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Solution of a two-dimensional parabolic model problem in a degenerate angular domain

In this paper, the boundary value problem of heat conduction in a domain was considered, boundary of which changes with time, as well as there is no the problem solution domain at the initial time, that is, it degenerates into a point. To solve the problem, the method of heat potentials was used, which makes it possible to reduce it to a singular Volterra type integral equations of the second kind. The peculiarity of the obtained integral equation is that it fundamentally differs from the classical Volterra integral equations, since the Picard method is not applicable to it and the corresponding homogeneous integral equation has a nonzero solution.

Keywords: heat equation, boundary value problem, degenerate domain, Volterra singular integral equation, regularization.

Introduction

Recently, in connection with the intensive development of modern contact technology and due to the high speed of electrical devices, more reliable measurement of the temperature field of the contact system has become relevant. And, no less important, it is necessary to study the dynamics of its change over time. At the same time, the temperature field of high-current contacts must be studied taking into account the change in the size of the contact area, which changes both due to the action of electrodynamic forces and due to the melting of the contact material at high temperatures.

When the electrodes are opened on the contact surface, the melting temperature is reached and a liquid metal bridge appears between them. As a result of further opening, this bridge is divided into two parts and the contact material is transferred from one electrode to another, that is, bridge erosion occurs, which can eventually disrupt their normal operation.

To solve this kind of heat conduction problems, it is necessary to use generalized heat potentials and further reduce the original boundary value problem to singular Volterra type integral equations. From a mathematical point of view, the peculiarity of the problems under consideration is that, firstly, the domain in which solutions are sought has a moving boundary, and secondly, at the initial moment of time, the contacts are in a closed state and the problem solution domain degenerates into a point [1–14].

The problem considered in this paper is called a model one, since the case is studied when the boundary of the domain in which the solution of the problem is sought moves according to the linear law $x = t$. In the future, it is planned to study this problem in the case when the boundary of the domain will change according to an arbitrary law $x = \gamma(t)$, $\gamma(0) = 0$.

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1 Statement of the boundary value problem

We consider the following two-dimensional boundary value problem in spatial variables in a cone $Q = \left\{(x, y, t)|\ \sqrt{x^2 + y^2} < t,\ t > 0, \right\}$ with a lateral surface $\Gamma = \left\{(x, y, t)|\ \sqrt{x^2 + y^2} = t,\ t > 0\right\}$ for the equation

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - a^2\beta \left(\frac{1}{x} \cdot \frac{\partial u}{\partial x} + \frac{1}{y} \cdot \frac{\partial u}{\partial x}\right) \tag{1}$$

with a boundary condition

$$u(x, y, t)|_{\Gamma} = g(x, y, t), \tag{2}$$

where $0 < \beta < 1$, $g(x, y, t)$ is a given function. It is necessary to find a function $u(x, y, t)$ satisfying the equation (1) in Q and the boundary condition (2).

Such boundary value problems in domains that change with time and degenerate into a point arise, for example: when describing the heat transfer process in a moving medium velocity of which is a function of the coordinates; in mathematical modeling of thermophysical processes in the electric arc of high-current disconnecting devices, while taking into account the effect of contracting the axial section of the arc into a contact spot in the cathode field. They are also relevant in the creation of new technologies in metallurgy, the production of crystals, laser technologies, etc.

Passing in (1),(2) to cylindrical coordinates, in the domain $Q = \{(r, t)|\ 0 < r < t,\ t > 0\}$, we obtain the following boundary value problem for the axisymmetric case:

$$\frac{\partial u}{\partial t} = a^2 \cdot \frac{1 - 2\beta}{r} \cdot \frac{\partial u}{\partial r} + a^2 \cdot \frac{\partial^2 u}{\partial r^2},\quad 0 < \beta < 1, \tag{3}$$

$$u(r, t)|_{r=0} = g_1(t), \quad t > 0, \tag{4}$$

$$u(r, t)|_{r=t} = g_2(t), \quad t > 0. \tag{5}$$

2 Representation of a solution of the boundary problem (3)–(5) using heat potentials

The fundamental solution for the equation (3) is the function

$$G(r, \xi, t - \tau) = \frac{1}{2a^2} \cdot \frac{r^\beta \cdot \xi^{1-\beta}}{t - \tau} \cdot \exp\left[-\frac{r^2 + \xi^2}{4a^2(t - \tau)}\right] \cdot I_\beta \left(\frac{r\xi}{2a^2(t - \tau)}\right),$$

where ξ is a parameter, $I_\beta(z)$ is the modified Bessel function of order β. We will seek the solution of the problem (3) – (5) as the sum of double layer heat potentials

$$u(r, t) = \int_0^t \frac{\partial G(r, \xi, t - \tau)}{\partial \xi}\bigg|_{\xi=\tau} \mu(\tau)d\tau + \int_0^t \frac{\partial G(r, \xi, t - \tau)}{\partial \xi}\bigg|_{\xi=0} \nu(\tau)d\tau, \tag{6}$$

where $\mu(t)$ and $\nu(t)$ are potential densities to be determined.

Let's transform the function (6), for this we calculate the derivative:

$$\frac{\partial G(r, \xi, t - \tau)}{\partial \xi} = \frac{1}{4a^4} \cdot \frac{r^\beta \cdot \xi^{1-\beta}}{(t - \tau)^2} \cdot \exp\left[-\frac{r^2 + \xi^2}{4a^2(t - \tau)}\right] \cdot \left\{rI_{\beta-1}\left(\frac{r\xi}{2a^2(t - \tau)}\right) - \xi I_\beta \left(\frac{r\xi}{2a^2(t - \tau)}\right)\right\} +$$

$$+\frac{1}{2a^2} \cdot \frac{r^\beta(1 - 2\beta)}{(t - \tau)\xi^\beta} \cdot \exp\left[-\frac{r^2 + \xi^2}{4a^2(t - \tau)}\right] \cdot I_\beta \left(\frac{r\xi}{2a^2(t - \tau)}\right),$$

where we have used the relation [15; 975]:

$$I'_\beta(z) = I_{\beta-1}(z) - \frac{\beta}{z} I_\beta(z).$$

Next we find

$$\left.\frac{\partial G(r,\xi,t-\tau)}{\partial \xi}\right|_{\xi=0} = \frac{1}{(2a^2)^{\beta+1}} \cdot \frac{r^{2\beta}}{2^\beta (t-\tau)^{\beta+1}} \cdot \frac{1}{\beta\Gamma(\beta)} \cdot \exp\left[-\frac{r^2}{4a^2(t-\tau)}\right] \tag{7}$$

and

$$\left.\frac{\partial G(r,\xi,t-\tau)}{\partial \xi}\right|_{\xi=\tau} =$$
$$= \frac{1}{4a^4} \cdot \frac{r^\beta \cdot \tau^{1-\beta}}{(t-\tau)^2} \cdot \exp\left[-\frac{r^2+\tau^2}{4a^2(t-\tau)}\right] \cdot \left\{ r I_{\beta-1}\left(\frac{r\tau}{2a^2(t-\tau)}\right) - \tau I_\beta\left(\frac{r\tau}{2a^2(t-\tau)}\right)\right\} +$$
$$+ \frac{1}{2a^2} \cdot \frac{r^\beta (1-2\beta)}{(t-\tau)\tau^\beta} \cdot \exp\left[-\frac{r^2+\tau^2}{4a^2(t-\tau)}\right] \cdot I_\beta\left(\frac{r\tau}{2a^2(t-\tau)}\right).$$

We transform the last equality as follows:

$$\left.\frac{\partial G(r,\xi,t-\tau)}{\partial \xi}\right|_{\xi=\tau} =$$
$$= \frac{r^\beta \tau^{1-\beta}(r-\tau)}{4a^4(t-\tau)^2} \cdot \exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right] \cdot \exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right] I_\beta\left(\frac{r\tau}{2a^2(t-\tau)}\right) +$$
$$+ \frac{r^{\beta+1}\tau^{1-\beta}}{4a^4(t-\tau)^2} \cdot \exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right] \ \exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right] I_{\beta-1,\beta}\left(\frac{r\tau}{2a^2(t-\tau)}\right) +$$
$$+ \frac{r^\beta(1-2\beta)}{2a^2(t-\tau)\tau^\beta} \cdot \exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right] \cdot \exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right] \cdot I_\beta\left(\frac{r\tau}{2a^2(t-\tau)}\right), \tag{8}$$

where we introduced the notation

$$I_{\beta-1,\beta}(z) = I_{\beta-1}(z) - I_\beta(z).$$

We substitute the obtained relations (7), (8) into the equality (6), and then we obtain the integral representation of the solution for the equation (9):

$$u(r,t) = \int_0^t \left\{ \frac{r^\beta \tau^{1-\beta}(r-\tau)}{4a^4(t-\tau)^2} \exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right] \exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right] I_\beta\left(\frac{r\tau}{2a^2(t-\tau)}\right) + \right.$$
$$+ \frac{r^{\beta+1}\tau^{1-\beta}}{4a^4(t-\tau)^2} \exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right] \exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right] I_{\beta-1,\beta}\left(\frac{r\tau}{2a^2(t-\tau)}\right) +$$
$$\left. + \frac{r^\beta(1-2\beta)}{2a^2(t-\tau)\tau^\beta} \exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right] \exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right] \cdot I_\beta\left(\frac{r\tau}{2a^2(t-\tau)}\right)\right\} \mu(\tau)d\tau +$$
$$+ \int_0^t \frac{1}{(2a^2)^{\beta+1}} \cdot \frac{r^{2\beta}}{2^\beta(t-\tau)^{\beta+1}} \cdot \frac{1}{\beta\Gamma(\beta)} \cdot \exp\left[-\frac{r^2}{4a^2(t-\tau)}\right] \cdot \nu(\tau)d\tau, \tag{9}$$

where

$$t^{-\beta} e^{\frac{t}{4a^2}} \mu(t) \in L_\infty(0,\infty).$$

3 Reduction of the boundary value problem (3) – (5) to a singular Volterra type integral equation

We require that the function $u(r,t)$ defined by the equality (9) satisfy the boundary conditions (4),(5), which will allow us to define the functions $\mu(t)$ and $\nu(t)$.

$$\lim_{r\to 0} u(r,t) = \lim_{r\to 0}\left[\int_0^t \left\{ \frac{r^{\beta}\tau^{1-\beta}(r-\tau)}{4a^4(t-\tau)^2}\exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right]\exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right] I_{\beta}\left(\frac{r\tau}{2a^2(t-\tau)}\right) + \right.\right.$$

$$+\frac{r^{\beta+1}\tau^{1-\beta}}{4a^4(t-\tau)^2}\exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right]\exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right] I_{\beta-1,\beta}\left(\frac{r\tau}{2a^2(t-\tau)}\right)+$$

$$\left. + \frac{r^{\beta}(1-2\beta)}{2a^2(t-\tau)\tau^{\beta}}\exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right]\exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right]\cdot I_{\beta}\left(\frac{r\tau}{2a^2(t-\tau)}\right)\right\}\mu(\tau)d\tau +$$

$$\left. + \int_0^t \frac{1}{(2a^2)^{\beta+1}}\cdot\frac{r^{2\beta}}{2^{\beta}(t-\tau)^{\beta+1}}\cdot\frac{1}{\beta\Gamma(\beta)}\cdot\exp\left[-\frac{r^2}{4a^2(t-\tau)}\right]\cdot\nu(\tau)d\tau\right] =$$

$$= \frac{1}{(2a^2)^{\beta+1}}\cdot\frac{1}{2^{\beta}}\cdot\frac{1}{\beta\Gamma(\beta)}\cdot\lim_{r\to 0}\int_0^t \frac{r^{2\beta}}{(t-\tau)^{\beta+1}}\cdot\exp\left[-\frac{r^2}{4a^2(t-\tau)}\right]\cdot\nu(\tau)d\tau =$$

$$= \left\| \frac{r^2}{4a^2(t-\tau)} = z \right\| = \frac{1}{(2a^2t)^{\beta+1}}\cdot\frac{1}{2^{\beta}}\cdot\frac{1}{\beta\Gamma(\beta)}\times$$

$$\times \lim_{r\to 0}\int_{\frac{r^2}{4a^2t}}^{\infty} \frac{r^{2\beta}\cdot(4a^2)^{\beta+1}\cdot z^{\beta+1}}{r^{2\beta+2}}\cdot\frac{r^2}{4a^2z^2}\cdot e^{-z}\cdot\nu\left(t-\frac{r^2}{4a^2z}\right)dz =$$

$$= \frac{1}{(2a^2)^{\beta+1}}\cdot\frac{1}{2^{\beta}}\cdot\frac{1}{\beta\Gamma(\beta)}\cdot\frac{(4a^2)^{\beta+1}}{4a^2}\cdot\lim_{r\to 0}\int_{\frac{r^2}{4a^2t}}^{\infty} z^{\beta-1}\cdot e^{-z}\cdot\nu\left(t-\frac{r^2}{4a^2z}\right)dz =$$

$$= \frac{1}{2a^2}\cdot\frac{1}{\beta\Gamma t(\beta)}\cdot\nu(t)\cdot\int_0^{\infty} z^{\beta-1}\cdot e^{-z}dz = \frac{1}{2a^2}\cdot\frac{1}{\beta\Gamma(\beta)}\cdot\Gamma(\beta)\cdot\psi(t) = \frac{1}{2a^2\beta}\cdot\nu(t) = g_1(t).$$

from here one of the sought-for densities $\nu(t)$ is directly determined

$$\nu(t) = 2a^2\beta g_1(t).$$

Therefore,

$$u(r,t) = \sum_{i=1}^{3} u_1(r,t) + \widetilde{g_1}(r,t), \tag{10}$$

where

$$u_1(r,t) = \int_0^t \frac{r^{\beta}\tau^{1-\beta}(r-\tau)}{4a^4(t-\tau)^2}\cdot e^{-\frac{(r-\tau)^2}{4a^2(t-\tau)}}\cdot e^{-\frac{r\tau}{2a^2(t-\tau)}}\cdot I_{\beta}\left(\frac{r\tau}{2a^2(t-\tau)}\right)\mu(\tau)d\tau,$$

$$u_2(r,t) = \int_0^t \frac{r^{\beta+1}\tau^{1-\beta}}{4a^4(t-\tau)^2}\cdot e^{-\frac{(r-\tau)^2}{4a^2(t-\tau)}}\cdot e^{-\frac{r\tau}{2a^2(t-\tau)}}\cdot I_{\beta-1,\beta}\left(\frac{r\tau}{2a^2(t-\tau)}\right)\mu(\tau)d\tau,$$

$$u_3(r,t) = \int_0^t \frac{r^{\beta}(1-2\beta)}{2a^2(t-\tau)\tau^{\beta}}\cdot e^{-\frac{(r-\tau)^2}{4a^2(t-\tau)}}\cdot e^{-\frac{r\tau}{2a^2(t-\tau)}}\cdot I_{\beta}\left(\frac{r\tau}{2a^2(t-\tau)}\right)\mu(\tau)d\tau,$$

$$\widetilde{g_1}(r,t) = \frac{1}{(2a^2)^{\beta}}\cdot\frac{1}{2^{\beta}}\cdot\frac{1}{\Gamma(\beta)}\int_0^t \frac{r^{2\beta}}{(t-\tau)^{\beta+1}}\cdot\exp\left[-\frac{r^2}{4a^2(t-\tau)}\right]\cdot g_1(\tau)d\tau.$$

Remark 1. If $g_1(t)$ is bounded, then $\widetilde{g_1}(r,t)$ is also bounded.

Indeed,

$$\widetilde{g_1}(r,t) \le \frac{1}{(2a^2)^\beta} \cdot \frac{1}{2^\beta} \cdot \frac{1}{\Gamma(\beta)} \cdot |g_1(t)| \int_0^t \frac{r^{2\beta}}{(t-\tau)^{\beta+1}} \cdot \exp\left[-\frac{r^2}{4a^2(t-\tau)}\right] d\tau =$$

$$= \left\| \frac{r^2}{4a^2(t-\tau)} = z \right\| = \frac{1}{(2a^2)^\beta} \cdot \frac{1}{2^\beta} \cdot \frac{1}{\Gamma(\beta)} \cdot 4^\beta \cdot a^{2\beta} \cdot |g_1(t)| \int_{\frac{r^2}{4a^2t}}^\infty z^{\beta-1} \cdot e^{-z} dz =$$

$$= |g_1(t)| \cdot \frac{\Gamma\left(\beta, \frac{r^2}{4a^2t}\right)}{\Gamma(\beta)} < |g_1(t)|, \qquad \forall (r,t) \in G.$$

Now let us satisfy the boundary condition (5).

$$u(r,t)|_{r=t} = \lim_{r\to t-0} u(r,t) = g_2(t) = \widetilde{g_1}(t,t)+$$

$$+\int_0^t \left\{ \frac{t^{\beta+1}\tau^{1-\beta}}{4a^4(t-\tau)^2} \exp\left[-\frac{t-\tau}{4a^2}\right] \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_{\beta-1,\beta}\left(\frac{t\tau}{2a^2(t-\tau)}\right) + \right.$$

$$\left. + \frac{t^\beta(1-2\beta)}{2a^2(t-\tau)\tau^\beta} \exp\left[-\frac{t-\tau}{4a^2}\right] \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] \cdot I_\beta\left(\frac{t\tau}{2a^2(t-\tau)}\right)\right\} \mu(\tau)d\tau - \frac{\mu(t)}{2a^2}.$$

As a result, we obtain the following integral equation for the unknown density $\mu(t)$:

$$\mu(t) - \int_0^t \left\{ \frac{t^\beta(1-2\beta)}{(t-\tau)\tau^\beta} \exp\left[-\frac{t-\tau}{4a^2}\right] \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] \cdot I_\beta\left(\frac{t\tau}{2a^2(t-\tau)}\right) + \right.$$

$$+ \frac{t^{\beta+1}\tau^{1-\beta}}{2a^2(t-\tau)^2} \exp\left[-\frac{t-\tau}{4a^2}\right] \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_{\beta-1,\beta}\left(\frac{t\tau}{2a^2(t-\tau)}\right) +$$

$$\left. + \frac{t^\beta\tau^{1-\beta}}{2a^2(t-\tau)} \exp\left[-\frac{t-\tau}{4a^2}\right] \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_\beta\left(\frac{t\tau}{2a^2(t-\tau)}\right)\right\} \mu(\tau)d\tau = F(t). \tag{11}$$

where

$$F(t) = -2a^2 g_2(t) + 2a^2 \widetilde{g_1}(t,t).$$

We introduce the following notation

$$t^{1-\beta} \exp\left[\frac{t}{4a^2}\right] \mu(t) = \mu_1(t), \quad t^{1-\beta} \exp\left[\frac{t}{4a^2}\right] F(t) = F_1(t).$$

Then the last integral equation is transformed into the following equation:

$$\mu_1(t) - \int_0^t N(t,\tau)\mu_1(\tau)d\tau = F_1(t), \tag{12}$$

kernel of which has the form:

$$N(t,\tau) = \sum_{i=1}^2 N_i(t,\tau),$$

and, moreover,

$$N_1(t,\tau) = \frac{t(1-2\beta)}{\tau(t-\tau)} \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] \cdot I_\beta\left(\frac{t\tau}{2a^2(t-\tau)}\right) +$$

$$+ \frac{t^2}{2a^2(t-\tau)^2} \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_{\beta-1,\beta}\left(\frac{t\tau}{2a^2(t-\tau)}\right),$$

$$N_2(t,\tau) = \frac{t}{2a^2(t-\tau)} \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_\beta\left(\frac{t\tau}{2a^2(t-\tau)}\right).$$

A feature of this integral equation follows from

Remark 2. The kernel of the integral equation (11) satisfies the equality

$$\lim_{t\to 0}\int_0^t N(t,\tau)d\tau = \frac{1-\beta}{\beta},$$

moreover, $\forall t>0$, $\forall\beta\in(0;1)$:

$$\int_0^t N_1(t,\tau)d\tau = \frac{1-\beta}{\beta}, \qquad \lim_{t\to 0}\int_0^t N_2(t,\tau)d\tau = 0.$$

Indeed,

$$\int_0^t N_1(t,\tau)d\tau = \int_0^t \left\{\frac{t(1-2\beta)}{\tau(t-\tau)}\exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right]\cdot I_\beta\left(\frac{t\tau}{2a^2(t-\tau)}\right)+\right.$$

$$\left.+\frac{t^2}{2a^2(t-\tau)^2}\exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right]I_{\beta-1,\beta}\left(\frac{t\tau}{2a^2(t-\tau)}\right)d\tau\right\} = \left\|\frac{t\tau}{2a^2(t-\tau)}=z\right\| =$$

$$= \int_0^\infty (1-2\beta)\cdot\frac{1}{z^2}\cdot z\cdot e^{-z}\cdot I_\beta(z)dz + \int_0^\infty e^{-z}\cdot\{I_{\beta-1}(z)-I_\beta(z)\}\,dz =$$

$$= (1-2\beta)\int_0^\infty \frac{1}{z}\cdot e^{-z}\cdot I_\beta(z)dz + 1 = \|(2.15.4.3)\ [16;272]\| =$$

$$= \frac{(1-2\beta)}{\sqrt{\pi}}\cdot\Gamma\left[\begin{array}{cc}\beta, & \frac{1}{2}\\ 1+\beta & \end{array}\right]+1 = \frac{1-\beta}{\beta};$$

$$\int_0^t N_2(t,\tau)d\tau = \int_0^t \frac{t}{2a^2(t-\tau)}\exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right]I_\beta\left(\frac{t\tau}{2a^2(t-\tau)}\right)d\tau =$$

$$= \left\|\frac{t\tau}{2a^2(t-\tau)}=z\right\| = \int_0^\infty \frac{t}{t+2a^2z}\cdot e^{-z}\cdot I_\beta(z)dz \le \frac{t}{2a^2}\int_0^\infty \frac{1}{z}\cdot e^{-z}\cdot I_\beta(z)dz =$$

$$= \frac{t}{2a^2}\cdot\frac{1}{\sqrt{\pi}}\cdot\Gamma\left[\begin{array}{cc}\beta, & \frac{1}{2}\\ 1+\beta & \end{array}\right] = \frac{t}{2a^2\beta}\underset{t\to 0}{\longrightarrow}0.$$

4 Solution of the characteristic integral equation

In order to find a solution of the integral equation (10), we first study the following characteristic integral equation:

$$\mu_1(t) - \int_0^t N_1(t,\tau)\mu_1(\tau)d\tau = \Phi(t). \tag{13}$$

Remark 3. Remark (12) implies that for $\frac{1}{2}<\beta<1$, $\left(0<\frac{1-\beta}{\beta}<1\right)$ the integral equation (13) in the class of essentially bounded functions has a unique solution that can be found by the method of successive approximations.

By Remark 2 for $0<\beta\le\frac{1}{2}$ $\left(\frac{1-\beta}{\beta}\ge 1\right)$ equation (13) is indeed characteristic for the equation (11).

Instead of the variables t, τ we introduce new variables x, y:

$$t=\frac{1}{y},\quad \tau=\frac{1}{x}; \qquad \mu_1(t)=\mu_1\left(\frac{1}{y}\right)=\mu_2(y), \qquad \Phi(t)=\Phi\left(\frac{1}{y}\right)=\Phi_1(y), \tag{14}$$

Then the equation (13) reduces to the following integral equation with respect to the unknown function $\mu_2(y)$:

$$\mu_2(y) - \int_y^\infty M_-(y-x)\mu_2(x)dx = \Phi_1(y), \tag{15}$$

where

$$M_-(y-x) = \frac{1-2\beta}{x-y} \cdot \exp\left(-\frac{1}{2a^2(x-y)}\right) \cdot I_\beta\left(\frac{1}{2a^2(x-y)}\right) +$$

$$+\frac{1}{2a^2(x-y)^2} \cdot \exp\left(-\frac{1}{2a^2(x-y)}\right) \cdot I_{\beta-1,\beta}\left(\frac{1}{2a^2(x-y)}\right).$$

Remark 4. If we find a solution to the equation (13), then we will obtain a solution to the equation (11) by applying the equivalent regularization method to the solution of the characteristic equation [17, 18].

5 *Solution of the homogeneous characteristic equation*

The equation (15) differs fundamentally from the Volterra equations of the second kind, for which the solution exists and is unique. The solution of the corresponding homogeneous equation

$$\mu_2(y) - \int_y^\infty M_-(y-x)\mu_2(x)dx = 0, \tag{16}$$

in the general case may also be non-trivial. The eigenfunctions of the integral equation (16) are determined by the roots of the following transcendental equation [18; 569] with respect to the parameter p:

$$\widehat{M_-}(-p) = \int_0^\infty M_-(z) \cdot e^{pz} dz = 1, \quad \mathrm{Re} p < 0, \tag{17}$$

since, by applying the Laplace transform to the equation (16), we obtain

$$\widehat{\mu_2}(p) \cdot \left[1 - \widehat{M_-}(-p)\right] = 0, \quad \mathrm{Re} p < 0. \tag{18}$$

In order to find the image of the function $\widehat{M_-}(-p)$ we use:
1) the formula (29.169) [19; 350];
2) the property: let $f(t) \risingdotseq \hat{f}(p)$, then $\frac{1}{t}f(t) \risingdotseq \int_p^\infty \hat{f}(p)dp$ [20; 506]. Thus, we have

$$\widehat{M_-}(-p) = 2(1-2\beta)K_\beta\left(\frac{\sqrt{-p}}{a}\right) I_\beta\left(\frac{\sqrt{-p}}{a}\right) +$$

$$+\frac{1}{a^2}\int_{-\infty}^p \left[K_{\beta-1}\left(\frac{\sqrt{-q}}{a}\right) I_{\beta-1}\left(\frac{\sqrt{-q}}{a}\right) - K_\beta\left(\frac{\sqrt{-q}}{a}\right) I_\beta\left(\frac{\sqrt{-q}}{a}\right)\right] dq.$$

To calculate the last integral, we use the formula (1.12.4.3) [16; 44]:

$$\frac{1}{a^2}\int_{-\infty}^p \left[K_{\beta-1}\left(\frac{\sqrt{-q}}{a}\right) I_{\beta-1}\left(\frac{\sqrt{-q}}{a}\right) - K_\beta\left(\frac{\sqrt{-q}}{a}\right) I_\beta\left(\frac{\sqrt{-q}}{a}\right)\right] dq =$$

$$= \left\|\frac{\sqrt{-q}}{a} = z\right\| = 2\int_{\frac{\sqrt{-p}}{a}}^\infty z\left[K_{\beta-1}(z)I_{\beta-1}(z) - K_\beta(z)I_\beta(z)\right]dz =$$

$$= z^2\left[\left(1 + \frac{(\beta-1)^2}{z^2}\right) I_{\beta-1}(z)K_{\beta-1}(z) - I'_{\beta-1}(z)K'_{\beta-1}(z)\right]\Bigg|_{\frac{\sqrt{-p}}{a}}^\infty -$$

$$- z^2\left[\left(1+\frac{\beta^2}{z^2}\right)I_\beta(z)K_\beta(z) - I'_\beta(z)K'_\beta(z)\right]\Big|_{\frac{\sqrt{-p}}{a}}^{\infty} =$$

$$= z^2\left[\left(1+\frac{(\beta-1)^2}{z^2}\right)I_{\beta-1}(z)K_{\beta-1}(z) -\right.$$

$$\left.- \left\{I_\beta(z)+\frac{\beta-1}{z}I_{\beta-1}(z)\right\}\left\{-K_\beta(z)+\frac{\beta-1}{z}K_{\beta-1}(z)\right\}\right]\Big|_{\frac{\sqrt{-p}}{a}}^{\infty} -$$

$$- z^2\left[\left(1+\frac{\beta^2}{z^2}\right)I_\beta(z)K_\beta(z) - \left\{I_{\beta-1}(z)-\frac{\beta}{z}I_\beta(z)\right\}\left\{-K_{\beta-1}(z)-\frac{\beta}{z}K_\beta(z)\right\}\right]\Big|_{\frac{\sqrt{-p}}{a}}^{\infty} =$$

$$= \left[\left(z^2+(\beta-1)^2\right)I_{\beta-1}(z)K_{\beta-1}(z) + z^2 I_\beta(z)K_\beta(z) -\right.$$

$$\left.-z\ (\beta-1)I_\beta(z)K_{\beta-1}(z) + z(\beta-1)I_\beta(z)K_\beta(z) - (\beta-1)^2 I_{\beta-1}(z)K_{\beta-1}(z)\right]\Big|_{\frac{\sqrt{-p}}{a}}^{\infty} -$$

$$- \left[(z^2+\beta^2)I_\beta(z)K_\beta(z) + z^2 I_{\beta-1}(z)K_{\beta-1}(z) + z\beta I_{\beta-1}(z)K_\beta(z) -\right.$$

$$\left.-z\beta I_\beta(z)K_{\beta-1}(z) - \beta^2 I_\beta(z)K_\beta(z)\right]\Big|_{\frac{\sqrt{-p}}{a}}^{\infty} =$$

$$= \left[zI_\beta(z)K_{\beta-1}(z) - zI_{\beta-1}(z)K_\beta(z)\right]\Big|_{\frac{\sqrt{-p}}{a}}^{\infty} =$$

$$= \left[2zI_\beta(z)K_{\beta-1}(z) - (zI_\beta(z)K_{\beta-1}(z) + zI_{\beta-1}(z)K_\beta(z))\right]\Big|_{\frac{\sqrt{-p}}{a}}^{\infty} =$$

$$= 2zI_\beta(z)K_{\beta-1}(z)\Big|_{\frac{\sqrt{-p}}{a}}^{\infty} = 1 - 2\frac{\sqrt{-p}}{a}I_\beta\left(\frac{\sqrt{-p}}{a}\right)K_{\beta-1}\left(\frac{\sqrt{-p}}{a}\right),$$

where we used the following relations:

$$K'_\beta(z) = -K_{\beta-1}(z) - \frac{\beta}{z}K_\beta(z),$$

$$K'_{\beta-1}(z) = -K_\beta(z) + \frac{\beta-1}{z}K_{\beta-1}(z),$$

$$I'_\beta(z) = I_{\beta-1}(z) - \frac{\beta}{z}I_\beta(z),$$

$$I'_{\beta-1}(z) = I_\beta(z) + \frac{\beta-1}{z}I_{\beta-1}(z).$$

Then the equation (17) will take the form:

$$2I_\beta\left(\frac{\sqrt{-p}}{a}\right)\left[(1-2\beta)K_\beta\left(\frac{\sqrt{-p}}{a}\right) - \frac{\sqrt{-p}}{a}K_{\beta-1}\left(\frac{\sqrt{-p}}{a}\right)\right] = 0, \qquad \mathrm{Re}p < 0,$$

where $K_\beta\left(\frac{\sqrt{-p}}{a}\right)$ is the Macdonald function.

Let's assume that $I_\beta\left(\frac{\sqrt{-p}}{a}\right) = 0$. According to the definition of the Bessel function for the imaginary argument $I_\beta\left(\frac{\sqrt{-p}}{a}\right) = e^{-\frac{\pi}{2}\beta i}J_\beta\left(\frac{i\sqrt{-p}}{a}\right)$, where $J_\beta\left(\frac{i\sqrt{-p}}{a}\right)$ is the Bessel function – cylinder function of the first kind. The function $J_\beta(z)$ for any real β has an infinite set of real roots; for $\beta > -1$ all its roots are real and equal $iz_k = \alpha_k$, $z_k = -i\alpha_k$, $\alpha_k \in \mathbb{R}$, $k \in \mathbb{Z}\backslash\{0\}$ [21], i.e. in our case $\frac{i\sqrt{-p_k}}{a} = \alpha_k$, where $\alpha_k \in \mathbb{R}$. Hence $p_k = a^2\alpha_k^2$, which contradicts the condition $\mathrm{Re}p < 0$.

Thus, it is necessary to find the roots of the equation for $0 < \beta \leq \frac{1}{2}$

$$(1-2\beta)K_\beta\left(\frac{\sqrt{-p}}{a}\right)-\frac{\sqrt{-p}}{a}K_{\beta-1}\left(\frac{\sqrt{-p}}{a}\right)=0,\quad \mathrm{Re}p<0. \tag{19}$$

It should be noted that for $\frac{1}{2}<\beta<1$ this equation has no roots. This means that in the equation (18)

$$1-\widehat{M_-}(-p)\neq 0,$$

whence it follows that $\widehat{\mu_2}(p)=0$. That is, the homogeneous integral equation (16) has only a zero solution in this case. For $0<\beta\leq\frac{1}{2}$ the equation (19) has a unique real root $p_0\leq 0$, and the root $p_0=0$ corresponds to the case $\beta=\frac{1}{2}$. And for $0<\beta<\frac{1}{2}$ the root is $p_0<0$. This means that the equation (16) for $0<\beta<\frac{1}{2}$ has a non-zero solution $\mu_2(y)=Ce^{p_0y}$, $p_0<0$. Then, returning to the original variables (14), we obtain that the homogeneous integral equation corresponding to the equation (13), for $0<\beta<\frac{1}{2}$ has an eigenfunction

$$\mu^{(0)}(t)=C\cdot\frac{1}{t^{1-\beta}}\cdot e^{\frac{p_0}{t}-\frac{t}{4a^2}},\quad p_0<0,\quad C=\mathrm{const}.$$

Accordingly, for $\beta=\frac{1}{2}$, the eigenfunction has the form:

$$\mu^{(0)}(t)=C\cdot\frac{1}{\sqrt{t}}\cdot e^{-\frac{t}{4a^2}},\qquad C=\mathrm{const}.$$

6 *Solution of an inhomogeneous characteristic integral equation. Construction of the resolvent.*

The equation (15) cannot be solved by directly applying the Laplace transform, since the convolution theorem is not applicable here. Let's apply the method of model solutions [18; 561]. Then the solution of the equation (15) has the form

$$\varphi_1(y)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\widehat{\Phi_1}(p)}{1-\widehat{M_-}(-p)}dp=\Phi_1(y)+\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\widehat{R_-}(-p)\widehat{\Phi_1}(p)e^{pz}dp,$$

where

$$\widehat{\Phi_1}(p)=\int_0^\infty\Phi_1(y)e^{-py}dy,\quad \widehat{R_-}(-p)=\frac{\widehat{M_-}(-p)}{1-\widehat{M_-}(-p)},\quad \mathrm{Re}p<0;$$

$$\widehat{M_-}(-p)=1+2I_\beta\left(\frac{\sqrt{-p}}{a}\right)\left[(1-2\beta)K_\beta\left(\frac{\sqrt{-p}}{a}\right)-\frac{\sqrt{-p}}{a}K_{\beta-1}\left(\frac{\sqrt{-p}}{a}\right)\right],\ \mathrm{Re}p<0.$$

If $\widehat{R_-}(-p)\risingdotseq R_-(y)$, then the solution of the equation (15) has the form

$$\varphi_1(y)=\Phi_1(y)+\frac{1}{2\pi i}\int_y^\infty R_-(y-x)\Phi_1(x)dx. \tag{20}$$

To find the resolvent $R_-(y)$, we write its image in the following form:

$$\widehat{R_-}\left(\frac{\sqrt{-p}}{a}\right)=\frac{1-2I_\beta\left(\frac{\sqrt{-p}}{a}\right)\left[\frac{\sqrt{-p}}{a}K_{\beta-1}\left(\frac{\sqrt{-p}}{a}\right)-(1-2\beta)K_\beta\left(\frac{\sqrt{-p}}{a}\right)\right]}{2I_\beta\left(\frac{\sqrt{-p}}{a}\right)\left[\frac{\sqrt{-p}}{a}K_{\beta-1}\left(\frac{\sqrt{-p}}{a}\right)-(1-2\beta)K_\beta\left(\frac{\sqrt{-p}}{a}\right)\right]},\quad \mathrm{Re}p<0,$$

and use the following properties [19; 191]:

1. If $\varphi(t)\risingdotseq\widehat{\varphi}(p)$, then

$$\varphi(\alpha t)\risingdotseq\frac{1}{\alpha}\widehat{\varphi}\left(\frac{p}{\alpha}\right),\ \alpha>0.$$

2. If $\widehat{\varphi}(p) \risingdotseq \varphi(t)$, then

$$\widehat{\varphi}\left(\sqrt{p}\right) = \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{t^{\frac{3}{2}}} \int_0^\infty \tau \cdot e^{-\frac{\tau^2}{4t}} \varphi(\tau) d\tau.$$

For convenience, we introduce the notation $\frac{\sqrt{-p}}{a} = z$ and find the original expression

$$\widehat{R^*}(z) = \frac{1 - 2I_\beta(z)\left[zK_{\beta-1}(z) - (1-2\beta)K_\beta(z)\right]}{2I_\beta(z)\left[zK_{\beta-1}(z) - (1-2\beta)K_\beta(z)\right]}.$$

According to [20; 519]:

$$\widehat{R^*}(z) = \frac{A(z)}{B(z)} \risingdotseq \sum_{-\infty}^{+\infty} \frac{A(z_k)}{B'(z_k)} \cdot e^{-z_k y},$$

where z_k are zeros of the function

$$B(z) = 2I_\beta(z)\left[zK_{\beta-1}(z) - (1-2\beta)K_\beta(z)\right].$$

1) Let $y_\beta(z) = zK_{\beta-1}(z) - (1-2\beta)K_\beta(z) = 0$. This equation, as noted earlier, has one root z_0 for $0 < \beta < \frac{1}{2}$.

2) Let $I_\beta(z) = e^{-\frac{\pi}{2}\beta i} J_\beta(iz) = 0$. Therefore, $iz_k = \alpha_k$ or $z_k = -i\alpha_k$, where $\alpha_k \in \mathbb{R}$.

Then

$$\widehat{R^*}(z) = \frac{A(z)}{B(z)} \risingdotseq \sum_{-\infty}^{+\infty} \frac{A(z_k)}{B'(z_k)} \cdot e^{-z_k y} = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{A(z_k)}{B'(z_k)} \cdot e^{-z_k y} + \frac{A(z_0)}{B'(z_0)} \cdot e^{-z_0 y} = R^*_-(y),$$

where

$$B(z) = 2I_\beta(z)\left[zK_{\beta-1}(z) - (1-2\beta)K_\beta(z)\right]$$

$$B'(z) = 2I_{\beta-1}(z)\left[zK_{\beta-1}(z) - (1-2\beta)K_\beta(z)\right] + 2(1-2\beta)I_\beta(z)K_{\beta-1}(z) + \\ + \left(\frac{4\beta(1-2\beta)}{z} - 2z\right) I_\beta(z)K_\beta(z).$$

Thus, we obtain that for $0 < \beta < \frac{1}{2}$:

$$R^*_-(y) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{e^{-z_k y}}{2I_{\beta-1}(z_k)\left[z_k K_{\beta-1}(z_k) - (1-2\beta)K_\beta(z_k)\right]} + \\ + \frac{e^{-z_0 y}}{2I_\beta(z_0)K_{\beta-1}(z_0)\left[1 - \frac{1}{1-2\beta} z_0^2\right]}. \tag{21}$$

We introduce the following notations:

$$A_{\beta,k} = \frac{1}{2I_{\beta-1}(z_k)\left[z_k K_{\beta-1}(z_k) - (1-2\beta)K_\beta(z_k)\right]}, \quad A_{\beta,0} = \frac{1}{2I_\beta(z_0)K_{\beta-1}(z_0)\left[1 - \frac{1}{1-2\beta} z_0^2\right]}.$$

From equality (21) we have

$$\widehat{R_-}\left(\frac{\sqrt{-p}}{a}\right) \risingdotseq R_-(y) = \frac{a^2}{2\sqrt{\pi} y^{\frac{3}{2}}} \cdot \sum_{k\in\mathbb{Z}\setminus\{0\}} A_{\beta,k} \cdot \int_0^\infty x e^{-\frac{x^2}{4y} - ia^2\alpha_k x} dx + \\ + \frac{a^2}{2\sqrt{\pi} y^{\frac{3}{2}}} \cdot A_{\beta,0} \cdot \int_0^\infty x e^{-\frac{x^2}{4y} - z_0 a^2 x} dx.$$

Lemma 1. The resolvent $R_-(y)$ satisfies the estimate

$$R_-(y) \le \frac{A}{\sqrt{y}}.$$

Proof.

$$R_-(y) \le \left| \frac{a^2}{2\sqrt{\pi} y^{\frac{3}{2}}} \left(\sum_{k\in\mathbb{Z}\setminus\{0\}} A_{\beta,k} \int_0^\infty x e^{-\frac{x^2}{4y} - i\alpha_k a^2 x} dx + A_{\beta,0} \int_0^\infty x e^{-\frac{x^2}{4y} - z_0 a^2 x} dx \right) \right| \le$$

$$\le \frac{a^2}{2\sqrt{y}} \cdot \left\{ \left| \sum_{k\in\mathbb{Z}\setminus\{0\}} A_{\beta,k} \right| + |A_{\beta,0}| \right\}.$$

Since $|A_{\beta,0}| = C_\beta = \text{const}$, we estimate the sum $\left|\sum_{k\in\mathbb{Z}\setminus\{0\}} A_{\beta,k}\right|$:

$$\left| \sum_{k\in\mathbb{Z}\setminus\{0\}} A_{\beta,k} \right| = \left| \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{2I_{\beta-1}(z_k)\left[z_k K_{\beta-1}(z_k) - (1-2\beta)K_\beta(z_k)\right]} \right| =$$

$$= \left\| \begin{array}{c} K_\beta(z) = \frac{\pi i}{2} e^{\frac{\pi}{2}\beta i} H_\beta^{(1)}(iz); \qquad I_\beta(z) = e^{-\frac{\pi}{2}\beta i} J_\beta(iz); \\ z_k = -i\alpha_k; \qquad z_{-k} = i\alpha_k; \\ J_\beta(-z) = e^{\beta\pi i} J_\beta(z); \qquad H_\beta^{(1)}(-z) = -e^{-\beta\pi i} H_\beta^{(2)}(z) \end{array} \right\| =$$

$$= \frac{1}{\pi} \left| \sum_{k=1}^{\infty} \left(\frac{1}{J_{\beta-1}(\alpha_k)\left[\alpha_k H_{\beta-1}^{(1)}(\alpha_k) + (1-2\beta) H_\beta^{(1)}(\alpha_k)\right]} - \right.\right.$$

$$\left.\left. - \frac{1}{e^{(\beta-1)\pi i} J_{\beta-1}(\alpha_k)\left[-\alpha_k e^{-(\beta-1)\pi i} H_{\beta-1}^{(2)}(\alpha_k) + (1-2\beta) e^{-\beta\pi i} H_\beta^{(2)}(\alpha_k)\right]} \right) \right| =$$

$$= \left\| \begin{array}{c} H_\beta^{(1)}(z) = J_\beta(z) + iN_\beta(z); \qquad H_\beta^{(2)}(z) = J_\beta(z) - iN_\beta(z) \\ J_\beta(\alpha_k) = 0 \end{array} \right\| =$$

$$= \frac{2}{\pi} \left| \sum_{k=1}^{\infty} \frac{\alpha_k}{\left(\alpha_k J_{\beta-1}(\alpha_k)\right)^2 + \left(\alpha_k N_{\beta-1}(\alpha_k)\right)^2 + 2\alpha_k(1-2\beta) N_{\beta-1}(\alpha_k) N_\beta(\alpha_k) + \left((1-2\beta)N_\beta(\alpha_k)\right)^2} \right| \le$$

$$\le \frac{2}{\pi} \left| \sum_{k=1}^{\infty} \frac{\alpha_k}{\left(\alpha_k J_{\beta-1}(\alpha_k)\right)^2 + \left(\alpha_k N_{\beta-1}(\alpha_k)\right)^2} \right| = \frac{2}{\pi} \left| \sum_{k=1}^{\infty} \frac{1}{\alpha_k H_{\beta-1}^{(1)}(\alpha_k) H_{\beta-1}^{(2)}(\alpha_k)} \right| \le$$

$$\le \frac{2}{\pi} \left| \int_{\alpha_1}^{\infty} \frac{d(\alpha_n)}{\alpha_k \cdot H_{\beta-1}^{(1)}(\alpha_k) \cdot H_{\beta-1}^{(2)}(\alpha_k)} \right| = \|(1.10.3.3)\ [16; 42]\| =$$

$$= \frac{2}{\pi} \left| -\frac{\pi}{4i} \cdot \ln \frac{H_{\beta-1}^{(2)}(\alpha_k)}{H_{\beta-1\nu}^{(1)}(\alpha_k)} \right|_{\alpha_1}^{\infty} \Bigg| = \frac{1}{2} \cdot \left| \left\{ \ln \left| \frac{H_{\beta-1}^{(2)}(\alpha_k)}{H_{\beta-1}^{(1)}(\alpha_k)} \right| + i \cdot \arg \frac{H_{\beta-1}^{(2)}(\alpha_k)}{H_{\beta-1}^{(1)}(\alpha_k)} \right\} \Bigg|_{\alpha_1}^{\infty} \right| =$$

$$= \frac{1}{2} \cdot \left| \arg H_{\beta-1}^{(2)}(\alpha_k) - \arg H_{\beta-1\nu}^{(1)}(\alpha_k) \right| \le \frac{\pi}{2}.$$

Thus, we get

$$R_-(y) \leq \frac{a^2}{2\sqrt{y}} \cdot \left\{ \left| \sum_{k\in\mathbb{Z}\setminus\{0\}} A_{\beta,k} \right| + |A_{\beta,0}| \right\} \leq \frac{a^2(\pi + 2C_\beta)}{4\sqrt{y}} = \frac{C_\beta{}^{(1)}}{\sqrt{y}},\ C_\beta{}^{(1)} = \text{const}.$$

$$R_-(y) \leq \frac{a^2}{2\sqrt{y}} \cdot \left| \sum_{k\in\mathbb{Z}\setminus\{0\}} A_{\beta,k} \right| \leq \frac{a^2\pi}{4\sqrt{y}}.$$

Lemma is proved.

7 *Solution of the characteristic equation*

We found a solution of the equation

$$\mu_2(y) - \int_y^\infty M_-(y-x)\mu_2(x)dx = \Phi_1(y),$$

which for $0 < \beta < \frac{1}{2}$ has the form

$$\mu_2(y) = \Phi_1(y) + \int_y^\infty R_-(x-y)\Phi_1(x)dx + Ce^{p_0 y}.$$

Returning to the original variables, we write the solution of the characteristic equation (20) as follows:

$$\mu_1(t) = \Phi(t) + \int_0^t \frac{R_-(t,\tau)}{\tau^2}\Phi(\tau)d\tau + Ce^{\frac{p_0}{t}}.$$

For the convergence of the last integral it is necessary that

$$\Phi_2(t) = \frac{1}{t} \cdot \Phi(t) \in L_\infty(0,\infty).$$

Then we write the solution of the characteristic equation (20) as

$$\mu_1(t) = t \cdot \Phi_2(t) + \int_0^t \tilde{R}(t,\tau) \cdot \Phi_2(\tau)d\tau + Ce^{\frac{p_0}{t}},$$

where

$$\tilde{R}(t,\tau) \leq C \cdot \frac{\sqrt{t}}{\sqrt{\tau}\cdot\sqrt{t-\tau}}.$$

The last inequality follows from Lemma 1.

8 *Solution of the initial integral equation. The Carleman-Vekua regularization*

Theorem 1. Initial integral equation (11) for any function $t^{-\beta}e^{\frac{t}{4a^2}} \cdot F(t) \in L_\infty(0,\infty)$ $\left(0 < \beta < \frac{1}{2}\right)$ has the unique solution in the class of functions

$$t^{-\beta}\exp\left[\frac{t}{4a^2}\right]\mu(t) \in L_\infty(0,\infty), \qquad \left(0 < \beta < \frac{1}{2}\right)$$

which can be found by the method of successive approximations.

Proof. We rewrite the initial integral equation (11) as

$$\mu_1(t) - \int_0^t N_1(t,\tau)\mu_1(\tau)d\tau = F_1(t) + \int_0^t N_2(t,\tau)\mu_1(\tau)d\tau. \tag{22}$$

Assuming the right-hand side of the equation (22) to be temporarily known, we write it in the following form:

$$[1-\mathcal{M}]\mu_2(t) \equiv$$

$$\equiv \mu_2(t) - \int_0^t M(t,\tau)\mu_2(\tau)d\tau = \frac{1}{t}F(t) + \frac{1}{t}\int_0^t \tilde{R}(t,\tau)\cdot\frac{F(\tau)}{\tau}d\tau + \frac{C}{t}e^{\frac{p_0}{t}}, \tag{23}$$

where

$$\mu_2(t) = \frac{1}{t}\mu_1(t), M(t,\tau) = \frac{\tau}{t}N_2(t,\tau) + \frac{\tau}{t}\cdot\int_\tau^t \tilde{R}(t,\xi)\cdot\frac{N_2(\xi,\tau)}{\xi}d\xi.$$

The following estimate for the kernel $M(t,\tau)$

$$M(t,\tau) \le \frac{\widetilde{D_1}}{\sqrt{t-\tau}} + \widetilde{D_2}, \quad \widetilde{D_1},\ \widetilde{D_2} = \text{const}$$

holds. Thus, we show that equation (23) for each $C \neq 0$ has a unique solution

$$\mu_2(t) = \mu_{2,\text{part}}(t) + C\mu_{2,\text{hom}}(t),$$

where

$$\mu_{2,\text{hom}}(t) = [1-\mathcal{M}]^{-1}\mu^{(0)}(t), \mu_2(t) = t^{-\beta}e^{\frac{t}{4a^2}}\mu(t).$$

At the same time, if $F(t)=0$, then integral equation (23) has a solution $\mu_2^{(0)}(t) = C\cdot[1-\mathcal{M}]^{-1}\mu^{(0)}(t)$. The theorem is proved.

9 *Solution of the boundary value problem (3)–(5)*

Theorem 2. If the conditions $g_1(t) \in L_\infty(0,\infty)$, $t^{-\beta}g_2(t) \in L_\infty(0,\infty)$ $\left(0<\beta<\frac{1}{2}\right)$ are satisfied, then the boundary value problem (3) – (5) has a solution $u(r,t) \in L_\infty(G)$.

Proof. From the integral representation (10) of the boundary value problem (3)–(5) we have

$$u(r,t) = \sum_{i=1}^{3} u_i(r,t) + \widetilde{g_1}(r,t),$$

where

$$u_1(r,t) = \int_0^t \frac{r^\beta \tau^{1-\beta}(r-\tau)}{4a^4(t-\tau)^2}\cdot e^{-\frac{(r-\tau)^2}{4a^2(t-\tau)}}\cdot e^{-\frac{r\tau}{2a^2(t-\tau)}}\cdot I_\beta\left(\frac{r\tau}{2a^2(t-\tau)}\right)\mu(\tau)d\tau,$$

$$u_2(r,t) = \int_0^t \frac{r^{\beta+1}\tau^{1-\beta}}{4a^4(t-\tau)^2}\cdot e^{-\frac{(r-\tau)^2}{4a^2(t-\tau)}}\cdot e^{-\frac{r\tau}{2a^2(t-\tau)}}\cdot I_{\beta-1,\beta}\left(\frac{r\tau}{2a^2(t-\tau)}\right)\mu(\tau)d\tau,$$

$$u_3(r,t) = \int_0^t \frac{r^\beta(1-2\beta)}{2a^2(t-\tau)\tau^\beta}\cdot e^{-\frac{(r-\tau)^2}{4a^2(t-\tau)}}\cdot e^{-\frac{r\tau}{2a^2(t-\tau)}}\cdot I_\beta\left(\frac{r\tau}{2a^2(t-\tau)}\right)\mu(\tau)d\tau,$$

$$\widetilde{g_1}(r,t) = \frac{1}{(2a^2)^\beta}\cdot\frac{1}{2^\beta}\cdot\frac{1}{\Gamma(\beta)}\int_0^t \frac{r^{2\beta}}{(t-\tau)^{\beta+1}}\cdot\exp\left[-\frac{r^2}{4a^2(t-\tau)}\right]\cdot g_1(\tau)d\tau.$$

Let $t^{-\alpha}e^{\frac{t}{4a^2}}\cdot\mu(t)\in L_\infty(0,\infty)$. Let us find out for what values of α the solution of the problem $u(r,t)$ will satisfy the condition $u(r,t)\in L_\infty(G)$. First, we estimate the first term.

$$u_1(r,t)=\int_0^t\frac{r^\beta\tau^{1-\beta}(r-\tau)}{4a^4(t-\tau)^2}\exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right]\exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right]I_\beta\left(\frac{r\tau}{2a^2(t-\tau)}\right)\mu(\tau)d\tau\leq$$

$$=\left\|\begin{array}{c}\exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right]\leq\exp\left[-\frac{t-\tau}{4a^2}\right],\\ \frac{r\tau}{2a^2(t-\tau)}=\xi\end{array}\right\|=$$

$$=C_1e^{-\frac{t}{4a^2}}\cdot\frac{r^\beta t^{1-\beta+\alpha}}{(2a^2)^2}\int_0^\infty\frac{\xi^{1-\beta+\alpha}}{\left(\frac{r}{2a^2}+\xi\right)^{2-\beta+\alpha}}\cdot e^{-\xi}I_\beta(\xi)d\xi\leq$$

$$\leq C_1e^{-\frac{t}{4a^2}}\cdot\frac{r^\beta t^{1-\beta+\alpha}}{(2a^2)^2}\int_0^\infty\frac{1}{\xi}\cdot e^{-\xi}I_\beta(\xi)d\xi=e^{-\frac{t}{4a^2}}\cdot\frac{r^\beta t^{1-\beta+\alpha}}{(2a^2)^2}\cdot\Gamma\left[\begin{array}{cc}\beta, & \frac{1}{2}\\ 1+\beta & \end{array}\right]=$$

$$=C_1e^{-\frac{t}{4a^2}}\cdot\frac{\sqrt{\pi}t^{1+\alpha}}{(2a^2)^2\beta}\cdot\left(\frac{r}{t}\right)^\beta\leq\frac{C_1\sqrt{\pi}}{(2a^2)^2\beta}\cdot e^{-\frac{t}{4a^2}}\cdot t^{1+\alpha}\leq\widetilde{C_1}=\text{const}\quad\forall(r,t)\in Q.$$

Now we estimate the second term.

$$u_2(r,t)=\int_0^t\frac{r^{\beta+1}\tau^{1-\beta}}{4a^4(t-\tau)^2}\cdot e^{-\frac{(r-\tau)^2}{4a^2(t-\tau)}}\cdot e^{-\frac{r\tau}{2a^2(t-\tau)}}\cdot I_{\beta-1,\beta}\left(\frac{r\tau}{2a^2(t-\tau)}\right)\mu(\tau)d\tau=$$

$$=\left\|\begin{array}{c}\exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right]\leq\exp\left[-\frac{t-\tau}{4a^2}\right],\\ \frac{r\tau}{2a^2(t-\tau)}=\xi\end{array}\right\|\leq$$

$$\leq C_2e^{-\frac{t}{4a^2}}\cdot\frac{r^\beta t^{\alpha-\beta}}{2a^2}\int_0^\infty\frac{\xi^{1-\beta+\alpha}}{\left(\frac{r}{2a^2}+\xi\right)^{1-\beta+\alpha}}\cdot e^{-\xi}I_{\beta-1,\beta}(\xi)d\xi\leq$$

$$\leq C_2e^{-\frac{t}{4a^2}}\cdot\frac{r^\beta t^{\alpha-\beta}}{2a^2}\int_0^\infty e^{-\xi}I_{\beta-1,\beta}(\xi)d\xi\leq C_2e^{-\frac{t}{4a^2}}\cdot\frac{t^\alpha}{2a^2}\leq\widetilde{C_2}=\text{const},\quad\forall(r,t)\in Q.$$

And, finally, we estimate the third term.

$$u_3(r,t)=\int_0^t\frac{r^\beta(1-2\beta)}{2a^2(t-\tau)\tau^\beta}\cdot e^{-\frac{(r-\tau)^2}{4a^2(t-\tau)}}\cdot e^{-\frac{r\tau}{2a^2(t-\tau)}}\cdot I_\beta\left(\frac{r\tau}{2a^2(t-\tau)}\right)\mu(\tau)d\tau=$$

$$=\left\|\begin{array}{c}\exp\left[-\frac{(r-\tau)^2}{4a^2(t-\tau)}\right]\leq\exp\left[-\frac{t-\tau}{4a^2}\right],\\ \frac{r\tau}{2a^2(t-\tau)}=\xi\end{array}\right\|\leq$$

$$\leq C_3e^{-\frac{t}{4a^2}}\cdot\frac{r^\beta(1-2\beta)t^{\alpha-\beta}}{2a^2}\int_0^\infty\frac{\xi^{\alpha-\beta}}{\left(\frac{r}{2a^2}+\xi\right)^{1+\alpha-\beta}}\cdot e^{-\xi}I_\beta(\xi)\,d\xi\leq$$

$$\leq C_3e^{-\frac{t}{4a^2}}\cdot\frac{r^\beta(1-2\beta)t^{\alpha-\beta}}{2a^2}\int_0^\infty\frac{1}{\xi}\cdot e^{-\xi}I_\beta(\xi)d\xi=C_3e^{-\frac{t}{4a^2}}\cdot\frac{r^\beta(1-2\beta)t^{\alpha-\beta}}{2a^2}\cdot\Gamma\left[\begin{array}{cc}\beta, & \frac{1}{2}\\ 1+\beta & \end{array}\right]=$$

$$=C_3e^{-\frac{t}{4a^2}}\cdot\frac{(1-2\beta)\sqrt{\pi}t^\alpha}{2a^2\beta}\cdot\left(\frac{r}{t}\right)^\beta\leq C_3\frac{(1-2\beta)\sqrt{\pi}}{2a^2\beta}\cdot e^{-\frac{t}{4a^2}}\cdot t^\alpha\leq\widetilde{C_3}=\text{const},\quad\forall(r,t)\in Q.$$

Hence it is clear that for $\alpha\geq 0$ the solution of the problem $u(r,t)\in L_\infty(G)$.

The estimate for the fourth term follows from Remark 1. This implies the validity of the main result, Theorem 2.

The results of this work will be used in solving a similar problem in a funnel-shaped degenerate domain, that is, when the boundary of the domain changes according to the law $r=\gamma(t)$, $\gamma(0)=0$.

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References

1 Kavokin A.A. Application of thermal potentials to the solution of the problem of heat conduction in a region degenerates at the initial moment / A.A. Kavokin, A.T. Kulakhmetova, Y.R. Shpadi // Filomat. — 2018. — 32. — No. 3. — P. 825–836. https://doi.org/10.2298/FIL1803825K

2 Amangaliyeva M.M. Boundary value problems for a spectrally loaded heat operator with load line approaching the time axis at zero or infinity / M.M. Amangaliyeva, D.M. Akhmanova, M.T. Dzhenaliev, M.I. Ramazanov // Differential Equations. — 2011. — 47. — No. 2. — P. 231–243. https://doi.org/10.1134/S0012266111020091

3 Jenaliyev M. On a Volterra equation of the second kind with "incompressible" kernel / M. Jenaliyev, M. Amangaliyeva, M. Kosmakova, M. Ramazanov // Advances in Difference Equations. — 2015. — 71. — P. 1–14. https://doi.org/10.1186/s13662-015-0418-6

4 Amangaliyeva M.M. On one homogeneous problem for the heat equation in an infinite angular domain / M.M. Amangaliyeva, M.T. Jenaliyev, M.T. Kosmakova, M.I. Ramazanov // Sib. Math. Jour. — 2015. — 56. — No. 6. — P. 982–995. https://doi.org/10.1134/S0037446615060038

5 Jenaliyev M. On a homogeneous parabolic problem in an infinite corner domain / M. Jenaliyev, M. Ramazanov // Filomat. — 2018. — 32. — No. 3. — P. 965–974. https://doi.org/10.2298/FIL1803965J

6 Рамазанов М.И. О сингулярном интегральном уравнении Вольтерра краевой задачи теплопроводности в вырождающейся области / М.И. Рамазанов, Н.К. Гульманов // Вестн. Удмурт. ун-та. Математика. Механика. Компьютерные науки. — 2021. — 31. — № 2. — С. 241–252. https://doi.org/10.35634/vm210206

7 Ramazanov M.I. Solution of the boundary value problem of heat conduction in a cone / M.I. Ramazanov, M.T. Jenaliyev, N.K. Gulmanov // Opuscula Mathematica. — 2022. — 42. — No. 1. — P. 75–91. https://doi.org/10.7494/OpMath.2022.42.1.75

8 Ramazanov M.I. Solution of a two-dimensional boundary value problem of heat conduction in a degenerating domain / M.I. Ramazanov, N.K. Gulmanov // Journal of Mathematics, Mechanics and Computer Science. — 2021. — 111. — No. 3. — P. 65–78. https://doi.org/10.26577/JMMCS.2021.v111.i3.06

9 Pskhu A.V. Boundary value problem for fractional diffusion equation in a curvilinear angle domain / A.V. Pskhu, M.I. Ramazanov, N.K. Gulmanov, S.A. Iskakov // Bulletin of the Karaganda university. Mathematics series. — 2022. — No. 1(105). — P. 83–95. https://doi.org/10.31489/2022M1/83-95

10 Jenaliyev M.T. To the solution of one pseudo-Volterra integral equation / M.T. Jenaliyev, M.I. Ramazanov, M.T. Kosmakova, A.O. Tanin // Bulletin of the Karaganda university. Mathematics series. — 2019. — No. 1(93). — P. 19–30. https://doi.org/10.31489/2019M1/19-30

11 Jenaliyev M.T. On the Solvability of the Burgers Equation with Dynamic Boundary Conditions in a Degenerating Domain / M.T. Jenaliyev, A.A. Assetov, M.G. Yergaliyev // Lobachevskii Journal of Mathematics. — 2021. — 42. — P. 3661–3674. https://doi.org/10.1134/S199508022203012X

12 Kosmakova M.T. To solving the fractionally loaded heat equation / M.T. Kosmakova, S.A. Iskakov, L.Zh. Kasymova // Bulletin of the Karaganda university. Mathematics series. — 2021. — No. 1(101). — P. 65–77. https://doi.org/10.31489/2021M1/65-77

13 Zarifzoda S.K. Volterra-Type Integro-Differential Equations with Two-Point Singular Differential Operator / S.K. Zarifzoda, T.K. Yuldashev, I.Djumakhon // Lobachevskii Journal of Mathematics. — 2021. — 42. — P. 3784–3792. https://doi.org/10.1134/S1995080222030234

14 Yumagulov M.G. First Approximation Formulas in the Problem of Perturbation of Definite and Indefinite Multipliers of Linear Hamiltonian Systems / M.G. Yumagulov, L.S. Ibragimova, A.S. Belova // Lobachevskii Journal of Mathematics. — 2021. — 42. — P. 3773–3783. https://doi.org/10.1134/S1995080222030222

15 Gradshteyn I.S. Table of Integrals, Series, and Products / I.S. Gradshteyn, I.M. Ryzhik. — Academic Press, 2014. — 1220 p.

16 Прудников А.П. Интегралы и ряды / А.П. Прудников, Ю.А. Брычков, О.И. Маричев. – М., 2003. — Т. 2. — 688 с.

17 Гахов Ф.Д. Уравнения типа свертки / Ф.Д. Гахов, Ю.И. Черский. — М.: Наука, 1978. — 296 с.

18 Polyanin A.D. Handbook of integral equations / A.D. Polyanin, A.V. Manzhirov. — Boca Raton: CRC Press, 2008. — 796 p.

19 Диткин В.А. Справочник по операционному исчислению / В.А. Диткин, А.П. Прудников. — М.: Высш. шк., 1965. — 468 с.

20 Лаврентьев М.А. Методы теории функций комплексного переменного / М.А. Лаврентьев, Б.В. Шабат.— 4-е изд., перераб. и доп. — М.: Наука, 1973. — 749 с.

21 Ватсон Дж.Н. Теория бесселевых функций / Дж.Н. Ватсон; пер. с англ. — М.: Изд-во иностранной литературы, 1949. — 800 с.

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Бұрыштық жойылатын облыста модельдік екі өлшемді параболалық есепті шешу

Жұмыста шекарасы уақыттың өзгеруімен қозғалатын жылу өткізгіштіктің шекаралық есебі зерттелген, сонымен қатар есепті шешу облысы уақыттың бастапқы сәтінде болмайды, яғни нүктеге айналады. Берілген есепті шешу үшін жылу потенциалдары әдісі қолданылған, бұл оны екінші ретті Вольтерра типті сингулярлық интегралдық теңдеуге түрлендіруге мүмкіндік береді. Алынған интегралдық теңдеудің ерекшелігі — ол классикалық Вольтерра интегралдық теңдеулерінен түбегейлі ерекшеленеді, өйткені оған Пикар әдісі қолданылмайды және сәйкес біртекті интегралдық теңдеудің нөлдік емес шешімі бар.

Кілт сөздер: жылу өткізгіштік теңдеуі, шекаралық есеп, жойылатын облыс, Вольтерраның сингулярлық интегралдық теңдеуі, регуляризация.

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Решение модельной двумерной параболической задачи в угловой вырождающейся области

В работе исследована краевая задача теплопроводности в области, граница которой преобразуется с изменением времени, а также область решения задачи отсутствует в начальный момент времени, то есть вырождается в точку. Для решения поставленной задачи использован метод тепловых потенциалов, что позволяет редуцировать ее к сингулярному интегральному уравнению типа Вольтерра второго рода. Особенность полученного интегрального уравнения заключается в том, что оно принципиально отличается от классических интегральных уравнений Вольтерра, так как к нему неприменим метод Пикара и соответствующее однородное интегральное уравнение имеет ненулевое решение.

Ключевые слова: уравнение теплопроводности, краевая задача, вырождающаяся область, сингулярное интегральное уравнение Вольтерра, регуляризация.

References

1 Kavokin, A.A., Kulakhmetova, A.T., & Shpadi, Y.R. (2018). Application of thermal potentials to the solution of the problem of heat conduction in a region degenerates at the initial moment. *Filomat, 32*(3), 825–836. https://doi.org/10.2298/FIL1803825K

2 Amangaliyeva, M.M., Akhmanova, D.M., Dzhenaliev M.T., & Ramazanov, M.I. (2011). Boundary value problems for a spectrally loaded heat operator with load line approaching the time axis at zero or infinity. *Differential Equations, 47*(2), 231–243. https://doi.org/10.1134/S0012266111020091

3 Jenaliyev, M., Amangaliyeva, M., Kosmakova, M., & Ramazanov, M. (2015). On a Volterra equation of the second kind with "incompressible" kernel. *Advances in Difference Equations, 71*, 1–14. https://doi.org/10.1186/s13662-015-0418-6

4 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2015). On one homogeneous problem for the heat equation in an infinite angular domain. *Sib. Math. Jour., 56*(6), 982–995. https://doi.org/10.1134/S0037446615060038

5 Jenaliyev, M., & Ramazanov, M.(2018). On a homogeneous parabolic problem in an infinite corner domain. *Filomat, 32*(3), 965–974. https://doi.org/10.2298/FIL1803965J

6 Ramazanov, M.I., & Gulmanov, N.K. (2021). O singuliarnom integralnom uravnenii Volterra kraevoi zadachi teploprovodnosti v vyrozhdaiushcheisia oblasti [On the singular Volterra integral equation of the boundary value problem for heat conduction in a degenerating domain]. *Vestnik Udmurtskogo universiteta. Matematika. Mekhanika. Kompiuternye nauki — The Bulletin of Udmurt University. Mathematics. Mechanics. Computer Science, 31*(2), 241–252 [in Russian]. https://doi.org/10.35634/vm210206

7 Ramazanov, M.I., Jenaliyev, M.T., & Gulmanov, N.K. (2022). Solution of the boundary value problem of heat conduction in a cone. *Opuscula Mathematica, 42*(1), 75–91. https://doi.org/10.7494/OpMath.2022.42.1.75

8 Ramazanov, M.I., & Gulmanov, N.K. (2021). Solution of a two-dimensional boundary value problem of heat conduction in a degenerating domain. *Journal of Mathematics, Mechanics and Computer Science, 111*(3), 65–78. https://doi.org/10.26577/JMMCS.2021.v111.i3.06

9 Pskhu, A.V., Ramazanov, M.I., Gulmanov, N.K. & Iskakov, S.A. (2022). Boundary value problem for fractional diffusion equation in a curvilinear angle domain. *Bulletin of the Karaganda university. Mathematics series, 1*(105), 83–95. https://doi.org/10.31489/2022M1/83-95

10 Jenaliyev, M.T., Ramazanov, M.I., Kosmakova, M.T., & Tanin, A.O. (2019). To the solution of one pseudo-Volterra integral equation. *Bulletin of the Karaganda university. Mathematics series, 1*(93), 19–30. https://doi.org/10.31489/2019M1/19-30

11 Jenaliyev, M.T., Assetov, A.A., & Yergaliyev, M.G. (2021). On the Solvability of the Burgers Equation with Dynamic Boundary Conditions in a Degenerating Domain. *Lobachevskii Journal of Mathematics, 42*, 3661–3674. https://doi.org/10.1134/S199508022203012X

12 Kosmakova, M.T., Iskakov, S.A., & Kasymova, L.Zh. (2021). To solving the fractionally loaded heat equation. *Bulletin of the Karaganda university. Mathematics series, 1*(101), 65–77. https://doi.org/10.31489/2021M1/65-77

13 Zarifzoda, S.K., Yuldashev, T.K., & Djumakhon, I. (2021). Volterra-Type Integro-Differential Equations with Two-Point Singular Differential Operator. *Lobachevskii Journal of Mathematics, 42*, 3784–3792. https://doi.org/10.1134/S1995080222030234

14 Yumagulov, M.G., Ibragimova, L.S., & Belova, A.S. (2021). First Approximation Formulas in the Problem of Perturbation of Definite and Indefinite Multipliers of Linear Hamiltonian Systems. *Lobachevskii Journal of Mathematics, 42*, 3773–3783. https://doi.org/10.1134/S1995080222030222

15 Gradshteyn, I.S., & Ryzhik, I.M. (2014). *Table of Integrals, Series, and Products.* Academic Press.

16 Prudnikov, A.P., Brychkov, Yu.A., & Marichev, O.I. (2003). *Integraly i riady [Integrals and series].* Moscow: Fizmatlit [in Russian].

17 Gakhov, F.D., & Chersky, Yu.I. (1978). *Uravneniia tipa svertki [Convolution type equations].* Moscow: Nauka [in Russian].

18 Polyanin, A.D., & Manzhirov, A.V. (2008). *Handbook of integral equations.* Boca Raton: CRC Press.

19 Ditkin, V.A., & Prudnikov, A.P. (1965). *Spravochnik po operatsionnomu ischisleniiu [Handbook of Operational Calculus].* Moscow: Vysshaia shkola [in Russian].

20 Lavrentev, M.A., & Shabat, B.V. (1973). *Metody teorii funktsii kompleksnogo peremennogo [Methods of the theory of functions of a complex variable].* Moscow: Nauka [in Russian].

21 Watson, Dzh.N. (1949). *Teoriia besselevykh funktsii [Theory of Bessel functions].* Moscow: Izdatelstvo inostrannoi literatury [in Russian].