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Well-posedness criteria for one family of boundary value problems

This paper considers a family of linear two-point boundary value problems for systems of ordinary differential equations. The questions of existence of its solutions are investigated and methods of finding approximate solutions are proposed. Sufficient conditions for the existence of a family of linear two-point boundary value problems for systems of ordinary differential equations are established. The uniqueness of the solution of the problem under consideration is proved. Algorithms for finding an approximate solution based on modified of the algorithms of the D.S. Dzhumabaev parameterization method are proposed and their convergence is proved. According to the scheme of the parameterization method, the problem is transformed into an equivalent family of multipoint boundary value problems for systems of differential equations. By introducing new unknown functions we reduce the problem under study to an equivalent problem, a Volterra integral equation of the second kind. Sufficient conditions of feasibility and convergence of the proposed algorithm are established, which also ensure the existence of a unique solution of the family of boundary value problems with parameters. Necessary and sufficient conditions for the well-posedness of the family of linear boundary value problems for the system of ordinary differential equations are obtained.

Keywords: Family of linear boundary value problems, multipoint boundary value problem, existence of solution, singular solution, well-posedness, necessary and sufficient condition.

Introduction

Problem statement and research methods

This paper is devoted to the study of a family of linear boundary value problems for differential equations

$$\frac{\partial v}{\partial t} = A(x,t)v + f(x,t), \qquad (x,t) \in [0,\omega] \times (0,T), \tag{1}$$

$$B_1(x)v(x,0) + B_2(x)v(x,T) = d(x), \qquad x \in [0,\omega], \tag{2}$$

where $(n \times n)$-matrix $A(x,t)$ e n-vector-function $f(x,t)$ are continuous on $[0,\omega] \times [0,T]$, $B_1(x)$, $B_2(x)$ and n-vector-function $d(x)$ are continuous on $[0,\omega]$, x is a parameter of the family ($x \in [0,\omega]$); $\|A(x,t)\| \le a_0$, $\|v(x,t)\| = \max\limits_{i=\overline{1,n}} \|v_i(x,t)\|$.

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In the present paper problem (1), (2) is investigated by the parameterization method [1].

The originallity of the parameterization method lies in the simple idea of introducing parameters at some points of the set on which the boundary value problem is considered, which subsequently allows us to construct an algorithm for finding a solution, obtain sufficient solvability conditions, establish solvability criteria for linear and nonlinear two-point boundary value problems, multipoint boundary value problems, boundary value problems with impulse influence, singular boundary value problems, nonlocal boundary value problems for differential equations, loaded differential equations, integro-differential Fredholm equations, differential equations with delayed argument, partial differential equations and others. These results are presented in the works of Dzhumabaev and his students (Assanova [2], Temesheva [3–7], Orumbayeva [8–10], Uteshova [11, 12], Iskakova [13, 14], Imanchiyev [15, 16], Bakirova [17], Kadirbayeva [18], Tleulessova [19], Abildayeva [20], Abdimanapova [21]).

Dzhumabaev and Assanova [22] studied a nonlocal boundary value problem for systems of linear hyperbolic equations with mixed derivative. A special substitution allowed to reduce this problem to an equivalent boundary value problem, which can be considered as a family of two-point boundary value problems for systems of ordinary differential equations, where the spatial variable servers as a parameter of the family.

This approach can also be used to study the linear nonlocal boundary value problem for a system of partial differential equations ($m = 1, 2, \ldots$)

$$\frac{\partial^{m+1}u}{\partial t \partial x^m} = A(x,t)\frac{\partial^m u}{\partial x^m} + f\,(x,t)\,, \quad u \in \mathbb{R}^n, \quad (x,t) \in [0,\omega] \times (0,T),$$

$$\left.\frac{\partial^k u}{\partial x^k}\right|_{x=0} = \psi_k(t), \quad t \in [0,T], \quad k = 0,1,\ldots,m-1, \quad \frac{\partial^0 u}{\partial x^0} = 0,$$

$$B_1(x)\left.\frac{\partial^m u(x,t)}{\partial x^m}\right|_{t=0} + B_2(x)\left.\frac{\partial^m u(x,t)}{\partial x^m}\right|_{t=T} = d(x).$$

This fact motivated us to investigate problem (1), (2).

In this paper problem (1), (2) is investigated by the parameterization method with a modified algorithm. Sufficient conditions for the existence of a unique solution are obtained. The well-posedness criteria for problem (1), (2) are established.

Notation

- N is a natural number;
- ν is a natural number;
- $\Omega_r = [0,\omega] \times [(r-1)h, rh)$, $h = T/N$, $r = \overline{1,N}$;
- $C([0,\omega],\mathbb{R}^n)$ is the space of continuous functions $d : [0,\omega] \to \mathbb{R}^n$ with the norm $\|d\|_0 = \max\limits_{x\in[0,\omega]} \|d(x)\|$;
- $C([0,\omega] \times [0,T],\, \mathbb{R}^n)$ is the space of continuous functions $v : [0,\omega] \times [0,T] \to \mathbb{R}^n$ with the norm $\|v\|_1 = \max\limits_{(x,t)\in[0,\omega]\times[0,T]} \|v(x,t)\|$;
- the index r takes on the values $1, 2, \ldots, N$;
- the index s takes on the values $1, 2, \ldots, N+1$;
- $C([0,\omega]\times[0,T],\, \Omega_r, \mathbb{R}^{nN})$ is the space of systems of functions $v(x,[t]) = (v_1(x,t), v_2(x,t), \ldots, v_N(x,t))$ with the norm $\|v\|_2 = \max\limits_{r=\overline{1,N}} \sup\limits_{(x,t)\in\Omega_r} \|v_r(x,t)\|$, where the function $v_r : \Omega_r \to \mathbb{R}^n$ is continuous and has a finite limit at $t \to rh - 0$ uniformly with respect to $x \in [0,\, \omega]$ for all r;
- $C([0,\, \omega],\, \mathbb{R}^{n(N+1)})$ is the space of functions $\lambda(x) = (\lambda_1(x), \lambda_2(x), \ldots, \lambda_{N+1}(x))$ with the norm $\|\lambda\|_3 = \max\limits_{s=\overline{1,N+1}} \max\limits_{x\in[0,\omega]} \|\lambda_s(x)\|$, where $\lambda_s : [0,\omega] \to \mathbb{R}^n$ are continuous for all s;

- $C([0,T],\mathbb{R}^n)$ is the space of continuous functions $v:[0,T]\to\mathbb{R}^n$ with the norm $\|v\|_4=\max\limits_{t\in[0,T]}\|v(t)\|$;
- I is the identity matrix of size n;
- O is the zero matrix of size $n\times n$;
- $O^{\langle 1\rangle}$ is the first column of the matrix O.

1 Solvability of a family problems (1), (2)

Definition 1. $v^*(x,t)\in C([0,\omega]\times[0,T],\mathbb{R}^n)$, continuously differentiable with respect to t and satisfying equation (1) and boundary conditions (2) for each fixed $x\in[0,\omega]$, is called a solution of the problem (1), (2).

Problem (1), (2) is investigated by the parameterization method [1]. For a fixed N, we make the partition $[0,\omega]\times[0,T)=\bigcup\limits_{r=1}^{N}\Omega_r$.

According to the scheme of the parameterization method, the problem (1), (2) is transformed into the equivalent family of multipoint boundary value problems with parameter for systems of differential equations

$$\frac{\partial\widetilde{v}_r}{\partial t}=A(x,t)(\widetilde{v}_r+\lambda_r(x))+f(x,t), \tag{3}$$

$$\widetilde{v}_r(x,(r-1)h)=0, \tag{4}$$

$$B_1(x)\lambda_1(x)+B_2(x)\lambda_{N+1}(x)=d(x), \tag{5}$$

$$\lambda_r(x)+\lim_{t\to rh-0}\widetilde{v}_r(x,t)-\lambda_{r+1}(x)=0,\ r=\overline{1,N}, \tag{6}$$

where $(x,t)\in\Omega_r$, $x\in[0,\omega]$, $\lambda_r(x)=v(x,(r-1)h)$, $\lambda_{N+1}(x)=\lim\limits_{t\to T-0}v(x,t)$, $\widetilde{v}_r(x,t)=v(x,t)-\lambda_r(x)$, $r=\overline{1,N}$. A solution of problem (3)–(6) is a pair $(\lambda^*(x),\widetilde{v}^*(x,[t]))$ $\Big(\lambda^*(x)\in C([0,\omega],\mathbb{R}^{n(N+1)})$, $\widetilde{v}^*(x,[t])\in C([0,\omega]\times[0,T],\Omega_r,\mathbb{R}^{nN})\Big)$ such that for each r is continuous and continuously differentiable with respect to t on Ω_r function $\widetilde{v}_r^*(x,t)$ at $\lambda_r(x)=\lambda_r^*(x)$ satisfies equation (3), condition (4), and $\lambda_1^*(x)$, $\lambda_{N+1}^*(x)$, $\lambda_r^*(x)$, $\lim\limits_{t\to rh-0}\widetilde{v}_r^*(x,t)$, satisfy (5), (6).

If the family of pairs $(\lambda^*(x),\widetilde{v}^*(x,[t]))$ is a solution of the family of problems (3)–(6), then the family of functions

$$v^*(x,t)=\begin{cases}\lambda_r^*(x)+\widetilde{v}_r^*(x,t) & \text{for}\quad (x,t)\in\Omega_r,\quad r=\overline{1,N},\\ \lambda_{N+1}^*(x) & \text{for}\quad x\in[0,\omega],\quad t=T\end{cases}$$

is a solution to the family of boundary value problems (1), (2).

If the family of systems of functions $\widehat{v}(x,[t])=(\widehat{v}_1(x,t),\widehat{v}_2(x,t),\dots,\widehat{v}_N(x,t))$ is a solution to problem (1)-(2), then the solution to problem (3)–(6) is the pair $(\widehat{\lambda}(x),\widehat{\widetilde{v}}(x,[t]))$ with elements $\widehat{\lambda}(x)=(\widehat{\lambda}_1(x),\widehat{\lambda}_2(x),\dots,\widehat{\lambda}_{N+1}(x))$, $\widehat{\lambda}_r(x)=\widehat{v}_r(x,(r-1)h)$, $r=\overline{1,N}$, $\widehat{\lambda}_{N+1}(x)=\lim\limits_{t\to T-0}\widehat{v}_N(x,t)$, $x\in[0,\omega]$, $\widehat{\widetilde{v}}(x,[t])=(\widehat{\widetilde{v}}_1(x,t),\widehat{\widetilde{v}}_2(x,t),\dots,\widehat{\widetilde{v}}_2(x,t))$, $\widehat{\widetilde{v}}_r=\widehat{v}_r(x,t)-\widehat{v}_r(x,(r-1)h)$, $(x,t)\in\Omega_r$, $r=\overline{1,N}$.

In problem (3)–(6), the initial conditions (4) appeared for elements of the family of systems of functions $\widetilde{v}(x,[t])$. For a known $\lambda_r(x)$, the Cauchy problem (3), (4) on Ω_r is equivalent to the family of Volterra integral equations of the second kind:

$$\widetilde{v}_r(x,t)=\int\limits_{(r-1)h}^{t}A(x,\tau)\widetilde{v}_r(x,\tau)d\tau+\int\limits_{(r-1)h}^{t}A(x,\tau)d\tau\cdot\lambda_r(x)+\int\limits_{(r-1)h}^{t}f(x,\tau)d\tau. \tag{7}$$

In (7), replacing $\widetilde{v}_r(x,\tau)$ by the right hand side of (7) and repeating this process ν times, we obtain the following representation of the function $\widetilde{v}_r(x,t)$:

$$\widetilde{v}_r(x,t) = D_{\nu,r}(x,t)\cdot\lambda_r(x) + F_{\nu,r}(x,t) + G_{\nu,r}(x,t,\widetilde{v}), \tag{8}$$

where

$$D_{\nu,r}(x,t) = \int\limits_{(r-1)h}^{t} A(x,\tau_1)d\tau_1 + \int\limits_{(r-1)h}^{t} A(x,\tau_1)\int\limits_{(r-1)h}^{\tau_1} A(x,\tau_2)d\tau_2 d\tau_1 + \ldots +$$
$$+ \int\limits_{(r-1)h}^{t} A(x,\tau_1)\int\limits_{(r-1)h}^{\tau_1} A(x,\tau_2)\ldots\int\limits_{(r-1)h}^{\tau_{\nu-1}} A(x,\tau_\nu)d\tau_\nu\ldots d\tau_2 d\tau_1,$$

$$F_{\nu,r}(x,t) = \int\limits_{(r-1)h}^{t} f(x,\tau_1)d\tau_1 + \int\limits_{(r-1)h}^{t} A(x,\tau_1)\int\limits_{(r-1)h}^{\tau_1} f(x,\tau_2)d\tau_2 d\tau_1 + \ldots +$$
$$+ \int\limits_{(r-1)h}^{t} A(x,\tau_1)\ldots\int\limits_{(r-1)h}^{\tau_{\nu-2}} A(x,\tau_{\nu-1})\int\limits_{(r-1)h}^{\tau_{\nu-1}} f(x,\tau_\nu)d\tau_\nu d\tau_{\nu-1}\ldots d\tau_1,$$

$$G_{\nu,r}(t,x,\widetilde{v}) = \int\limits_{(r-1)h}^{t} A(x,\tau_1)\ldots\int\limits_{(r-1)h}^{\tau_{\nu-1}} A(x,\tau_\nu)\widetilde{v}_r(x,\tau_\nu)d\tau_\nu\ldots d\tau_1,$$

$t\in[(r-1)h, rh)$, $r=\overline{1,N}$.

Determining from (8) the limits

$$\lim_{t\to rh-0}\widetilde{v}_r(x,t) = D_{\nu,r}(x,rh)\cdot\lambda_r(x) + F_{\nu,r}(x,rh) + G_{\nu,r}(rh,x,\widetilde{v}),\quad x\in[0,\omega],\quad r=\overline{1,N},$$

substituting them into (5), (6) and multiplying (5) by $h>0$, we obtain the family of systems of linear algebraic equations with respect to $\lambda_r(x)$, $x\in[0,\omega]$:

$$hB_1(x)\lambda_1(x) + hB_2(x)\lambda_{N+1}(x) = hd(x), \tag{9}$$

$$(I + D_{\nu,r}(x,rh))\lambda_r(x) - \lambda_{r+1}(x) = -F_{\nu,r}(x,rh) - G_{\nu,r}(rh,x,\widetilde{v}),\quad r=\overline{1,N}. \tag{10}$$

We write system (9), (10) in the form:

$$Q_\nu(h,x)\lambda(x) = -F_\nu(h,x) - G_\nu(h,x,\widetilde{v}),\quad \lambda(x)\in C([0,\omega],\mathbb{R}^{n(N+1)}),$$

where

$$Q_\nu(h,x) = \begin{pmatrix} hB_1(x) & O & O & \ldots & O & hB_2(x)\\ I+D_{\nu,1}(x,h) & -I & O & \ldots & O & O\\ O & I+D_{\nu,2}(x,2h) & -I & \ldots & O & O\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ O & O & O & \ldots & -I & O\\ O & O & O & \ldots & I+D_{\nu,N}(x,Nh) & -I \end{pmatrix},$$

$$F_\nu(h,x)=(-hd(x),F_{\nu,1}(h,x),F_{\nu,2}(2h,x),\dots,F_{\nu,N}(Nh,x)),$$
$$G_\nu(h,x,\widetilde{v})=(O^{\langle 1\rangle},G_{\nu,1}(h,x,\widetilde{v}),G_{\nu,2}(2h,x,\widetilde{v}),\dots,G_{\nu,N}(Nh,x,\widetilde{v})).$$

As can be seen, the process of finding a solution to problem (1), (2) is reduced to solving a family of systems of linear algebraic equations (10) for some $\widetilde{v}(x,[t])$ and solving the family of Cauchy problems (3), (4) on Ω_r when $\lambda_r(x)$, $r=\overline{1,N}$ is found.

Let us describe the algorithm for finding a solution to problem (3)–(6). Let the matrix $Q_\nu(h,x)$ be reversible for all $x\in[0,\omega]$.

Step 0. (*a*) The family of parameters $\lambda^{(1)}(x)$ is found from the equation $Q_\nu(h,x)\lambda(x)=-F_\nu(h,x)$.

(*b*) We determine the components of the system of functions $\widetilde{v}^{(0)}(x,[t])$ by solving the Cauchy problems (3), (4) on Ω_r at $\lambda_r(x)=\lambda_r^{(0)}(x)$, $r=\overline{1,N}$.

(*c*) On $[0,\omega]\times[0,T]$ we define the function

$$v^{(0)}(x,t)=\begin{cases}\lambda_r^{(0)}(x)+\widetilde{v}_r^{(0)}(x,t) & \text{for} \quad (x,t)\in\Omega_r, \quad r=\overline{1,N},\\ \lambda_{N+1}^{(0)}(x) & \text{for} \quad x\in[0,\omega], \quad t=T.\end{cases}$$

Step 1. (*a*) The family of parameters $\lambda^{(1)}(x)$ is found from the equation $Q_\nu(h,x)\lambda(x)=-F_\nu(h,x)-G_\nu(h,x,\widetilde{v}^{(0)})$.

(*b*) We determine the components of the system of functions $\widetilde{v}^{(1)}(x,[t])$ by solving the Cauchy problems (3), (4) on Ω_r at $\lambda_r(x)=\lambda_r^{(1)}(x)$, $r=\overline{1,(N+1)}$.

(*c*) On $[0,\omega]\times[0,T]$ we define the function

$$v^{(1)}(x,t)=\begin{cases}\lambda_r^{(1)}(x)+\widetilde{v}_r^{(1)}(x,t) & \text{for} \quad (x,t)\in\Omega_r, \quad r=\overline{1,N},\\ \lambda_{N+1}^{(1)}(x) & \text{for} \quad x\in[0,\omega], \quad t=T.\end{cases}$$

At the k-th step, we find the pair $(\lambda^{(k)}(x),\widetilde{v}^{(k)}(x,[t]))$, $k=0,1,2,\dots$. On $\bar{\Omega}$ we define the piecewise continuous function

$$v^{(k)}(x,t)=\begin{cases}\lambda_r^{(k)}(x)+\widetilde{v}_r^{(k)}(x,t) & \text{for} \quad (x,t)\in\Omega_r, \quad r=\overline{1,N},\\ \lambda_{N+1}^{(k)}(x) & \text{for} \quad x\in[0,\omega], \quad t=T.\end{cases}$$

Condition 1. For some $h>0: Nh=T$, ν and for any $x\in[0,\omega]$ the matrix $Q_\nu(h,x):\mathbb{R}^{n(N+1)}\to\mathbb{R}^{n(N+1)}$ is invertible and the following inequalities are satisfied:

$$\|(Q_\nu(h,x))^{-1}\|\le\gamma_\nu(h,x)\le\gamma_\nu(h),$$
$$q_\nu(h)=\gamma_\nu(h)\Big\{e^{a_0h}-\sum_{j=0}^{\nu}\frac{(a_0h)^j}{j!}\Big\}<1. \tag{11}$$

The following statement establishes sufficient conditions for the feasibility and convergence of the proposed algorithm. It should be noted that this statement ensures the existence of a unique solution of the family of boundary value problems with parameters (3)–(6).

Theorem 1. Let Condition 1 be met. Then the sequence of pairs $(\lambda^{(k)}(x),\widetilde{v}^{(k)}(x,[t]))$ converges to the unique solution $(\lambda^*(x),\widetilde{v}^*(x,[t]))$ of problem (3)–(6) and the following estimates hold true:

$$\|\lambda^*-\lambda^{(k)}\|_3\leqslant\frac{q_\nu(h)}{1-q_\nu(h)}\|\lambda^{(k)}-\lambda^{(k-1)}\|_3, \tag{12}$$

$$\|\widetilde{v}_r^*(x,t)-\widetilde{v}_r^{(k)}(x,t)\|\le\big(e^{a_0(t-(r-1)h)}-1\big)\|\lambda_r^*(x)-\lambda_r^{(k)}(x)\|, \tag{13}$$

where $k=1,2,\dots$, $(x,t)\in\Omega_r$, $r=\overline{1,N}$.

Proof. The continuity of the matrices $A(x,t)$ and $B_1(x)$, $B_2(x)$ on $[0,\omega]\times[0,T]$ and $[0,\omega]$, respectively, implies the continuity of the matrix $Q_\nu(h,x):\mathbb{R}^{n(N+1)}\to\mathbb{R}^{n(N+1)}$ on $[0,\omega]$. Let us fix $\widetilde{x}$, $\widehat{x}\in[0,\omega]$. The matrix $(Q_\nu(h,x))^{-1}:\mathbb{R}^{n(N+1)}\to\mathbb{R}^{n(N+1)}$ is continuous for all $x\in[0,\omega]$, since the inequality $\|(Q_\nu(h,\widetilde{x}))^{-1}-(Q_\nu(h,\widehat{x}))^{-1}\|\le\gamma_\nu^2(h))\|Q_\nu(h,\widehat{x})-Q_\nu(h,\widetilde{x})\|$ holds.

The solution of problem (3)-(6) is found by the algorithm. Solving the equation $Q_\nu(h,x)\lambda(x)=-F_\nu(h,x)$, we find $\lambda^{(0)}(x)$. Since the matrix $(Q_\nu(h,x))^{-1}$ and the vector $F_\nu(h,x)$ are continuous for all $x\in[0,\omega]$, we have $\lambda^{(0)}(x)\in C([0,\omega],\mathbb{R}^{n(N+1)})$ and

$$\|\lambda^{(0)}\|_3\le\gamma_\nu(h)h\max\Big\{1,\sum_{j=0}^{\nu-1}\frac{(a_0h)^j}{j!}\Big\}\max\{\|d\|_0,\|f\|_1\}.$$

For any r and $x\in[0,\omega]$, we find the function $\widetilde{v}_r^{(0)}(x,t)$ from the Cauchy problem (3), (4) with $\lambda_r(x)=\lambda_r^{(0)}(x)$:

$$\frac{\partial\widetilde{v}_r}{\partial t}=A(x,t)\widetilde{v}_r+A(x,t)\lambda_r^{(0)}(x)+f(x,t),\quad \widetilde{v}_r(x,(r-1)h)=0,\quad r=\overline{1,N}.$$

Then for $\widetilde{v}_r^{(0)}(x,t)$ we have the estimate

$$\|\widetilde{v}_r^{(0)}(x,t)\|\le\big(e^{a_0(t-(r-1)h)}-1\big)\|\lambda_r^{(0)}(x)\|+(t-(r-1)h)e^{a_0(t-(r-1)h)}\|f\|_1,$$

whence it follows that

$$\|\widetilde{v}^{(0)}\|_2\le\big(e^{a_0h}-1\big)\|\lambda^{(0)}\|_3+he^{a_0h}\|f\|_1.$$

Then, following the algorithm, we solve the equation $Q_\nu(h,x)\lambda(x)=-F_\nu(h,x)-G_\nu(h,x,\widetilde{v}^{(0)})$ and find $\lambda^{(1)}(x)$. We have

$$\|\lambda^{(1)}-\lambda^{(0)}\|_3=\|-(Q_\nu(h,x))^{-1}\cdot G_\nu(h,x,\widetilde{v}^{(0)})\|\le\gamma_\nu(h)\max_{r=\overline{1,N}}\|G_{\nu,r}(rh,x,\widetilde{v}^{(0)})\|\Big\}\le$$

$$\le\gamma_\nu(h)\max_{r=\overline{1,N}}\Big\{\int_{(r-1)h}^{rh}a_0\dots\int_{(r-1)h}^{\tau_{\nu-1}}a_0\|\widetilde{v}_r^{(0)}(x,\tau_\nu)\|d\tau_\nu\dots d\tau_1\Big\}\le\gamma_\nu(h)\frac{(a_0h)^\nu}{\nu!}\|\widetilde{v}^{(0)}\|_2.$$

We define the components of the system of functions $\widetilde{v}^{(1)}(x,[t])=(\widetilde{v}_1^{(1)}(x,t),\widetilde{v}_2^{(1)}(x,t),\dots,\widetilde{v}_N^{(1)}(x,t))$ by solving the Cauchy problem (3), (4) with $\lambda_r(x)=\lambda_r^{(1)}(x)$:

$$\frac{\partial\widetilde{v}_r}{\partial t}=A(x,t)\widetilde{v}_r+A(x,t)\lambda_r^{(1)}(x)+f(x,t),\quad \widetilde{v}_r(x,(r-1)h)=0,\quad r=\overline{1,N}.$$

The difference $(\widetilde{v}_r^{(1)}(x,t)-\widetilde{v}_r^{(0)}(x,t))$ is estimated as follows:

$$\|\widetilde{v}_r^{(1)}(x,t)-\widetilde{v}_r^{(0)}(x,t)\|\le\big(e^{a_0(t-(r-1)h)}-1\big)\|\lambda_r^{(1)}(x)-\lambda_r^{(0)}(x)\|.$$

We assume that the pair $(\lambda^{(k-1)}(x),\widetilde{v}^{(k-1)}(x,[t]))$ is determined and for all $(x,t)\in\Omega_r$ the following inequalities hold:

$$\|\lambda^{(k-1)}-\lambda^{(k-2)}\|_3\le q_\nu(h)\|\lambda^{(k-2)}-\lambda^{(k-3)}\|_3,$$

$$\|\widetilde{v}_r^{(k-1)}(x,t)-\widetilde{v}_r^{(k-2)}(x,t)\|\le\big(e^{a_0(t-(r-1)h)}-1\big)\|\lambda_r^{(k-1)}(x)-\lambda_r^{(k-2)}(x)\|.\qquad(14)$$

At the k-th step of the algorithm, solving the equation $Q_\nu(h,x)\lambda(x)=-F_\nu(h,x)-G_\nu(h,x,\widetilde{v}^{(k-1)})$, we find $\lambda^{(k)}(x)$. Taking into account (14), we establish that

$$\|\lambda^{(k)}-\lambda^{(k-1)}\|_3\le q_\nu(h)\|\lambda^{(k-1)}-\lambda^{(k-2)}\|_3,\quad k=2,3,\dots.\qquad(15)$$

We define the components of the system of functions $\widetilde{v}^{(k)}(x,[t]) = (\widetilde{v}_1^{(k)}(x,t), \widetilde{v}_2^{(k)}(x,t), \ldots, \widetilde{v}_N^{(k)}(x,t))$ by solving the Cauchy problem (3), (4) with $\lambda_r(x) = \lambda_r^{(k)}(x)$:

$$\frac{\partial \widetilde{v}_r}{\partial t} = A(x,t)\widetilde{v}_r + A(x,t)\lambda_r^{(k)}(x) + f(x,t), \quad \widetilde{v}_r(x,(r-1)h) = 0, \quad r = \overline{1,N}.$$

For all $(x,t) \in \Omega_r$, $r = \overline{1,N}$ $(k = 1,2,3,\ldots)$ we estimate the difference $(\widetilde{v}_r^{(k)}(x,t) - \widetilde{v}_r^{(k-1)}(x,t))$:

$$\|\widetilde{v}_r^{(k)}(x,t) - \widetilde{v}_r^{(k-1)}(x,t)\| \le \big(e^{a_0(t-(r-1)h)} - 1\big)\|\lambda_r^{(k)}(x) - \lambda_r^{(k-1)}(x)\|. \tag{16}$$

By the condition of Theorem, $q_\nu(h) < 1$, so it follows from (15), (16) that the pair $(\lambda^{(k)}(x), \widetilde{v}^{(k)}(x,[t]))$, $k = 0,1,2,\ldots$, converges to $(\lambda^*(x), \widetilde{v}^*(x,[t]))$, the solution of problem (3)–(6) in $C([0,\omega],\mathbb{R}^{n(N+1)}) \times C([0,\omega]\times[0,T],\Omega_r,\mathbb{R}^{nN})$.

It is not difficult to establish the validity of the inequalities:

$$\|\lambda^{(k+\ell)} - \lambda^{(k)}\|_3 \leqslant \frac{q_\nu(h)}{1-q_\nu(h)}\|\lambda^{(k)} - \lambda^{(k-1)}\|_3, \tag{17}$$

$$\|\lambda^{(k)} - \lambda^{(0)}\|_3 \leqslant \frac{1-q_\nu^k(h)}{1-q_\nu(h)}\gamma_\nu(h)\frac{(a_0h)^\nu}{\nu!}\|\widetilde{v}^{(0)}\|_2,$$

$$\begin{aligned}
\|\widetilde{v}_r^{(k+\ell)}(x,t) - \widetilde{v}_r^{(k)}(x,t)\| &\le \big(e^{a_0(t-(r-1)h)} - 1\big)\|\lambda_r^{(k+\ell)}(x) - \lambda_r^{(k)}(x)\|, \\
\|\widetilde{v}_r^{(k)}(x,t) - \widetilde{v}_r^{(0)}(x,t)\| &\le \big(e^{a_0(t-(r-1)h)} - 1\big)\|\lambda_r^{(k)}(x) - \lambda_r^{(0)}(x)\|,
\end{aligned} \tag{18}$$

$(x,t) \in \Omega_r$, $r = \overline{1,N}$, $k = 1,2,\ldots$. In the inequalities (17), (18), letting $\ell \to \infty$, we establish the validity of the estimates (12), (13).

Let us show the uniqueness of the solution of problem (3)–(6). Let $v^*(x,t)$ and $\widehat{v}(x,t)$ be two solutions of problem (1), (2). Then the pairs $(\lambda^*(x), \widetilde{v}^*(x,[t]))$ and $(\widehat{\lambda}(x), \widehat{\widetilde{v}}(x,[t]))$ are solutions to the boundary value problem (3)–(6), here

$$\lambda^*(x) \in C([0,\,\omega],\,\mathbb{R}^{n(N+1)}), \quad \lambda_s^*(x) = v^*(x,(s-1)h), s = \overline{1,N+1},$$

$$\widetilde{v}_r^*(x,[t]) \in C([0,\omega]\times[0,T],\,\Omega_r,\mathbb{R}^{nN}),$$

$$\widetilde{v}_r^*(x,t) = v^*(x,t) - v^*(x,(r-1)h), \quad (x,t) \in \Omega_r, \quad r = \overline{1,N},$$

$$\widehat{\lambda}(x) \in C([0,\,\omega],\,\mathbb{R}^{n(N+1)}), \quad \widehat{\lambda}_s(x) = \widehat{v}(x,(s-1)h), \quad s = \overline{1,N+1},$$

$$\widehat{\widetilde{v}}(x,[t]) \in C([0,\omega]\times[0,T],\,\Omega_r,\mathbb{R}^{nN}),$$

$$\widehat{\widetilde{v}}_r(x,t) = \widehat{v}(x,t) - \widehat{v}(x,(r-1)h), \quad (x,t) \in \Omega_r, \quad r = \overline{1,N}.$$

Under our assumptions, the following equations hold:

$$\widetilde{v}_r^*(x,t) = \int\limits_{(r-1)h}^{t} A(x,\tau)\widetilde{v}_r^*(x,\tau)d\tau + \int\limits_{(r-1)h}^{t} A(x,\tau)d\tau\cdot\lambda_r^*(x) + \int\limits_{(r-1)h}^{t} f(x,\tau)d\tau,$$

$$\widehat{\widetilde{v}}_r(x,t) = \int\limits_{(r-1)h}^{t} A(x,\tau)\widehat{\widetilde{v}}_r(x,\tau)d\tau + \int\limits_{(r-1)h}^{t} A(x,\tau)d\tau\cdot\widehat{\lambda}_r(x) + \int\limits_{(r-1)h}^{t} f(x,\tau)d\tau,$$

$$Q_\nu^{-1}(h,x)\lambda^*(x) = -\big(F_\nu(h,x) + G_\nu(h,x,\widetilde{v}^*)\big),$$

$$Q_\nu^{-1}(h,x)\widehat{\lambda}(x) = -\big(F_\nu(h,x) + G_\nu(h,x,\widehat{\widetilde{v}})\big).$$

Then the following inequalities are true

$$\|\widetilde{v}^* - \widehat{\widetilde{v}}\|_2 \le \Big(e^{a_0 h} - 1\Big) \cdot \|\lambda^* - \widehat{\lambda}\|_3, \tag{19}$$

$$\|\lambda^* - \widehat{\lambda}\|_3 \le q_\nu(h)\|\lambda^* - \widehat{\lambda}\|_3.$$

Hence, by virtue of inequality (11), $\lambda^*(x) = \widehat{\lambda}(x)$. Then from (19) we obtain that $v^*(x,t) = \widehat{v}(x,t)$ for $(x,t) \in [0,\omega] \times [0,T]$. Theorem 1 is proved.

Since problem (1), (2) and problem (3)–(6) are equivalent, the following statement holds true.

Corollary 1. Let Condition 1 be met. Then the sequence $v^{(k)}(x,t)$ $(k = 0,1,2,\ldots)$ converges to the unique solution $v^*(x,t)$ of problem (1), (2) and the following estimates are true:

$$\|v^* - v^{(0)}\|_1 \le \frac{\gamma_\nu(h)e^{a_0 h}}{1 - q_\nu(h)} \cdot \frac{(a_0 h)^\nu}{\nu!} \left(\big(e^{a_0 h} - 1\big) \max_{s=\overline{1,N+1}} \max_{x\in[0,\omega]} \|v^{(0)}(x,(s-1)h)\| + he^{a_0 h}\|f\|_1 \right).$$

2 Well-posedness criteria for the family of problems (1), (2)

Definition 2. The boundary value problem (1), (2) is called well-posed if for any $f(x,t) \in C([0,\omega] \times [0,T], \mathbb{R}^n)$, $d(x) \in C([0,\omega], \mathbb{R}^n)$ it has a unique solution $v(x,t)$ and

$$\|v\|_1 \le K \max\Big\{\|d\|_1, \|f\|_1\Big\},$$

where K is a constant, independent of $f(x,t)$ and $d(x)$. The number K is called the well-posedness constant of problem (1), (2).

Let us consider the equation

$$\frac{1}{h}Q_*(h,x)\lambda(x) = -F_*(h,A,f,d,x), \qquad \lambda(x) \in C([0,\omega], \mathbb{R}^{n(N+1)}),$$

where $Q_*(h,x) = \lim\limits_{\nu\to\infty} Q_\nu(h,x)$, $F_*(h,A,f,d,x) = \lim\limits_{\nu\to\infty} \dfrac{1}{h}F_\nu(h,x)$.

Theorem 2. The boundary value problem (1), (2) is well-posed for all $x \in [0,\omega]$ if and only if there exists $h_0 \in (0,T]$ such that for any $h \in (0,h_0] : Nh = T$ there is a number $\nu = \nu(h)$, such that the matrix $Q_\nu(h,x) : \mathbb{R}^{n(N+1)} \to \mathbb{R}^{n(N+1)}$ is invertible and the following inequalities hold:

$$\|(Q_\nu(h,x))^{-1}\| \le \gamma_\nu(h), \tag{20}$$

$$q_\nu(h) = \gamma_\nu(h)\Big\{e^{a_0 h} - \sum_{j=0}^{\nu} \frac{(a_0 h)^j}{j!}\Big\} < 1. \tag{21}$$

Proof. The sufficiency of the conditions of Theorem 2 for the well-posedness of problem (1), (2) follows from Corollary 1.

Necessity. Let problem (1), (2) be well-posed with a constant K. Problem (1), (2) for every fixed $\widehat{x} \in [0,\omega]$ is a linear two-point boundary value problem for the ordinary differential equation:

$$\frac{d\widehat{v}}{dt} = \widehat{A}(t)\widehat{v} + \widehat{f}(t), \quad t \in (0,T), \quad \widehat{v} \in \mathbb{R}^n, \tag{22}$$

$$\widehat{B}_1\widehat{v}(0) + \widehat{B}_2\widehat{v}(T) = \widehat{d}. \tag{23}$$

Here $\widehat{v}(t) = v(\widehat{x}, t)$, $\widehat{A}(t) = A(\widehat{x}, t)$, $\widehat{f}(t) = f(\widehat{x}, t)$,$\widehat{B}_1 = B_1(\widehat{x})$, $\widehat{B}_2 = B_2(\widehat{x})$, $\widehat{d} = d(\widehat{x})$.

Since for $f(x,t) = \widehat{f}(t)$, $d(x) = \widehat{d}$ we have:

$$\|\widehat{v}^*\|_4 = \max_{t\in[0,T]} \|v^*(\widehat{x},t)\| \le \max_{(x,t)\in[0,\omega]\times[0,T]} \|v^*(x,t)\| \le K\max\{\|d\|_0, f\|_1\} = K\max\{\|\widehat{d}\|, \|\widehat{f}\|_4\},$$

then the correct solvability of problem (1), (2) follows from the correct solvability of problem (22), (23) with constant K for every fixed $\widehat{x} \in [0,\omega]$.

For any $\varepsilon > 0$ there is $h_0 \in (0,T]$, satisfying the inequality

$$\frac{1}{a_0h_0}(e^{a_0h_0} - 1 - a_0h_0) \le \frac{\varepsilon}{(2+\varepsilon)(1+\varepsilon)}.$$

Then, by Theorem 3 [1; p. 42], we obtain the following estimate for all $h \in (0, h_0] : Nh = T$:

$$\|(Q_*(h,\widehat{x}))^{-1}\| \leqslant \frac{(1+\varepsilon)K}{h}.$$

In view of the arbitrariness of $\widehat{x} \in [0,\omega]$, we obtain

$$\|(Q_*(h,x))^{-1}\| \leqslant \frac{(1+\varepsilon)K}{h}, \quad \forall x \in [0,\omega].$$

Let us choose ν_1 such that:

$$\frac{2(1+\varepsilon)K}{h}\Big\{e^{a_0h} - \sum_{j=0}^{\nu_1}\frac{(a_0h)^j}{j!}\Big\} < 1.$$

For any ν, we have there is the inequality

$$\|Q_*(h,x) - Q_\nu(h,x)\| \leqslant \sum_{j=\nu+1}^{\infty}\frac{(a_0h)^j}{j!} = \Big\{e^{a_0h} - \sum_{j=0}^{\nu}\frac{(a_0h)^j}{j!}\Big\}.$$

Then it follows from the theorem on small perturbations of boundedly invertible operators that for all $\nu \ge \nu_1$ the matrix $Q_\nu(h,x) : \mathbb{R}^{n(N+1)} \to \mathbb{R}^{n(N+1)}$ is invertible and

$$\|(Q_\nu(h,x))^{-1}\| \leqslant \frac{\|(Q_*(h,x))^{-1}\|}{1 - \|(Q_*(h,x))^{-1}\| \cdot \|Q_*(h,x) - Q_\nu(h,x)\|} < \frac{2(1+\varepsilon)K}{h}.$$

Thus, for all $\nu \ge \nu_1$, $h \in (0,h_0] : Nh = T$ and $x \in [0,\omega]$, taking $\gamma_\nu(h) = \dfrac{2(1+\varepsilon)K}{h}$, we obtain that the inequalities (20), (21). Theorem 2 is proved.

Theorem 3. The boundary value problem (1), (2) is well-posed if and only if for any ν there exists $h = h(\nu) : Nh = T$, such that the matrix $Q_\nu(h,x) : \mathbb{R}^{n(N+1)} \to \mathbb{R}^{n(N+1)}$ is invertible for all $x \in [0,\omega]$ and the inequalities (20), (21) are true.

Proof. Sufficiency. The well-posedness of problem (1), (2) under the conditions of Theorem follows from Corollary 1.

Necessity. Let the problem (1), (2) be well-posed with constant K. Reasoning as in the proof of Theorem 2, for a given $\varepsilon > 0$ we find $h_0 = h_0(\varepsilon)$ such that for all $h \in (0,h_0] : Nh = T$ and $x \in [0,\omega]$ the matrix $Q_*(h,x) : \mathbb{R}^{n(N+1)} \to \mathbb{R}^{n(N+1)}$ is invertible and

$$\|(Q_*(h,x))^{-1}\| \leqslant \frac{(1+\varepsilon)K}{h}.$$

We choose $h_1 \in (0, h_0]$ such that the relation is satisfied:

$$\frac{2(1+\varepsilon)K}{h_1}\Big\{e^{a_0 h_1} - \sum_{j=0}^{\nu}\frac{(a_0 h_1)^j}{j!}\Big\} < 1. \tag{24}$$

Since $\|(Q_*(h,x))^{-1}\| \cdot \|Q_*(h,x) - Q_\nu(h,x)\| < 0.5$, then, by virtue of (24), by the small perturbation theorem of boundedly reversible operators, for all $h \in (0, h_1] : Nh = T$ and $x \in [0, \omega]$ the inequality holds $\|(Q_\nu(h,x))^{-1}\| < \dfrac{2(1+\varepsilon)K}{h}$.

Taking $\gamma_\nu(h) = \dfrac{2(1+\varepsilon)K}{h}$, by virtue of choosing $h \in (0, h_1] : Nh = T$, we obtain the fulfillment of inequalities (20) and (21). Theorem 3 is proved.

Theorem 4. Let for some ν there exist $h_0 = h_0(\nu)$ such that for all $h \in (0, h_0] : Nh = T$ and $x \in [0, \omega]$ the matrix $Q_\nu(h,x) : \mathbb{R}^{n(N+1)} \to \mathbb{R}^{n(N+1)}$ is invertible and

$$\|(Q_\nu(h,x))^{-1}\| \le \frac{\gamma}{h},$$

where γ is a constant, independent of h and x. Then problem (1), (2) is well-posed with constant $K = \gamma$.

Proof. For any $\varepsilon > 0$ there is $h_0 \in (0, T]$ satisfying the inequality

$$\frac{1}{a_0 h_0}(e^{a_0 h_0} - 1 - a_0 h_0) \le \frac{\varepsilon}{(2+\varepsilon)(1+\varepsilon)}.$$

We choose $h_1 \in (0, h_0] : Nh_1 = T$ such that the following inequality is satisfied:

$$\frac{\gamma}{h_1}\Big\{e^{a_0 h_1} - \sum_{j=0}^{\nu}\frac{(a_0 h_1)^j}{j!}\Big\} < 1.$$

Then $q_\nu(h) \le q_\nu(h_1) < 1$ for all $h \in (0, h_1] : Nh = T$ and, by Corollary 1, the problem (1), (2) has a unique solution $v^*(x,t)$ and

$$\max_{(x,t)\in[0,\omega]\times[0,T]}\|v^*(x,t)\| \le e^{a_0 h}\Big(\Big(\frac{\gamma}{1-q_\nu(h)}\cdot\frac{(a_0 h)^\nu}{\nu!}\cdot\frac{e^{a_0 h}-1}{h}+1\Big)\times$$
$$\times\,\gamma\max\Big\{1, \sum_{j=0}^{\nu-1}\frac{(a_0 h)^j}{j!}\Big\} + \frac{\gamma}{1-q_\nu(h)}\frac{(a_0 h)^\nu}{\nu!}e^{a_0 h}\Big)\max\{\|d\|_0, \|f\|_1\} + he^{a_0 h}\|f\|_1.$$

Letting $h \to 0$ in the above inequality, we obtain that

$$\max_{(x,t)\in[0,\omega]\times[0,T]}\|v^*(x,t)\| \le \gamma\max\{\|d\|_0, \|f\|_1\}.$$

Theorem 4 is proved.

Theorem 5. Let problem (1), (2) be well-posed with constant K. Then for any ν and $\varepsilon > 0$ there exists $h_0 = h_0(\nu, \varepsilon)$ such that for all $h \in (0, h_0] : Nh = T$ and $x \in [0, \omega]$ the matrix $Q_\nu(h,x) : \mathbb{R}^{n(N+1)} \to \mathbb{R}^{n(N+1)}$ is invertible and

$$\|(Q_\nu(h,x))^{-1}\| \le \frac{(1+\varepsilon)K}{h}.$$

Proof. For a given $\varepsilon > 0$, find $h_0 = h_0(\varepsilon)$ such that for all $h \in (0, h_0] : Nh = T$ and $x \in [0, \omega]$ the matrix $Q_*(h, x) : \mathbb{R}^{n(N+1)} \to \mathbb{R}^{n(N+1)}$ is invertible and the following estimate holds true:

$$\|(Q_*(h,x))^{-1}\| \leqslant \frac{(2+\varepsilon)K}{2h}.$$

Let us choose $h_1 \in (0, h_0]$ satisfying the inequality:

$$\frac{(2+\varepsilon)K}{h_1}\Big\{e^{a_0 h_1} - \sum_{j=0}^{\nu} \frac{(a_0 h_1)^j}{j!}\Big\} < \frac{\varepsilon}{1+\varepsilon}.$$

Since $\|(Q_*(h,x))^{-1}\| \cdot \|Q_*(h,x) - Q_\nu(h,x)\| < \frac{1}{2} \cdot \frac{\varepsilon}{1+\varepsilon}$ then, the theorem on small perturbations of boundedly invertible operators, for all $h \in (0, h_1] : Nh = T$ and $x \in [0, \omega]$ the following estimate holds $\|(Q_\nu(h,x))^{-1}\| < \frac{(1+\varepsilon)K}{h} = \gamma_\nu(h)$ and, based on (24),

$$q_\nu(h) = \gamma_\nu(h)\Big\{e^{a_0 h} - \sum_{j=0}^{\nu} \frac{(a_0 h)^j}{j!}\Big\} < \frac{\varepsilon}{2+\varepsilon} < 1.$$

Then, according Corollary 1, there exists a unique solution $v^*(x,t)$ of problem (1), (2) and the following estimate holds:

$$\max_{(x,t)\in[0,\omega]\times[0,T]} \|v^*(x,t)\| \leq e^{a_0 h}\Big(\Big(\frac{(1+\varepsilon)K}{1-q_\nu(h)} \cdot \frac{(a_0 h)^\nu}{\nu!} \cdot \frac{e^{a_0 h}-1}{h} + 1\Big)(1+\varepsilon)K \times$$
$$\times \max\Big\{1, \sum_{j=0}^{\nu-1} \frac{(a_0 h)^j}{j!}\Big\} + \frac{(1+\varepsilon)K}{1-q_\nu(h)} \frac{(a_0 h)^\nu}{\nu!} e^{a_0 h}\Big) \max\{\|d\|_0, \|f\|_1\} + h e^{a_0 h} \|f\|_1.$$

Letting $h \to 0$, we obtain the estimate $\max\limits_{(x,t)\in[0,\omega]\times[0,T]} \|v^*(x,t)\| \leq (1+\varepsilon)K \max\{\|d\|_0, \|f\|_1\}$.
Theorem 5 is proved.

Conclusion

The paper proposes a modified algorithm of the parameterization method: an additional parameter is introduced and at the last point of the segment on which the boundary value problem is considered. This is the difference between the proposed modified algorithm and the classical algorithm of the parameterization method. This modification allows us to simplify the structure of the linear operator equation with respect to the introduced parameters. Sufficient conditions for the existence of a single solution of the problem (1), (2) and criteria of correct solvability of the family of linear boundary value problems for the system of ordinary differential equations are obtained. Note that the idea of the methodology used in this paper has wide prospects of development for the study of problems of solutions of linear and nonlinear boundary value problems.

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Шеттік есептің бір үйірінің қисынды шешімділік критерийлері туралы

Мақалада дифференциалдық теңдеулер жүйелері үшін сызықтық екі нүктелі шеттік есептер үйірі карастырылған. Оның шешімдерінің бар болу сұрақтары зерттеліп, жуық шешімді табу әдістері ұсынылған. Жәй дифференциалдық теңдеулер жүйесі үшін сызықтық екі нүктелі шеттік есептер үйірінің жеткілікті шарттары анықталған. Қарастырылған есептің шешімінің жалғыздығы дәлелденді. Д.С. Жұмабаевтың параметрлеу әдісінің алгоритмдерінің бір модификациясы негізінде зерттелетін есептің жуық шешімін табу алгоритмдері берілген және олардың жинақтылығы дәлелденген. Параметрлеу әдісінің схемасы бойынша есеп дифференциалдық теңдеулер жүйелері үшін көп нүктелі шеттік есептерінің эквивалентті үйіріне түрлендірілген. Жаңа белгісіз функцияларды енгізу арқылы біз зерттелетін есепті баламалы есепке, екінші текті Вольтерра интегралдық теңдеуіне келтіреміз. Параметрметрлі шеттік есептер үйірінің жалғыз шешімінің бар болуын қамтамасыз ететін ұсынылған

алгоритмнің орындылығы мен жинақтылығының жеткілікті шарттары анықталды. Жәй дифференциалдық теңдеулер жүйесі үшін сызықтық шеттік есептер үйірінің қисынды шешімділігінің қажетті және жеткілікті шарттары алынды.

Кілт сөздер: сызықтық шеттік есептер үйірі, көпнүктелі шеттік есеп, шешімнің бар болуы, жалғыз шешім, қисынды шешімділік, қажетті және жеткілікті шарт.

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О критериях корректной разрешимости одного семейства краевых задач

В статье рассмотрено семейство линейных двухточечных краевых задач для систем дифференциальных уравнений. Исследованы вопросы существования его решений и предложены методы нахождения приближенных решений. Установлены достаточные условия существования семейства линейных двухточечных краевых задач для системы обыкновенных дифференциальных уравнений. Доказана единственность решения рассматриваемой задачи. Даны алгоритмы нахождения приближенного решения исследуемой задачи, основанные на одной модификации алгоритмов метода параметризации Д.С. Джумабаева, и доказана их сходимость. По схеме метода параметризации задача будет преобразована в эквивалентное семейство многоточечных краевых задач для систем дифференциальных уравнений. Введя новые неизвестные функции, сведем исследуемую задачу к эквивалентной задаче, интегральному уравнению Вольтерра второго рода. Установлены достаточные условия осуществимости, сходимости предложенного алгоритма, вместе с тем обеспечивающие существование единственного решения семейства краевых задач с параметрами. Получены необходимые и достаточные условия корректной разрешимости семейства линейных краевых задач для системы обыкновенных дифференциальных уравнений.

Ключевые слова: семейство линейных краевых задач, многоточечная краевая задача, существование решения, единственное решение, корректная разрешимость, необходимое и достаточное условие.

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