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Strongly minimal Jonsson sets and their properties

This article introduced and considered the Johnson sets and their fragments. And respectively was considered strongly minimal Jonsson sets. On this basis, introduced the concept of the independence of special subsets of existentially closed submodel of the semantic model. The notion of independence leads to the concept of base and further we develop technique for Jonssonien analog of theorem on uncountable categoricity.

Key words: Johnson sets, thoery *T*, Johnson theory, semantic model.

Throughout this chapter, T will be a existentially complete Jonsson theory or the fragment of some Jonsson set and it is a subset of the semantic model of the fixing considered Jonsson theory in a countable language.

We say that T is uncountably categorical if it is κ - categorical for some uncountable κ.

We have the several examples of uncountably categorical Jonsson theories.

For example, the theory of algebraically closed fields of a fixed characteristic, the theory of (Z,s) and the theory of torsion-free divisible Abelian groups are κ-categorical for all uncountable κ but not $\aleph_0$-categorical.

On the other hand, the theory of an infinite Abelian group where every element has order 2 is κ-categorical for all infinite cardinals.

This article is devoted to the study of the concept Jonsson sets and its application. Jonsson sets concept defined in [1] and further results were obtained, which were presented in [2–4].

The concept of strongly minimality, as for the sets and for the theories played a decisive role in obtaining results which describe the uncountable categorical theories [5].

As is known, the main examples of the theories of algebras are examples of inductive theories, and they tend to represent an example of incomplete theories.

In modern model theory an technical apparatus developed mainly for complete theories, so today appliances study of incomplete theories are noticeably poorer than for complete theories.

On the one hand the Jonsson conditions are a natural algebraic requirements that arise in the study of a wide class of algebras.

On the other hand natural examples of Jonsson theories are many, it is, for example, the theory of boolean algebras, abelian groups, fields of fixed characteristics, polygons (S-Acts, where S is monoid), and etc.

All of these examples are important in an algebra and in the various areas of mathematics. As can be seen, the list of the following scope of application of the technique developed for studying Jonsson theories can be quite broad.

Thus, all of the above suggests that the study of model-theoretic properties of Jonsson theories is a topical task.

Studying the inductive theories [6], it follows that Jonsson theory, as a subclass of inductive theories are such a part where there are certain methods of investigation incomplete theories, namely the method of transfer of properties of first-order theory of Jonsson center on the its Jonsson theory.

On this method and on research in the study of Jonsson theories and unrelated to the material in this article, we refer the reader to the following [7–10].

As noted above, the basic technique associated with more subtle methods of studying the behavior of model elements, is the prerogative of the art study complete theories.

Therefore, even just trying to find a generalization of standard concepts from the arsenal of complete theories, we can come to a tautology or a concept that is not technically justified.

Therefore even been proposed Jonsson set.

Recall the basic definitions of [1], which are associated with these sets.

Suppose we are given an arbitrary language L.

The theory T is called Jonsson, if:

1) the theory T has infinite models;

2) the theory T of inductive;

3) the theory T has the joint embedding property (JEP);

4) the theory T has the property of amalgam (AP).

Jonsson theory T be a perfect theory, if its semantic model saturated.

Let T be perfect Jonsson theory complete for existential sentences in the language L and its semantic model is C.

We say that a set X be Σ-definable if it is definable by some existential formula.

a) The set X is said Jonsson in the theory T if it satisfies the following properties:

X is the Σ-definable subset of C;

dcl (X) is a support of some existentially closed submodel C.

b) The set X is said to be algebraically Jonsson in the theory T if it satisfies the following properties:

X is Σ-definable subset of C;

6) acl(X) is the support of some existentially closed submodel C.

From the definition of Jonsson sets can be seen that they work very simply in the sense of Morley rank [1]. It turns out that the elements of the set-theoretic difference(wells) of the closure and a Jonsson set have rank 0, i.e, they are algebraic. So, this is a case where we can work with the elements even in the case of incomplete.

The second point the utility of such a definition Jonsson set is that we closing a given set immediately obtain some existentially closed model. This in turn enables us first to determine Jonsson fragment from the set, and in principle and in an arbitrary theory.

At this point quite well studied are perfect Jonsson theory. For them, was proved a criterion of perfectness [7] , which provide to carry out many model-theoretic facts about Jonsson theory and its center. There are complete descriptions as the center of such theories and models of their classes.

If in the case of study of complete theories we mainly deal with two objects, it is the theory itself and its models, in the case of study of Jonsson theory we consider as models the class of existential closed models of the theory, as well as some additional condition is the completness of the theory in logical sense. At least, this theory must be existentially complete.

We give a definition of Jonsson fragment:

We say that all ∀ ∃-consequences of an arbitrary theory create Jonsson fragment of this theory, if the deductive closure of these ∀∃-consequences will be Jonsson theory.

Due to the fact that this is not always true, it would be interesting to be able to allocate in arbitrary theory this part that will Jonsson theory. Such a task the place to be if only because of the fact that morleyzation of a theory it provides us, moreover, the resulting theory is perfect [6].

Another way is to use such a fact that any countable model of inductive theory necessarily isomorphically embeds in some existentially closed model of this theory [6].

Next, consider all ∀ ∃-consequences which are true in this model. Then in the case of Jonsson theory is well known fact that ∀ ∃-consequences which true in this existentially closed model form a Jonsson theory.

To study the behavior of the elements of wells in the case of Jonsson sets, we can always consider the ∀ ∃-consequences which true in the above closures of Jonsson set. In view of the above, in this case, considered set of sentences would be Jonsson theory.

Obtained in this case Jonsson theory will be called the Jonsson fragment of corresponding Jonsson set. It is clear that we can carry out research on the relationship Jonsson fragments from the original theory, which is a new formulation of the problem of study Jonsson theory.

The main objective of this article is the following problem:

In the frame of these newly introduced definitions, consider and try to describe strongly minimal Jonsson sets.

This in turn will entail a number of new formulations of problems, such as refinement of Lachlan-Baldwin Theorem in the framework of the newly introduced subjects.

Recall that Jonsson theory T has a semantic model C in enough large cardinality. If this model is saturated, this theory called perfect Jonsson theory.

Semantic model of perfect Jonsson theory uniquely determined by their power.

Further, since we have to deal with perfect Jonsson theory, it is convenient to work within a large semantic existentially closed model containing all other existentially closed models of considered perfect Jonsson theory. We call this model of universal existential domain (UED).

It can also be characterized by the following conditions.

1. Each model of this theory is isomorphically embeddable in C.

2. Every isomorphism between two its submodels which are models of considered theory extends to an automorphism model C.

We will not consider all subsets of C, but only a Jonsson subset .

Let me recall the main previous results of the author of this article on strongly minimal Jonsson sets.

For any Σ - definable subsets of a semantic model we have that the following result is yields.

Lemma 1. Σ - definable subset of the semantic model is definable over a set of parameters from A if and only if it invariant under all automorphisms of model C, leaving in place each element of A.

It follows that the definable closure dcl (A) of Jonsson set A, i.e the set of all elements definable over A is the set of elements that are invariant under all automorphisms of A.

It follows from Lemma 1 that the element b is algebraic over A if and only if it has only a finite number of conjugate elements over A.

Let us consider Jonsson minimal sets. Further, under the structure of the model refers to the signature or the language L of Jonsson theory under consideration.

Let M a structure, and let $D \subseteq M^n$ infinite Σ - definable subset. We say that D is minimal in M, if for any Σ – definable $Y \subseteq D$ or Y is finite, or $D \setminus Y$ finite. If $\varphi(\bar{v}, \bar{a})$ is the formula that determines the D then we can also say that $\varphi(\bar{v}, \bar{a})$ is minimal.

We say that and φ be Jonsson strongly minimal, if φ is minimal in any existentially closed extension N of M.

We say that a theory T Jonsson strongly minimal if $\forall M \in E_T, M$ is Jonsson strongly minimal.

The following properties of the algebraic closure true for any algebraically Jonsson set D.

i) $acl(acl(A)) = acl(A) \supseteq A.$

ii) If $A \subseteq B$, then $acl(A) \subseteq acl(B)$.

iii) If $a \in acl(A)$, then $a \in acl(A_0)$ for some finite $A_0 \subseteq A$.

More subtle property holds if D Jonsson strongly minimal.

Lemma on a replacement. Suppose that D is a subset of the semantic model of the theory and it Jonsson strongly minimal, $A \subseteq D$ and $a, b \in D$. If $a \in acl(A \cup \{b\} \setminus acl(A))$, then $b \in acl(A \cup \{a\})$.

In any Jonsson strongly minimal set, we can define the concept of independence, which generalizes the linear independence in vector spaces and algebraic independence of algebraically closed fields.

We fix $\mathrm{M} \models T$ and D is Jonsson strongly minimal set in the M-existential closed submodel of semantic model of T where T is Jonsson theory.

Definition 1. We say that $A \subseteq D$ independent if $a \notin acl(A \setminus \{a\})$ for all $a \in A$. If $C \subset D$, we say that A independent over C, if $a \notin acl(C \cup A \setminus \{a\})$ for all $a \in A$.

Definition 2. We say that A is a basis for $Y \subseteq D$, if $A \subseteq Y$ independent and $acl(A) = acl(Y)$.

Obviously, that any maximal independent subset of Y is the basis for Y.

Let $I(E_T, \aleph_0)$ denotes the number of countable existentially closed models of Jonsson theory T

Using the technique of proofs for complete theories and concepts relevant to the changing techniques for Jonsson sets, we can prove Jonsson analogues of the results to appropriate spectrum of countable models [6].

Corollary 1. If T is Jonsson strongly minimal Jonsson theory, complete for existential sentences, then T is κ–categorical for $\kappa \geq \aleph_1$ and $I(E_T, \aleph_0) \leq \aleph_0$.

Corollary 2. If T Jonsson theory complete for the existential sentence uncountably categorical and there is Jonsson strongly minimal L -formula, then either T $\aleph_0$ -categorical or $I(E_T, \aleph_0) = \aleph_0$.

Theorem 1. If T Jonsson theory complete for the existential sentence is uncountably categorical, but not $\aleph_0$ -categorical, then $I(E_T, \aleph_0) \leq \aleph_0$.

Definition 3. Jonsson stability (J -stability). Let T is a Jonsson theory, $\mathrm{S}^{\mathrm{J}}(X)$ is the set of all existential complete n-type over X, in accordance with the T, for any finite n. We shall say that Jonsson theory T be J- λ stable if for any T-existentially closed model and for any its subset X $|X|\leq\lambda\Rightarrow|S^J(X)|\leq\lambda$.

Theorem 2. If T Jonsson superstable, but not $\aleph_0$ -categorical, then $I(E_T,\aleph_0)\geq\aleph_0$.

Let us consider the stability for fragments of Jonsson sets.

Let X Jonsson set and M is existentially closed model, where dcl (X) = M.

Consider the fragment of Jonsson set X as the theory $\mathrm{Th}_{\forall\exists}(M)=T_M$.

Lemma 2. T_M will Jonsson theory.

Theorem 3. Let T_M, as described above. If $\lambda\geq\omega$, then the following conditions are equivalent:

(1) T_M is $J-\lambda-$ stable;

(2) T^* is λ - stable, where T^* is the center of T.

Theorem 4. Then the following conditions are equivalent:

(1) $T_M^*-\omega$ — categorical;

(2) $T_M-\omega$ — categorical.

Definition 4. Let $A,B\in E_T$ and $A\subset B$. Then B is algebraically simple extension A in E_T, if for any model $C\in E_T$ so that if A isomorphically embedded in C, then B is isomorphically embedded in C.

Let X be algebraically Jonsson set , acl (X) = M, the formula that determines the set X is strongly minimal existential formula.

Theorem 5. Then the following conditions are equivalent

(1) $T_M^*-\omega_1$ categorical;

(2) Any countable model from E_{T_M} has algebraically simple extension in E_{T_M}.

Lemma 3. Let $A,B\subseteq D$ be independent with $A\subseteq acl(B)$.

i) Suppose that $A_0\subseteq A, B_0\subseteq B, A_0\cup B_0$ is a basis for *acl(B) and* $a\in A\setminus A_0$. Then, there is $b\in B_0$ such that $A_0\cup\{a\}\cup(B_0\setminus\{b\})$ is a basis for *acl(B).*

ii) $|\mathrm{A}|\leq|\mathrm{B}|$.

iii) If A and B are bases for $Y\subseteq D$, then $|\mathrm{A}|=|\mathrm{B}|$.

Definition 5. If $Y\subseteq D,$ then the dimension of Y is the cardinality of a basis for Y.

We let dim(Y) denote the dimension of Y.

Note that if D is uncountable, then $\dim(D)=|D|$ because our language is countable and acl(A) is countable for any countable $A\subseteq D$.

For strongly minimal theories, every model is determined up to isomorphism by its dimension.

Theorem 6. Suppose T is a fragment of strongly minimal Jonsson set.

If $M,N\models E_T$ where E_T where E_T be a class of all exsitentielly closed models of T, then $M\cong N$ if and only if $\dim(N)=\dim(N)$.

Corollary 3. If T is a fragment of strongly minimal Jonson set , then T^* is a center of T and $I(E_T,\aleph_0)$ is a countable spectrum of existentially closed models of T

Corollary 4. If T is uncountably categorical existential complete Jonsson theory and there is a strongly minimal Jonsson subset of its semantic model, then either T is $\aleph_0$ -categorical or $I(E_T,\aleph_0)=\aleph_0$

Theorem 7 (an Jonsson variant of Baldwin—Lachlan Theorem). If T is uncountably categorical fragment of Jonsson set but not $\aleph_0$ -categorical, then $I(E_T,\aleph_0)=\aleph_0$.

All undefined in this article definitions, as well as more detailed information about Jónsson theories can be found in [7].

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А.Р.Ешкеев

Қатты минималды йонсондық жиындар және олардың қасиеттері

Мақалада йонсондық жиындар және олардың фрагменттері енгізілген және қарастырылған. Сонымен қатар қатты минималды йонсондық жиындар зерттелген. Осы негізде семантикалық модельдің экзистенционалды тұйық ішкі моделінің бір арнайы ішкі жиынының тәуелсіздігі енгізілді. Тәуелсіздік ұғымы бойынша базиске ие боламыз, және осы базисқа арналған техника бойынша саналымды емес категориялылық туралы теоремасын йонсондық аясында дамытамыз.

А.Р.Ешкеев

Сильно минимальные йонсоновские множества и их свойства

В статье рассмотрены йонсоновские множества и их фрагменты. Изучены соответственно сильно минимальные йонсоновские множества. На этой основе вводится понятие независимости специальных подмножеств экзистенциально замкнутой подмодели семантической модели. Понятие независимости приводит к понятию базиса и далее мы развиваем технику для получения йонсоновского аналога теоремы о несчетной категоричности.

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