table 11

The Numerical Results for Example 2 with $n = 10$ and $\tau = 5$

x_i	u_i	S_i	$\lvert u_i - S_i\rvert$	$\lvert u_i - S_i\rvert^2$
0	1	$9.99828516 \times 10^{-1}$	$1.71483912 \times 10^{-4}$	$2.940676226 \times 10^{-8}$
$\frac{\pi}{10}$	$9.87412008 \times 10^{-8}$	$9.51847558 \times 10^{-1}$	0.95184745	0.90601358
$\frac{2\pi}{10}$	$2.01267331 \times 10^{-7}$	$8.09581066 \times 10^{-1}$	0.80951045	0.65530718
$\frac{3\pi}{10}$	$1.60929543 \times 10^{-7}$	$3.09309388 \times 10^{-1}$	0.30930922	$9.56721979 \times 10^{-1}$
$\frac{4\pi}{10}$	$6.07030052 \times 10^{-8}$	$2.05322609 \times 10^{-4}$	$2.05261906 \times 10^{-4}$	$4.21324500 \times 10^{-8}$
$\frac{5\pi}{10}$	$1.39447547 \times 10^{-9}$	$-3.08849005 \times 10^{-1}$	0.3088490064	$9.53877087 \times 10^{-1}$
$\frac{6\pi}{5}$	$2.82203126 \times 10^{-8}$	$-5.87699570 \times 10^{-1}$	0.58769959	0.34539081
$\frac{7\pi}{5}$	$9.68987137 \times 10^{-8}$	$-8.08974167 \times 10^{-1}$	0.80897426	0.65443935
$\frac{8\pi}{5}$	$1.14481596 \times 10^{-7}$	$-9.51067784 \times 10^{-1}$	0.95106875	0.904531772
9π	$2.97953270 \times 10^{-8}$	$-9.51915530 \times 10^{-1}$	0.951915559	0.90614323
π	$-1.$	-1.00001557	1.557×10^{-5}	2.424249×10^{-10}
LSE				4.5628859

Table 12

The Numerical Results for Example 2 with $n = 10$ and $\tau = 179.7764$

x_i	u_i	S_i	$\lvert u_i - S_i\rvert$	$\lvert u_i - S_i\rvert^2$
0	1	0	1.	1
$\frac{\pi}{10}$	$9.51056516 \times 10^{-1}$	0	9.5105651×10^{-1}	$9.04508497 \times 10^{-1}$
$\frac{2\pi}{10}$	$8.09016994 \times 10^{-1}$	0	$8.09016994 \times 10^{-1}$	$6.54508497 \times 10^{-1}$
$\frac{3\pi}{10}$	$5.87785252 \times 10^{-1}$	0	$5.87785252 \times 10^{-1}$	$3.45491503 \times 10^{-1}$
$\frac{4\pi}{10}$	$3.09016994 \times 10^{-1}$	0	$3.09016994 \times 10^{-1}$	$9.54915028 \times 10^{-2}$
$\frac{5\pi}{10}$	$6.12323400 \times 10^{-17}$	0	$6.12323400 \times 10^{-17}$	$3.74939946 \times 10^{-33}$
$\frac{6\pi}{5}$	$-3.09016994 \times 10^{-1}$	0	$3.09016994 \times 10^{-1}$	$9.54915028 \times 10^{-2}$
$\frac{7\pi}{5}$	$-5.87785252 \times 10^{-1}$	0	$5.87785252 \times 10^{-1}$	$3.45491503 \times 10^{-1}$
$\frac{8\pi}{5}$	$-8.09016994 \times 10^{-1}$	0	$8.09016994 \times 10^{-1}$	$6.54508497 \times 10^{-1}$
9π	$-9.51056516 \times 10^{-1}$	0	9.5105651×10^{-1}	$9.04508497 \times 10^{-1}$
π	$-1.$	0	1	1
LSE				5.9999999999

Table 13

The Numerical Results for Example 3 with $n = 5$ and $\tau = 1$

x_i	u_i	Q_i	$\lvert u_i - Q_i\rvert$	$\lvert u_i - Q_i\rvert^2$
0	1	0.99806294	0.00193706	$3.75221081 \times 10^{-6}$
0.2	1.016	1.01417071	0.00182929	$3.34628927 \times 10^{-6}$
0.4	1.128	1.12604257	0.00195743	$3.83151884 \times 10^{-6}$
0.6	1.432	1.42968468	0.00231532	$5.36072097 \times 10^{-6}$
0.8	2.024	2.02005699	0.00394301	$1.55473138 \times 10^{-5}$
1	3	3.00220523	0.00220523	$4.86302129 \times 10^{-6}$
LSE				$3.6701074960 \times 10^{-5}$

Table 14

The Numerical Results for Example 3 with $n = 5$ and $\tau = 5$

x_i	u_i	Q_i	$\lvert u_i - Q_i\rvert$	$\lvert u_i - Q_i\rvert^2$
0	1	0.99806294	0.00193706	$1.84295248 \times 10^{-7}$
0.2	1.016	1.01417071	0.00182929	$4.02037755 \times 10^{-7}$
0.4	1.128	1.12604257	0.00195743	$1.27690052 \times 10^{-6}$
0.6	1.432	1.42968468	0.00231532	$4.74011424 \times 10^{-6}$
0.8	2.024	2.02005699	0.00394301	$2.86127439 \times 10^{-5}$
1	3	3.00220523	0.00220523	$2.14332879 \times 10^{-6}$
LSE				$3.73594204345 \times 10^{-5}$

Table 15

The Numerical Results for Example 3 with $n = 5$ and $\tau = 179.7764$

x_i	u_i	Q_i	$\lvert u_i - Q_i\rvert$	$\lvert u_i - Q_i\rvert^2$
0	1	0.98349595	0.01650405	$2.72383633 \times 10^{-4}$
0.2	1.016	1.00261956	0.01338044	$31.79036304 \times 10^{-4}$
0.4	1.128	1.11802403	0.00997597	$9.95200562 \times 10^{-5}$
0.6	1.432	1.42833778	0.00366222	$1.34118860 \times 10^{-5}$
0.8	2.024	2.03365712	0.00965712	$9.32599357 \times 10^{-5}$
1	3	3.03761808	0.03761808	$1.41511981 \times 10^{-3}$
LSE				0.0020727316215

Table 16

The Numerical Results for Example 3 with $n = 10$ and $\tau = 1$

x_i	u_i	S_i	$\lvert u_i - S_i\rvert$	$\lvert u_i - S_i\rvert^2$
0	1	0.9997679	0.0002321	5.3870937910^{-8}
0.1	1.002	1.0017762	0.0002238	$5.00871848 \times 10^{-8}$
0.2	1.016	1.0157778	0.0002222	$4.93746997 \times 10^{-8}$
0.3	1.054	1.05377251	0.00022749	$5.17498550 \times 10^{-8}$
0.4	1.128	1.12775968	0.00024032	$5.77557608 \times 10^{-8}$
0.5	1.25	1.249738	0.000262	$6.86443139 \times 10^{-8}$
0.6	1.432	1.43170555	0.00029445	$8.67031218 \times 10^{-8}$
0.7	1.686	1.68565824	0.00034176	$1.16799251 \times 10^{-7}$
0.8	2.024	2.02360244	0.00039756	$1.58055550 \times 10^{-7}$
0.9	2.458	2.45741921	0.00058079	$3.37315181 \times 10^{-7}$
1	3	3.00023429	0.00023429	$5.48933512 \times 10^{-8}$
LSE				$1.085249207159 \times 10^{-6}$

Table 17

The Numerical Results for Example 3 with $n = 10$ and $\tau = 5$

x_i	u_i	S_i	$\|u_i - S_i\|$	$\|u_i - S_i\|^2$
0	1	0.99986069	$1.39312375 \times 10^{-4}$	$1.94079379 \times 10^{-8}$
0.1	1.002	1.0018592	$1.40802641 \times 10^{-4}$	$1.98253837 \times 10^{-8}$
0.2	1.016	1.01585209	$1.47913604 \times 10^{-4}$	$2.18784343 \times 10^{-8}$
0.3	1.054	1.0538375	$1.62496806 \times 10^{-4}$	$52.64052120 \times 10^{-8}$
0.4	1.128	1.12781298	$1.87016823 \times 10^{-4}$	$3.49752920 \times 10^{-8}$
0.5	1.25	1.24977517	$2.24826927 \times 10^{-4}$	$5.05471470 \times 10^{-8}$
0.6	1.432	1.43171956	$2.80436529 \times 10^{-4}$	$7.86446468 \times 10^{-8}$
0.7	1.686	1.68563868	$3.61320281 \times 10^{-4}$	$1.30552345 \times 10^{-7}$
0.8	2.024	2.02353461	$4.65394696 \times 10^{-4}$	$2.16592223 \times 10^{-7}$
0.9	2.458	2.45727901	$7.20992595 \times 10^{-4}$	$5.198303221 \times 10^{-7}$
1	3	3.00001603	$1.60253886 \times 10^{-5}$	$2.56813081 \times 10^{-10}$
LSE				$1.1189157570 \times 10^{-6}$

Table 18

The Numerical Results for Example 3 with $n = 10$ and $\tau = 179.7764$

x_i	u_i	S_i	$\|u_i - S_i\|$	$\|u_i - S_i\|^2$
0	1	0.99986069	$0.13930000 \times 10^{-4}$	$1.940727610 \times 10^{-8}$
0.1	1.002	1.00185920	$1.40800000 \times 10^{-4}$	$1.98246400 \times 10^{-8}$
0.2	1.016	1.01585209	$1.47910000 \times 10^{-4}$	$2.18773681 \times 10^{-8}$
0.3	1.054	1.05383750	$1.62500000 \times 10^{-4}$	$2.64062500 \times 10^{-8}$
0.4	1.128	1.12781298	$1.87020000 \times 10^{-4}$	$3.49764804 \times 10^{-8}$
0.5	1.25	1.24977517	$2.24830000 \times 10^{-4}$	$5.05485289 \times 10^{-8}$
0.6	1.432	1.43171956	$2.80440000 \times 10^{-4}$	$7.86465936 \times 10^{-8}$
0.7	1.686	1.68563868	$3.61320000 \times 10^{-4}$	$13.05521424 \times 10^{-8}$
0.8	2.024	2.02353461	$4.65390000 \times 10^{-4}$	$21.65878521 \times 10^{-8}$
0.9	2.458	2.45727900	$7.21000000 \times 10^{-4}$	$51.9841000 \times 10^{-8}$
1	3	3.00001603	$1.60000000 \times 10^{-4}$	$2.56000000 \times 10^{-8}$
LSE				$10.40243756 \times 10^{-8}$

Conclusion

This paper presents numerical solutions for Volterra-Fredholm integral equations and investigates the convergence analysis. Three test examples from previous studies [25–27] are considered. The numerical results from Tables 1-18, indicate that accuracy decreases as τ increases and as n decreases. Additionally, we found that when the exact solution is a linear function, the accuracy is significantly high.

Author Contributions

S.H. Salim did the main part of this research. The results were audited and reviewed by R.K. Saeed and K.H.F. Jwamer. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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