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On the Fourier transform of functions from a Lorentz space $L_{\overline{2},\overline{r}}$ with a mixed metric

The classical inequalities of Bochkarev play a very important role in harmonic analysis. The meaning of these inequalities lies in the connection between the metric characteristics of functions and the summability of their Fourier coefficients. One of the most important directions of harmonic analysis is the theory of Fourier series. His interest in this direction is explained by his applications in various departments of modern mathematics and applied sciences, as well as the availability of many unsolved problems. One of these problems is the study of the interrelationships of the integral properties of functions and the properties of the sum of its coefficients. The solution of these problems was dedicated to the efforts of many mathematicians. And further research in this area are important and interesting problems and can give new, unexpected effects. In the article we receive a two-dimensional analog of the Bochkarev type theorem for the Fourier transform.

Keywords: Lorentz Space, Hausdorff-Young-Riesz theorem, Bochkarev's theorem, Cauchy-Bunyakovsky inequality, Helder's inequality.

Introduction

This article is devoted to the Hardy-Littlewood inequalities for an anisotropic Lorentz space. This inequalities characterize the connection between the Fourier coefficients and integral properties of the function. The study of relationship between the integrality of a function and the summability of its Fourier coefficients has been the subject of many papers. There are well-known classical results in this direction, such as Parseval, Bessel, Riesz, Hardy-Littlewood, Palley, Stein [1, 2], also modern works [3–11] and others. However, the Hausdorff-Young-Riesz theorem does not extend to the spaces $L_{2,r}$, if $r \neq 2$.

In 1997 Bochkarev S.V. [12] established that, in contrast to the spaces $L_{p,r}$, $1 < p < 2, 1 \leq r \leq \infty$, in the Lorentz space $L_{2,r}$, $2 < r \leq \infty$ the direct analogue of the Hausdorff - Young - Riesz theorem is not satisfied. And he derived upper bounds for the Fourier coefficients of functions from $L_{2,r}$ replacing the Hausdorff-Young-Riesz theorem and proved that for some class of multiplicative systems these estimates are unstrengthened.

Theorem (S.V. Bochkarev). Let $\{\phi_n\}_{n=1}^{\infty}$ be an orthonormal system of complex-valued functions on $[0, 1]$,

$$\|\phi_n\| \leq M, \ n = 1, 2, ...$$

and let $f \in L_{2,r}, \quad 2 < r \leq \infty$. Then the inequality

$$\sup_{n \in N} \frac{1}{|n|^{\frac{1}{2}} \log(n+1)^{\frac{1}{2}-\frac{1}{r}}} \sum_{m=1}^{n} a_m^* \leq C\|f\|_{L_{2,r'}}$$

holds, where a_n are the Fourier coefficients of the system $\{\phi_n\}_{n=1}^{\infty}$.

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In 2015 an analogue of Bochkarev's theorem was received for the Fourier transform of a function from the space $L_{2,r}(\mathbb{R})$.

Theorem A [13]. *Let* $\Re_N = \{A = \bigcup\limits_{i=1}^{N} A_i$, *where* A_i *are segments in* $\mathbb{R}\}$, *then for any functions* $f \in L_{2,r}(\mathbb{R})$, $2 < r < \infty$ *the following inequality holds:*

$$\sup_{N\geq 8}\sup_{A\subset\Re_N}\frac{1}{|A|^{\frac{1}{2}}\log_2(1+N)^{\frac{1}{2}-\frac{1}{r}}}\left|\int\limits_A \hat{f}(\xi)d\xi\right| \leq 23\|f\|_{L_{2,r}}.$$

The aim of this article is to obtain a two-dimensional analog of the Bochkarev type theorem for the Fourier transform. To do this, we need to introduce the following definitions:

Definition 1 [14]. Let $\bar{p} = (p_1, p_2)$, $\bar{r} = (r_1, r_2)$ and satisfy the following conditions: $0 < \bar{p} \leq \infty$, $0 < \bar{r} \leq \infty$. The Lorentz Space $L_{\bar{p},\bar{r}}[0,1]^2$ with a mixed metric is defined as the set of all measurable functions defined on $[0,1]^2$, for which the norms are finite:

$$\|f\|_{L_{\bar{p},\bar{r}}} = \|\|f\|_{L_{p_1,r_1}}\|_{L_{p_2,r_2}} = \left(\int\limits_0^1\left(t_2^{\frac{1}{p_2}}\left(\int\limits_0^1\left(t_1^{\frac{1}{p_1}}f^{*_1}(t_1,\cdot)\right)^{r_1}\frac{dt_1}{t_1}\right)_{t_2}^{*_2}\right)^{\frac{r_2}{r_1}}\frac{dt_2}{t_2}\right)^{\frac{1}{r_2}}$$

in the case $0 < \bar{r} < \infty$, and

$$\|f\|_{L_{\bar{p},\infty}} = \sup_{t_1,t_2} t_1^{\frac{1}{p_1}} t_2^{\frac{1}{p_2}} f^{*_1 *_2}(t_1,t_2)$$

in the case $\bar{r} = \infty$.

Definition 2 [15]. Let $f \in L_1(\mathbb{R}^2)$. Its two-dimensional Fourier transform is defined by the following formula:

$$\hat{f}(\xi_1,\xi_2) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(x_1,x_2)e^{-2\pi i(x_1\xi_1+x_2\xi_2)}dx_1dx_2.$$

Main results

To prove the main theorem, it is necessary to prove an auxiliary lemma:

Lemma. Let $\frac{4}{3} < q_1, q_2 < 2$ and $f \in L_{\bar{q},\bar{2}}(\mathbb{R}^2)$. Then for any measurable sets A_1 and A_2 of finite measure in $\Re_N$ the inequality

$$\sup_{A_1\subset\Re_N}\sup_{A_2\subset\Re_N}\frac{1}{|A_1|^{\frac{1}{q_1}}|A_2|^{\frac{1}{q_2}}}\int\limits_{A_1}\int\limits_{A_2}\left|\hat{f}(\xi_1,\xi_2)\right|d\xi_1d\xi_2 \leq$$

$$\leq C\left(\frac{q_1}{2(q_1-1)}\right)^{\left(\frac{1}{q_1}-\frac{1}{2}\right)}\left(\frac{q_2}{2(q_2-1)}\right)^{\left(\frac{1}{q_2}-\frac{1}{2}\right)}\|f\|_{L_{\bar{q},\bar{2}}}$$

holds.

Proof. We consider the following inequality

$$\left|\int\limits_{A_1}\int\limits_{A_2}\hat{f}(\xi_1,\xi_2)d\xi_1d\xi_2\right| =$$

$$= \left| \int\limits_{A_1} \int\limits_{A_2} \left(\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} f(x_1, x_2) e^{-2\pi x_1 \xi_1 x_2 \xi_2} dx_1 dx_2 \right) d\xi_1 d\xi_2 \right| \leq$$

$$\leq |A_1||A_2| \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} |f(x_1, x_2)| dx_1 dx_2 = |A_1||A_2| \|f\|_{L_{\bar{1}}}, \quad (1)$$

and from the Cauchy-Bunyakovsky inequality and from the Plancherel theorem we have:

$$\left| \int\limits_{A_1} \int\limits_{A_2} \hat{f}(\xi_1, \xi_2) d\xi_1 d\xi_2 \right| \leq |A_1|^{\frac{1}{2}} |A_2|^{\frac{1}{2}} \left(\int\limits_{A_1} \int\limits_{A_2} \left(\hat{f}(\xi_1, \xi_2) \right)^2 d\xi_1 d\xi_2 \right)^{\frac{1}{2}} =$$

$$= |A_1|^{\frac{1}{2}} |A_2|^{\frac{1}{2}} \left(\int\limits_{A_1} \int\limits_{A_2} (f(x_1, x_2))^2 dx_1 dx_2 \right)^{\frac{1}{2}} = |A_1|^{\frac{1}{2}} |A_2|^{\frac{1}{2}} \|f\|_{L_{\bar{2}}}. \quad (2)$$

Consider representation (2) $f = f_{00} + f_{01} + f_{10} + f_{11}$ constructed like the following.

Let $0 < \tau_1, \tau_2 < \infty$, $\chi_{\Omega_{x_2}}(x_1)$ be a set characteristic function Ω_{x_2}.

$$\Omega_{x_2} = \{(x_1, x_2) : |f(x_1, x_2)| > f^{*_1}(\tau_1, x_2)\} \cup e_{x_2},$$

where e_{x_2} is a measurable subset $\{(x_1, x_2) : |f(x_1, x_2)| = f^{*_1}(\tau_1, x_2)\}$ such that:

$$\mu_1(\Omega_{x_2}) = \tau_1.$$

This set is always available, since for a fixed x_2

$$\mu_1\{(x_1, x_2) : |f(x_1, x_2)| > f^{*_1}(\tau_1, x_2)\} \leq \tau_1,$$

$$\mu_1\{(x_1, x_2) : |f(x_1, x_2)| \geq f^{*}(\tau_1, x_2)\}.$$

Denote by g_0 and g_1 the functions

$$g_0(x_1, x_2) = f(x_1, x_2) \chi_{\Omega_{x_2}}(x_1),$$

$$g_1(x_1, x_2) = f(x_1, x_2) - g_0(x_1, x_2).$$

In turn, each function g_0, g_1 can be represented as

$$g_0 = f_{00} + f_{01}, \; g_1 = f_{10} + f_{11}.$$

Let

$$W_0 = \{x_2 \in (0, \infty) : \|g_0(\cdot, x_2)\|_{L_1} > (\|g_0(\cdot, x_2)\|_{L_1})^{*_2}(\tau_2)\} \cup e_0,$$

where

$$e_0 \subset \{x_2 \in (0, \infty) : \|g_0(\cdot, x_2)\|_{L_1} = (\|g_0(\cdot, x_2)\|_{L_1})^{*_2}(\tau_2)\}, \; \mu_2(W_0) = \tau_2,$$

and

$$W_1 = \{x_2 \in (0, \infty) : \|g_1(\cdot, x_2)\|_{L_2} > (\|g_1(\cdot, x_2)\|_{L_2})^{*_2}(\tau_2)\} \cup e_1,$$

where

$$e_1 \subset \{x_2 \in (0, \infty) : \|g_1(\cdot, x_2)\|_{L_2} = (\|g_1(\cdot, x_2)\|_{L_2})\}, \; \mu_2(W_1) = \tau_2.$$

Then

$$f_{00}(x_1,x_2)=g_0(x_1,x_2)\chi_{W_0}(x_2),\ \ f_{01}(x_1,x_2)=g_0(x_1,x_2)-f_{00}(x_1,x_2),$$

$$f_{10}(x_1,x_2)=g_1(x_1,x_2)\chi_{W_1}(x_1),\ \ f_{11}(x_1,x_2)=g_1(x_1,x_2)-f_{10}(x_1,x_2).$$

Thus representation is constructed

$$f=f_{00}+f_{01}+f_{10}+f_{11}.$$

Then for an arbitrary $\tau=(\tau_1,\tau_2)\in(0,\infty)^2$, we get

$$\left|\int\limits_{A_1}\int\limits_{A_2}\hat f(\xi_1,\xi_2)d\xi_1d\xi_2\right|\le$$

$$\le\left|\int\limits_{A_1}\int\limits_{A_2}\hat f_{00}(\xi_1,\xi_2)d\xi_1d\xi_2\right|+\left|\int\limits_{A_1}\int\limits_{A_2}\hat f_{01}(\xi_1,\xi_2)d\xi_1d\xi_2\right|+$$

$$+\left|\int\limits_{A_1}\int\limits_{A_2}\hat f_{10}(\xi_1,\xi_2)d\xi_1d\xi_2\right|+\left|\int\limits_{A_1}\int\limits_{A_2}\hat f_{11}(\xi_1,\xi_2)d\xi_1d\xi_2\right|=I_1+I_2+I_3+I_4. \tag{3}$$

For I_1 we use inequality (1) and get the following estimate

$$I_1\le|A_1||A_2|M_1M_2\int\limits_0^\infty\int\limits_0^\infty|f_{00}(x_1,x_2)|dx_1dx_2.$$

Now let us estimate I_2

$$I_2\le|A_2|^{\frac12}\left(\int\limits_{A_2}\left(\int\limits_{A_1}\hat f_{01}(\xi_1,\xi_2)d\xi_1\right)^2d\xi_2\right)^{\frac12}\le$$

$$\le|A_2|^{\frac12}\left(\int\limits_{A_2}\left|\int\limits_{A_1}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}f_{01}(x_1,x_2)e^{-2\pi x_1\xi_1x_2\xi_2}dx_1dx_2d\xi_1\right|^2d\xi_2\right)^{\frac12}\le$$

$$\le|A_2|^{\frac12}\left(\int\limits_{A_2}\left|\int\limits_{-\infty}^{+\infty}\left(\int\limits_{A_1}\int\limits_{-\infty}^{+\infty}f_{01}(x_1,x_2)e^{-2\pi x_1\xi_1}dx_1d\xi_1\right)e^{-2\pi ix_2\xi_2}dx_2\right|^2d\xi_2\right)^{\frac12}.$$

Applying Plancherel theorem, we get

$$I_2\le|A_2|^{\frac12}\left(\int\limits_{-\infty}^{+\infty}\widehat{\left(\int\limits_{A_1}\int\limits_{-\infty}^{+\infty}f_{01}(x_1,x_2)e^{-2\pi ix_1\xi_1}dx_1d\xi_1\right)}_{k_2}^{2}dx_2\right)^{\frac12}=$$

$$=|A_2|^{\frac12}\left(\int\limits_{-\infty}^{+\infty}\left|\int\limits_{A_1}\int\limits_{-\infty}^{+\infty}f_{01}(x_1,x_2)e^{-2\pi ix_1\xi_1}dx_1d\xi_1\right|^2dx_2\right)^{\frac12}\le$$

$$\leq C|A_1||A_2|^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} |f_{01}(x_1,x_2)|dx_1 \right)^2 dx_2 \right)^{1/2}.$$

Let's estimate I_3

$$I_3 = \left| \int_{A_1} \int_{A_2} \widehat{f_{10}(\xi_1,\xi_2)} d\xi_1 d\xi_2 \right| =$$

$$= \left| \int_{A_1} \int_{A_2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{10}(x_1,x_2) e^{-2\pi i x_1 \xi_1 x_2 \xi_2} dx_1 dx_2 d\xi_1 d\xi_2 \right| \leq$$

$$\leq \int_{A_1} \int_{A_2} \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} f_{10}(x_1,x_2) e^{-2\pi i x_1 \xi_1} dx_1 \right| d\xi_1 e^{-2\pi i x_2 \xi_2} d\xi_2 dx_2 \leq$$

$$\leq C|A_2| \int_{A_1} \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} f_{10}(x_1,x_2) e^{-2\pi i x_1 \xi_1} dx_1 d\xi_1 \right| dx_2.$$

Using Cauchy-Bunyakovsky inequality, we obtain

$$I_3 \leq C|A_2||A_1|^{\frac{1}{2}} \left(\int_{A_1} \left(\int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} f_{10}(x_1,x_2) e^{-2\pi i x_1 \xi_1} dx_1 d\xi_1 \right| dx_2 \right)^2 \right)^{\frac{1}{2}} \leq$$

$$\leq C|A_2||A_1|^{\frac{1}{2}} \int_{-\infty}^{+\infty} \left(\int_{A_1} \left| \int_{-\infty}^{+\infty} f_{10}(x_1,x_2) e^{-2\pi i x_1 \xi_1} dx_1 d\xi_1 \right|^2 \right)^{\frac{1}{2}} dx_2.$$

Using Plancherel theorem, we get the following

$$I_3 \leq C|A_2||A_1|^{\frac{1}{2}} \int_{-\infty}^{+\infty} \left(\left| \int_{-\infty}^{+\infty} f_{10}(x_1,x_2) dx_1 \right|^2 \right)^{\frac{1}{2}} dx_2.$$

Applying for I_4 Cauchy-Bunyakovsky inequality and Plancherel equality, we get the following estimate

$$I_4 = \left| \int_{A_1} \int_{A_2} \widehat{f_{11}}(\xi_1,\xi_2) d\xi_1 d\xi_2 \right| \leq |A_1|^{1/2} |A_2|^{1/2} \left(\int_{A_1} \int_{A_2} \left| \hat{f}_{11}(\xi_1,\xi_2) d\xi_1 d\xi_2 \right|^2 \right)^{\frac{1}{2}} =$$

$$= |A_1|^{1/2} |A_2|^{1/2} \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f_{11}(x_1,x_2)|^2 dx_1 dx_2 \right)^{\frac{1}{2}}.$$

Substituting the obtained estimates into relation (3), we have

$$\left|\int\limits_{A_1}\int\limits_{A_2}\hat{f}(\xi_1,\xi_2)d\xi_1 d\xi_2\right| \leq |A_1||A_2|\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}|f_{00}(x_1,x_2)|dx_1dx_2+$$

$$+|A_1||A_2|^{\frac{1}{2}}\left(\int\limits_{-\infty}^{+\infty}\left(\int\limits_{-\infty}^{+\infty}|f_{01}(x_1,x_2)|dx_1\right)^2 dx_2\right)^{\frac{1}{2}}+$$

$$+|A_2||A_1|^{\frac{1}{2}}\int\limits_{-\infty}^{+\infty}\left(\int\limits_{-\infty}^{+\infty}|f_{10}(x_1,x_2)|^2\,dx_1\right)^{\frac{1}{2}}dx_2+$$

$$+|A_1|^{\frac{1}{2}}|A_2|^{\frac{1}{2}}\left(\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}|f_{11}(x_1,x_2)|^2dx_1dx_2\right)^{\frac{1}{2}}.$$

Further

$$\left|\int\limits_{A_1}\int\limits_{A_2}\hat{f}(\xi_1,\xi_2)d\xi_1 d\xi_2\right| \leq |A_1||A_2|\int\limits_{0}^{\infty}\left(\int\limits_{0}^{\infty}f^{*_1}(t_1,\cdot)dt_1\right)^{*_2}_{t_2}dt_2+$$

$$+|A_1||A_2|^{\frac{1}{2}}\left(\int\limits_{0}^{\infty}\left(\left(\int\limits_{\tau_1}^{\infty}f^{*_1}(t_1,\cdot)dt_1\right)^{*_2}_{t_2}\right)^2 dt_2\right)^{\frac{1}{2}}+$$

$$+|A_1|^{\frac{1}{2}}|A_2|\int\limits_{0}^{\infty}\left(\left(\int\limits_{0}^{\infty}(f^{*_1}(t_1,\cdot))^2dt_1\right)^{*_2}_{t_2}\right)^{\frac{1}{2}}dt_2+$$

$$+|A_1|^{\frac{1}{2}}|A_2|^{\frac{1}{2}}\left(\int\limits_{\tau_2}^{\infty}\left(\int\limits_{\tau_1}^{\infty}(f^{*_1}(t_1,\cdot))^2dt_1\right)^{*_2}_{t_2}dt_2\right)^{\frac{1}{2}}=$$

$$=J_1+J_2+J_3+J_4.$$

Now we estimate each term. To estimate J_1 we use Helder's inequality

$$J_1=|A_1||A_2|\int\limits_{0}^{\infty}t_2^{\frac{1}{q_2}}t_2^{\frac{1}{q_2'}}\left(\int\limits_{0}^{\infty}t_1^{\frac{1}{q_1}}t_1^{\frac{1}{q_1'}}f^{*_1}(t_1,\cdot)\frac{dt_1}{t_1}\right)^{*_2}_{t_2}\frac{dt_2}{t_2}\leq$$

$$\leq|A_1||A_2|\left(\int\limits_{0}^{\infty}t_2^{\frac{2}{q_2}}\left(\int\limits_{0}^{\infty}\left(t_1^{\frac{1}{q_1}}f^{*_1}(t_1,\cdot)\right)^2\frac{dt_1}{t_1}\right)^{*_2}_{t_2}\frac{dt_2}{t_2}\right)^{\frac{1}{2}}\times$$

$$\times\left(\int\limits_{0}^{\infty}t_2^{\frac{2}{q_2'}}\left(\int\limits_{0}^{\infty}t_1^{\frac{2}{q_1'}}\frac{dt_1}{t_1}\right)\frac{dt_2}{t_2}\right)^{\frac{1}{2}}=$$

$$= |A_1||A_2|\|f\|_{L_{\bar{q},\bar{2}}} \left(\int_0^\infty t_1^{\frac{2}{q_1'}} \frac{dt_1}{t_1}\right)^{\frac{1}{2}} \left(\int_0^\infty t_2^{\frac{2}{q_2'}} \frac{dt_2}{t_2}\right)^{\frac{1}{2}} =$$

$$= |A_1||A_2|\|f\|_{L_{\bar{q},\bar{2}}} \left(\frac{q_1'}{2}\right)^{\frac{1}{2}} \left(\frac{q_2'}{2}\right)^{\frac{1}{2}} \tau_1^{\frac{1}{q_1'}} \tau_2^{\frac{1}{q_2'}} =$$

$$= |A_1||A_2|\|f\|_{L_{\bar{q},\bar{2}}} \left(\frac{q_1}{2(q_1-1)}\right)^{\frac{1}{2}} \left(\frac{q_2}{2(q_2-1)}\right)^{\frac{1}{2}} \tau_1^{1-\frac{1}{q_1}} \tau_2^{1-\frac{1}{q_2}}.$$

To estimate the term J_2 we also use Helder's inequality

$$J_2 = |A_1||A_2|^{\frac{1}{2}} \left(\int_{\tau_2}^\infty \left(\left(\int_0^{\tau_1} f^{*_1}(t_1,\cdot)dt_1\right)^{*_2}_{t_2}\right)^2 dt_2\right)^{\frac{1}{2}} =$$

$$= |A_1||A_2|^{\frac{1}{2}} \left(\int_{\tau_2}^\infty \left(t_2^{\frac{1}{q_2}} t_2^{-\frac{1}{q_2}} \left(\int_0^{\tau_1} t_1^{\frac{1}{q_1}} t_1^{\frac{1}{q_1'}} f^{*_1}(t_1,\cdot)\frac{dt_1}{t_1}\right)^{*_2}_{t_2}\right)^2 \frac{t_2 dt_2}{t_2}\right)^{\frac{1}{2}} \le$$

$$\le |A_1||A_2|^{\frac{1}{2}} \left(\int_{\tau_2}^\infty t_2^{\frac{1}{q_2}} \left(\int_0^{\tau_1} \left(t_1^{\frac{1}{q_1}} f^{*_1}(t_1,\cdot)\right)^2 \frac{dt_1}{t_1}\right)^{*_2}_{t_2} \frac{dt_2}{t_2}\right)^{\frac{1}{2}} \times$$

$$\times \left(\int_{\tau_2}^\infty t_2^{1-\frac{2}{q_2}} \left(\int_0^{\tau_1} t_1^{\frac{2}{q_1}} \frac{dt_1}{t_1}\right) \frac{dt_2}{t_2}\right)^{1/2} =$$

$$= |A_1||A_2|^{\frac{1}{2}} \|f\|_{L_{\bar{q},\bar{2}}} \left(\int_{\tau_2}^\infty t_2^{1-\frac{2}{q_2}} \frac{dt_2}{t_2}\right)^{\frac{1}{2}} \left(\int_0^{\tau_1} \left(t_1^{\frac{2}{q_1'}}\right) \frac{dt_1}{t_1}\right)^{\frac{1}{2}}.$$

Since $q_2 < 2$, the second integral is estimated in terms of $\tau_2^{\frac{1}{2}-\frac{1}{q_2}}$. So

$$J_2 \le |A_1||A_2|^{\frac{1}{2}} \|f\|_{L_{\bar{q},\bar{2}}} \left(\frac{q_1'}{2}\right)^{\frac{1}{2}} \tau_1^{\frac{1}{q_1'}} \tau_2^{\frac{1}{2}-\frac{1}{q_2}} =$$

$$= |A_1||A_2|^{\frac{1}{2}} \|f\|_{L_{\bar{q},\bar{2}}} \left(\frac{q_1}{2(q_1-1)}\right)^{\frac{1}{2}} \tau_1^{1-\frac{1}{q_1}} \tau_2^{\frac{1}{2}-\frac{1}{q_2}}.$$

Now we estimate J_3

$$J_3 = |A_1|^{\frac{1}{2}} |A_2| \int_0^{\tau_2} \left(\left(\int_{\tau_1}^\infty (f^{*_1}(t_1,\cdot))^2 dt_1\right)^{*_2}_{t_2}\right)^{\frac{1}{2}} dt_2 =$$

$$= |A_1|^{\frac{1}{2}} |A_2| \int_0^{\tau_2} t_2^{-\frac{1}{q_2}} \left(\left(\int_{\tau_1}^\infty \left(t_1^{\frac{1}{q_1}} t_2^{\frac{1}{q_2}} f^{*_1}(t_1,\cdot) t_1^{-\frac{1}{q_1}}\right)^2 \frac{t_1 dt_1}{t_1}\right)^{*_2}_{t_2}\right)^{\frac{1}{2}} \frac{t_2 dt_2}{t_2^{\frac{1}{2}+\frac{1}{2}}}.$$

We use Helder's inequality

$$J_3 \le |A_1|^{\frac{1}{2}}|A_2| \sup_{\tau_1<t_1<\infty} t_1^{\frac{1}{2}-\frac{1}{q_1}} \left(\int\limits_0^{\tau_2} t_2^{-\frac{1}{q_2}} \left(\left(\left(\int\limits_{\tau_1}^{\infty} \left(t_1^{\frac{1}{q_1}} t_2^{\frac{1}{q_2}} f^{*_1}(t_1,\cdot)\right)^2 \frac{dt_1}{t_1}\right)^{*_2}_{t_2}\right) \frac{dt_2}{t_2}\right)^{\frac{1}{2}} \times$$

$$\times \left(\int\limits_0^{\tau_2} \left(t_2^{1-\frac{1}{q_2}}\right)^2 \frac{dt_2}{t_2}\right)^{\frac{1}{2}} =$$

$$= |A_1|^{\frac{1}{2}}|A_2| \|f\|_{L_{\bar{q},\bar{2}}} \sup_{\tau_1<t_1<\infty} t_1^{\frac{1}{2}-\frac{1}{q_1}} \left(\int\limits_0^{\tau_2} t_2^{2\left(1-\frac{1}{q_2}\right)} \frac{dt_2}{t_2}\right)^{\frac{1}{2}} =$$

$$= |A_1|^{\frac{1}{2}}|A_2| \|f\|_{L_{\bar{q},\bar{2}}} \tau_1^{\frac{1}{2}-\frac{1}{q_1}} \tau_2^{1-\frac{1}{q_2}} \left(\frac{q_2}{2(q_2-1)}\right)^{\frac{1}{2}}.$$

It remains to estimate the last integral

$$J_4 \le |A_1|^{\frac{1}{2}}|A_2|^{\frac{1}{2}} \left(\int\limits_0^{\tau_1} \left(\int\limits_{\tau_2}^{\infty} (f^{*_1}(t_1,\cdot))^2 t_1^{\frac{2}{q_1}-1} t_1^{1-\frac{2}{q_1}} t_2^{\frac{2}{q_2}-1} t_2^{1-\frac{2}{q_2}} dt_1\right)^{*_2}_{t_2} dt_2\right)^{\frac{1}{2}}.$$

Since $q_2 < 2$, we get

$$J_4 \le |A_1|^{\frac{1}{2}}|A_2|^{\frac{1}{2}} \left(\int\limits_0^{\tau_1} \left(\int\limits_{\tau_2}^{\infty} \left(t_1^{\frac{1}{q_1}} t_2^{\frac{1}{q_2}} f^{*_1}(t_1,\cdot)\right)^2 \frac{dt_1}{t_1}\right)^{*_2}_{t_2} \frac{dt_2}{t_2}\right)^{\frac{1}{2}} \tau_1^{\frac{1}{2}-\frac{1}{q_1}} \tau_2^{\frac{1}{2}-\frac{1}{q_2}} \le$$

$$\le |A_1|^{\frac{1}{2}}|A_2|^{\frac{1}{2}} \|f\|_{L_{\bar{q},\bar{2}}} \tau_1^{\frac{1}{2}-\frac{1}{q_1}} \tau_2^{\frac{1}{2}-\frac{1}{q_2}}.$$

Thus, we have

$$\left|\int\limits_{A_1}\int\limits_{A_2} \hat{f}(\xi_1,\xi_2)\right| \le \|f\|_{L_{\bar{q},\bar{2}}} \left(|A_1||A_2| \left(\frac{q_1}{2(q_1-1)}\right)^{\frac{1}{2}} \left(\frac{q_2}{2(q_2-1)}\right)^{\frac{1}{2}} \times\right.$$

$$\times \tau_1^{1-\frac{1}{q_1}} \tau_2^{1-\frac{1}{q_2}} +$$

$$+ |A_1||A_2|^{\frac{1}{2}} \left(\frac{q_1}{2(q_1-1)}\right)^{\frac{1}{2}} \tau_1^{1-\frac{1}{q_1}} \tau_2^{\frac{1}{2}-\frac{1}{q_2}} +$$

$$\left. + |A_1|^{\frac{1}{2}}|A_2| \left(\frac{q_2}{2(q_2-1)}\right)^{\frac{1}{2}} \tau_1^{\frac{1}{2}-\frac{1}{q_1}} \tau_2^{1-\frac{1}{q_2}} + |A_1|^{\frac{1}{2}}|A_2|^{\frac{1}{2}} \tau_1^{\frac{1}{2}-\frac{1}{q_1}} \tau_2^{\frac{1}{2}-\frac{1}{q_2}}\right).$$

Choosing $\tau_1 = \frac{\left(\frac{q_1}{2(q_1-1)}\right)^{-1}}{|A_1|}$ and $\tau_2 = \frac{\left(\frac{q_2}{2(q_2-1)}\right)^{-1}}{|A_2|}$, we get

$$\left|\int\limits_{A_1}\int\limits_{A_2} \hat{f}(\xi_1,\xi_2)\right| \le \|f\|_{L_{\bar{q}\bar{2}}} \left(|A_1||A_2| \left(\frac{q_1}{2(q_1-1)}\right)^{\frac{1}{2}} \left(\frac{q_2}{2(q_2-1)}\right)^{\frac{1}{2}} \times\right.$$

$$\times\left(\frac{q_1}{2(q_1-1)}\right)^{\frac{1}{q_1}-1}|A_1|^{\frac{1}{q_1}-1}\left(\frac{q_2}{2(q_2-1)}\right)^{\frac{1}{q_2}-1}|A_2|^{\frac{1}{q_2}-1}+$$

$$+|A_1||A_2|^{\frac{1}{2}}\left(\frac{q_1}{2(q_1-1)}\right)^{\frac{1}{2}}\left(\frac{q_1}{2(q_1-1)}\right)^{\frac{1}{q_1}-1}|A_1|^{\frac{1}{q_1}-1}\times$$

$$\times\left(\frac{q_2}{2(q_2-1)}\right)^{\frac{1}{q_2}-\frac{1}{2}}|A_2|^{\frac{1}{q_2}-\frac{1}{2}}+$$

$$+|A_1|^{\frac{1}{2}}|A_2|\left(\frac{q_2}{2(q_2-1)}\right)^{\frac{1}{2}}\left(\frac{q_1}{2(q_1-1)}\right)^{\frac{1}{q_1}-\frac{1}{2}}|A_1|^{\frac{1}{q_1}-\frac{1}{2}}\times$$

$$\times\left(\frac{q_2}{2(q_2-1)}\right)^{\frac{1}{q_2}-1}|A_2|^{\frac{1}{q_2}-1}+|A_1|^{\frac{1}{2}}|A_2|^{\frac{1}{2}}\left(\frac{q_1}{2(q_1-1)}\right)^{\frac{1}{q_1}-\frac{1}{2}}|A_1|^{\frac{1}{q_1}-\frac{1}{2}}\times$$

$$\left.\times\left(\frac{q_2}{2(q_2-1)}\right)^{\frac{1}{q_2}-\frac{1}{2}}|A_2|^{\frac{1}{q_2}-\frac{1}{2}}\right).$$

We get the following inequality:

$$\left|\int\limits_{A_1}\int\limits_{A_2}\hat{f}(\xi_1,\xi_2)\right|\leq$$

$$\leq M|A_1|^{\frac{1}{q_1}}|A_2|^{\frac{1}{q_2}}\left(\frac{q_1}{2(q_1-1)}\right)^{\left(\frac{1}{q_1}-\frac{1}{2}\right)}\left(\frac{q_2}{2(q_2-1)}\right)^{\left(\frac{1}{q_2}-\frac{1}{2}\right)}\|f\|_{L_{\bar{q},\bar{2}}},$$

or

$$\frac{1}{|A_1|^{\frac{1}{q_1}}|A_2|^{\frac{1}{q_2}}}\int\limits_{A_1}\int\limits_{A_2}\left|\hat{f}(\xi_1,\xi_2)d\xi_1d\xi_2\right|\leq$$

$$\leq\left(\frac{q_1}{2(q_1-1)}\right)^{\left(\frac{1}{q_1}-\frac{1}{2}\right)}\left(\frac{q_2}{2(q_2-1)}\right)^{\left(\frac{1}{q_2}-\frac{1}{2}\right)}\|f\|_{L_{\bar{q},\bar{2}}}.$$

Taking the least upper bound over all $A_1\subset\Re_N$ and $A_2\subset\Re_N$, we obtain the assertion of the lemma, that is,

$$\sup_{A_1\subset\Re_N}\sup_{A_2\subset\Re_N}\frac{1}{|A_1|^{\frac{1}{q_1}}|A_2|^{\frac{1}{q_2}}}\int\limits_{A_1}\int\limits_{A_2}\left|\hat{f}(\xi_1,\xi_2)\right|d\xi_1d\xi_2\leq$$

$$\leq C\left(\frac{q_1}{2(q_1-1)}\right)^{\left(\frac{1}{q_1}-\frac{1}{2}\right)}\left(\frac{q_2}{2(q_2-1)}\right)^{\left(\frac{1}{q_2}-\frac{1}{2}\right)}\|f\|_{L_{\bar{q},\bar{2}}},$$

where $|A|$ is the number of elements in A.

Theorem. Let $\Phi_{m_1,m_2}(x_1,x_2)=\varphi_{m_1}(x_1)\cdot\psi_{m_2}(x_2)$, $m_1,m_1\in\mathbb{N}$ be an orthonormal bounded system of functions. Then, for any $f\in L_{\bar{2},\bar{r}}(\mathbb{R}^2)$, where $2<r_1,r_2<\infty$ the inequality holds:

$$\sup_{\substack{|A_1|\geq 8\\ A_1\subset\mathbb{N}}}\sup_{\substack{|A_2|\geq 8\\ A_2\subset\mathbb{N}}}\frac{1}{|A_1|^{\frac{1}{2}}|A_2|^{\frac{1}{2}}(\log_2(|A_1|+1))^{\frac{1}{2}-\frac{1}{r_1}}(\log_2(|A_2|+1))^{\frac{1}{2}-\frac{1}{r_2}}}\times$$

$$\times\int\limits_{A_1}\int\limits_{A_2}\left|\hat{f}(\xi_1,\xi_2)\right|d\xi_1d\xi_2\leq\|f\|_{L_{\bar{2},\bar{r}}}.$$

Proof. Let $|A_1|, |A_2| \geq 8$. Then for any (q_1, q_2) such that $1 < q_1, q_2 < 2$ and $f \in L_{\bar{q},\bar{2}}$ the following estimate is true

$$\|f\|_{L_{\bar{q},\bar{2}}} \leq \|f\|_{L_{\bar{2},\bar{r}}} \|1\|_{L_{\bar{p},\bar{r}'}}, \tag{4}$$

where $\frac{1}{\bar{q}} = \frac{1}{2} + \frac{1}{\bar{p}}, \quad \frac{1}{\bar{r}'} = \frac{1}{2} - \frac{1}{\bar{r}}$. Now we consider

$$\|1\|_{L_{\bar{p},\bar{r}'}} = \left(\int_0^1 \left(\int_0^1 t_1^{\frac{r_1'}{p_1}} t_2^{\frac{r_1'}{p_2}} \frac{dt_1}{t_1} \right)^{\frac{r_2'}{r_1'}} \frac{dt_2}{t_2} \right)^{\frac{1}{r_2'}} =$$

$$= \left(\int_0^1 t_1^{r_1'\left(\frac{1}{q_1} - \frac{1}{2}\right)} \frac{dt_1}{t_1} \right)^{\frac{1}{r_1'}} \left(\int_0^1 t_2^{r_2'\left(\frac{1}{q_2} - \frac{1}{2}\right)} \frac{dt_2}{t_2} \right)^{\frac{1}{r_2'}} =$$

$$= \left(\frac{1}{r_1'\left(\frac{1}{q_1} - \frac{1}{2}\right)} \right)^{\frac{1}{r_1'}} \left(\frac{1}{r_2'\left(\frac{1}{q_2} - \frac{1}{2}\right)} \right)^{\frac{1}{r_2'}} \leq \left(\frac{2q_1}{2 - q_1} \right)^{\frac{1}{r_1'}} \left(\frac{2q_2}{2 - q_2} \right)^{\frac{1}{r_2'}}.$$

According to the previous inequality, we obtain

$$\|f\|_{L_{\bar{q},\bar{2}}} \leq \left(\frac{2q_1}{2 - q_1} \right)^{\frac{1}{r_1'}} \left(\frac{2q_2}{2 - q_2} \right)^{\frac{1}{r_2'}} \|f\|_{L_{\bar{2},\bar{r}}}.$$

Applying Lemma 1, we get

$$\frac{1}{|A_1|^{\frac{1}{q_1}} |A_2|^{\frac{1}{q_2}}} \int_{A_1} \int_{A_2} \left| \hat{f}(\xi_1, \xi_2) \right| d\xi_1 d\xi_2 \leq$$

$$\leq C \left(\frac{q_1}{2(q_1 - 1)} \right)^{\left(\frac{1}{q_1} - \frac{1}{2}\right)} \left(\frac{q_2}{2(q_2 - 1)} \right)^{\left(\frac{1}{q_2} - \frac{1}{2}\right)} \|f\|_{L_{\bar{q},\bar{2}}}.$$

Taking into account (4), we get the following inequality

$$\frac{1}{|A_1|^{\frac{1}{q_1}} |A_2|^{\frac{1}{q_2}}} \sum_{k_1 \in A_1} \sum_{k_2 \in A_2} \left| \hat{f}(\xi_1, \xi_2) \right| d\xi_1 d\xi_2 \leq C \left(\frac{q_1}{2(q_1 - 1)} \right)^{\left(\frac{1}{q_1} - \frac{1}{2}\right)} \left(\frac{q_2}{2(q_2 - 1)} \right)^{\left(\frac{1}{q_2} - \frac{1}{2}\right)} \times$$

$$\times \left(\frac{2q_1}{2 - q_1} \right)^{\frac{1}{r_1'}} \left(\frac{2q_2}{2 - q_2} \right)^{\frac{1}{r_2'}} \|f\|_{L_{\bar{2},\bar{r}}}.$$

Taking into account the arbitrariness of parameters q_1 and q_2, we set

$$q_1 = \frac{2 \log_2 |A_1|}{\log_2 |A_1| + 2} < 2,$$

$$q_2 = \frac{2 \log_2 |A_2|}{\log_2 |A_2| + 2} < 2,$$

$\frac{1}{\bar{q}} - \frac{1}{2} = \frac{1}{\log_2 N}, \quad |A_1|^{\frac{1}{q_1}} = |A_1|^{\frac{1}{\log_2 |A_1|} + \frac{1}{2}} = 2|A_1|^{\frac{1}{2}}, \quad |A_2|^{\frac{1}{q_2}} = 2|A_2|^{\frac{1}{2}}.$

$$\frac{1}{|A_1|^{\frac{1}{2}} |A_2|^{\frac{1}{2}}} \int_{A_1} \int_{A_2} \left| \hat{f}(\xi_1, \xi_2) \right| d\xi_1 d\xi_2 \leq$$

$$\leq C\left(\frac{1}{2\left(1-\frac{1}{q_1}\right)}\right)^{\left(\frac{1}{q_1}-\frac{1}{2}\right)}\left(\frac{1}{2\left(1-\frac{1}{q_2}\right)}\right)^{\left(\frac{1}{q_2}-\frac{1}{2}\right)}\times$$

$$\times\left(\frac{1}{\frac{1}{q_1}-\frac{1}{2}}\right)^{\frac{1}{r_1'}}\left(\frac{1}{\frac{1}{q_2}-\frac{1}{2}}\right)^{\frac{1}{r_2'}}\|f\|_{L_{\bar{2},\bar{r}}}\leq$$

$$\leq 4M\left(\frac{1}{2\left(\frac{1}{2}-\frac{1}{\log_2|A_1|}\right)}\right)^{\frac{1}{\log_2|A_1|}}\left(\frac{1}{2\left(\frac{1}{2}-\frac{1}{\log_2|A_2|}\right)}\right)^{\frac{1}{\log_2|A_2|}}\times$$

$$\times\left(\log_2|A_1|\right)^{\frac{1}{r_1'}}\left(\log_2|A_2|\right)^{\frac{1}{r_2'}}\|f\|_{L_{\bar{2},\bar{r}}}.$$

Considering $|A_1|, |A_2| \geq 8$, we get the following estimate

$$\frac{1}{|A_1|^{\frac{1}{2}}|A_2|^{\frac{1}{2}}\left(\log_2|A_1|\right)^{\frac{1}{2}-\frac{1}{r_1}}\left(\log_2|A_2|\right)^{\frac{1}{2}-\frac{1}{r_2}}}\int\limits_{A_1}\int\limits_{A_2}\left|\hat{f}(\xi_1,\xi_2)\right|d\xi_1 d\xi_2\leq C\|f\|_{L_{\bar{2},\bar{r}}}.$$

Taking the least upper bound over all A_1 and A_2 from $\mathbb{N}$, we get

$$\sup_{\substack{|A_1|\geq 8\\ A_1\subset\mathbb{N},}}\sup_{\substack{|A_2|\geq 8\\ A_2\subset\mathbb{N},}}\frac{1}{|A_1|^{\frac{1}{2}}|A_2|^{\frac{1}{2}}\left(\log_2(|A_1|+1)\right)^{\frac{1}{2}-\frac{1}{r_1}}\left(\log_2(|A_2|+1)\right)^{\frac{1}{2}-\frac{1}{r_2}}}\times$$

$$\times\int\limits_{A_1}\int\limits_{A_2}\left|\hat{f}(\xi_1,\xi_2)\right|d\xi_1 d\xi_2\leq C\|f\|_{L_{\bar{2},\bar{r}}}.$$

Conclusions

The results obtained in this study specifically the Bochkarev-type inequality in a space of a mixed metric, allow us to effectively address problems concerning Fourier series multipliers [16–18].

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Аралас метрикалы $L_{\overline{2},\overline{r}}$ Лоренц кеңістігіндегі Фурье функцияларының түрлендірулері жайлы

Гармоникалық талдауда классикалық Бочкарев теңсіздіктері өте маңызды рөл атқарады. Бұл теңсіздіктердің мәні функциялардың метрикалық сипаттамалары мен олардың Фурье коэффициенттерінің қосындысы арасындағы байланыста жатыр. Гармоникалық талдаудың маңызды бағыттарының бірі Фурье қатарларының теориясы. Оның бұл салаға деген қызығушылығы қазіргі математика мен қолданбалы ғылымдардың әртүрлі салаларында қолданылуына, сондай-ақ көптеген шешілмеген мәселелердің болуына байланысты. Осы мақсаттардың бірі функцияның интегралдық қасиеттері мен оның коэффициенттерінің қосындысының қасиеттері арасындағы байланысты зерттеу. Көптеген математиктердің еңбектері осы есептерді шешуге арналды. Бұл саладағы әрі қарайғы зерттеулер маңызды және қызықты зерттеу болып табылады және жаңа, күтпеген нәтижелерге әкелуі мүмкін. Мақалада Фурье түрленуі үшін Бочкарев типті теореманың екі өлшемді аналогы алынған.

Кілт сөздер: Лоренц кеңістігі, Хаусдорф–Янг–Рис теоремасы, Бочкарев теоремасы, Коши-Буняковский теңсіздігі, Хельдер теңсіздігі.

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О преобразовании Фурье функций в пространстве Лоренца $L_{\overline{2},\overline{r}}$ со смешанной метрикой

Классические неравенства Бочкарева играют очень важную роль в гармоническом анализе. Смысл этих неравенств заключается в связи между метрическими характеристиками функций и суммируемостью их коэффициентов Фурье. Одним из важнейших направлений гармонического анализа является теория рядов Фурье. Его интерес к этому направлению объясняется его приложениями в различных разделах современной математики и прикладных наук, а также наличием многих нерешенных проблем. Одной из таких задач является изучение взаимосвязей интегральных свойств функции и свойств суммы ее коэффициентов. Решению этих задач были посвящены усилия многих математиков. И дальнейшие исследования в этой области являются важными и интересными задачами и могут привнести новые, неожиданные эффекты. В данной статье мы получаем двумерный аналог теоремы типа Бочкарева для преобразования Фурье.

Ключевые слова: пространство Лоренца, теорема Хаусдорфа-Юнга-Рисса, теорема Бочкарева, неравенство Коши-Буняковского, неравенство Гельдера.

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