DOI 10.31489/2023M3/81-90
UDC 510.67

N.D. Markhabatov*

L.N. Gumilyov Eurasian National University, Astana, Kazakhstan;
Kazakh-British Technical University Almaty, Kazakhstan
(E-mail: markhabatov@gmail.com)

Ranks and approximations for families of cubic theories

In this paper, we study the rank characteristics for families of cubic theories, as well as new properties of cubic theories as pseudofiniteness and smooth approximability. It is proved that in the family of cubic theories, any theory is a theory of finite structure or is approximated by theories of finite structures. The property of pseudofiniteness or smoothly approximability allows one to investigate finite objects instead of complex infinite ones, or vice versa, to produce more complex ones from simple structures.

Keywords: approximation of a theory, cube, cubic structure, cubic theory, pseudofinite theory, smoothly approximated structure.

1 Introduction

Modern mathematical models, which are large relational structures (random graphs) and at the same time time-dependent dynamic models, such as the growth of the Internet, social networks and computer security, cannot be described and explored by infinite models in standard graph theory. However, if a set of models is algorithmically well defined, then these sets exhibit general patterns that are inherent in «almost all» models in the community. These general laws for well-defined systems can be investigated using statistical and model-theoretic methods. From a model-theoretic point of view, one can approach approximations [1], definability [2], and interpretability [3].

The ranks and degrees for families of complete theories [4], similar to the Morley rank and degree for a fixed theory, and the Cantor-Bendixson rank and degree, were introduced by S. Sudoplatov. The problem arises of describing ranks and degrees for natural theory families. Ranks and degrees for families of incomplete theories are examined in [5, 6], for families of permutation theories - in [7], and for families of all theories of arbitrary languages - in [8].

The [1] examines approximations of theories both in the general context and in relation to specific natural theory families. The problem of describing the approximation forms of the natural theory families arises.

This work is devoted to the description of the ranks and degrees of families of cubic theories, as well as approximation by theories of finite cubic structures. Pseudofinite structures are mathematical structures that resemble finite structures but are not actually finite. They are important in various areas of mathematics, including model theory and algebraic geometry. Further study of pseudofinite structures will continue to reveal new insights and applications in mathematics and beyond.

1.1 Preliminaries from cubic theories

Cubic structures are defined in [9], theoretical properties of the model are discussed and included in the monograph [10], applications in discrete mathematics are presented [11]. The following necessary terminology for cubic structures was taken from [9, 11] without specifying it.

*Corresponding author.
E-mail: markhabatov@gmail.com

Definition 1. An *n-dimensional cube* or an *n-cube* (where $n \in \omega$) is a graph isomorphic to the graph $\mathcal{Q}_n$ with universe $\{0,\ 1\}^n$ and such that any two vertices $(\delta_1, \ldots,\ \delta_n)$ and $(\delta'_1, \ldots,\ \delta'_n)$ are adjacent if and only if these vertices differ by exactly one coordinate.

Let λ be an infinite cardinal number. A λ-*dimensional cube* or a λ-*cube* is a graph isomorphic to a graph $\Gamma = \langle X; R\rangle$ that satisfies the following conditions:

(1) the universe $X \subseteq \{0;\ 1\}^\lambda$ is generated from an arbitrary function $f \in\ X$ by the operator $\langle f\rangle$ attaching, to the set $\{f\}$, all results of substitutions for any finite tuples $(f\,(i_1)\,, \ldots,\ f(i_m))$ by tuples $(1 - f\,(i_1)\,, \ldots, 1 - f(i_m))$;

(2) the relation R consists of edges connecting functions differing exactly in one coordinate.

The described graph $\mathcal{Q} \rightleftarrows \mathcal{Q}_f$ with the universe $\langle f\rangle$ is a canonical representative for the class of λ-cubes.

Note that the canonical representative of the class of n-cubes (as well as the canonical representatives of the class of λ-cubes) are generated by any its function: $\{0,1\}^n = \langle f\rangle$, where $f \in \{0,1\}^n$. Therefore the universes of canonical representatives $\mathcal{Q}_f$ of n-cubes like λ-cubes, will be denoted by $\langle f\rangle$.

Any graph $\Gamma =\ \langle X;\ R\rangle$, where any connected component is a cube, is called a *cubic structure.* A theory T of the graph language $\{R^{(2)}\}$ is *cubic* if $T\ =\ Th(\mathcal{M})$ for some cubic structure $\mathcal{M}$. In this case, the structure $\mathcal{M}$ is called a *cubic model* of T.

The *invariant* of a theory T is the function

$$Inv_T : \omega \cup \{\infty\} \to \omega \cup \{\infty\},$$

satisfying the following conditions:

(1) for any natural *n;* $Inv_T(n)$ is the number of connected components in any model of T, being n-cubes, if that number is finite, and $Inv_T(n)\ = \infty$ if that number is infinite;

(2) $Inv_T(\infty) =\ 0$ if models of T do not contain infinite-dimensional cubes (i. e., the dimensions of cubes are totally bounded), otherwise we set $Inv_T(\infty) = 1$.

The *diameter* $d(T)$ of a cubic theory T is the maximal distance between elements in models of T, if these distances are bounded, and we set $d(T) \rightleftharpoons \infty$ otherwise. The *support* (accordingly the ∞-*support*) $Supp(T)(Supp_\infty(T))$ of a theory T is the set $\{n \in \omega | Inv_T\,(n) \neq\ 0\}(\{n \in \omega | Inv_T(n)\ = \infty\})$.

If the diameter $d(T)$ is finite then there exists an upper estimate for dimensions of cubes, being in models of T. It means that $Supp(T)$ is finite, i. e., $Inv_T(\infty) =\ 0$. In this case the ∞-support is non-empty.

If $d(T) = \infty$ then $Inv_T(\infty) = 1$. In this case the support $Supp(T)$ can be either finite or infinite.

1.2 Preliminaries from model theory and approximations of theories

Historically, pseudofinite fields were first introduced by J. Ax and S. Kochen [12] in the form of non-principal ultraproducts of finite fields. Later, J. Ax in [13] connected the notion of pseudofiniteness and the construction of ultraproducts. The class of *pseudofinite fields* was defined in the work of J. Ax [13] and regardless of him in the work of Yu. Ershov [14] with an axiom system indicating this class.

In 1965 J. Ax [15] investigated fields F having the property that *every absolutely irreducible variety over F has an F-rational point.* It was shown that the non-principal ultraproduct of finite fields has such property. Yu. Ershov called such fields *regularly closed.* The notion of *pseudofiniteness* is credited to work in the 1968s by J. Ax [13]. He introduced the notion of *pseudofiniteness* to show the decidability of the theory of all finite fields, i.e. there is an algorithm to decide whether a given statement is true for all finite fields. It was proved that pseudofinite fields are exactly those infinite fields that have every elementary property common to all finite fields, that is, pseudofinite fields are infinite models of the theory of finite fields.

In the early 1990s, E. Hrushovski resumed research in the field of pseudofinite structures in meeting on Finite and Infinite Combinatorics in Sets and Logic [16], as well as in the joint works of E. Hrushovski and G. Cherlin and the following definition first occurs in [17], subsequently in [18]:

Definition 2. Let Σ be a language and $\mathcal{M}$ be a Σ-structure. A Σ-structure $\mathcal{M}$ is *pseudofinite* if for each Σ-sentences φ, $\mathcal{M} \models \varphi$ implies that there is a finite $\mathcal{M}_0$ such that $\mathcal{M}_0 \models \varphi$. The theory $T = Th(\mathcal{M})$ of a pseudofinite structure $\mathcal{M}$ is called pseudofinite.

In the work [1] S. Sudoplatov defined approximations relative given family $\mathcal{T}$ of complete theories.

Definition 3. [1] Let $\mathcal{T}$ be a family of theories and T be a theory such that $T \notin \mathcal{T}$. The theory T is said to be *$\mathcal{T}$-approximated,* or *approximated by the family* $\mathcal{T}$, or a *pseudo-$\mathcal{T}$-theory,* if for any formula $\varphi \in T$ there exists $T' \in \mathcal{T}$ for which $\varphi \in T'$.

If a theory T is $\mathcal{T}$-approximated, then $\mathcal{T}$ is said to be an *approximating family* for T, and theories $T' \in \mathcal{T}$ are said to be *approximations* for T. We put $\mathcal{T}_\varphi = \{T \in \mathcal{T} \mid \varphi \in T\}$. Any set $\mathcal{T}_\varphi$ is called the *φ-neighbourhood*, or simply a *neighbourhood*, for $\mathcal{T}$. A family $\mathcal{T}$ is called *e-minimal* if for any sentence $\varphi \in \Sigma(\mathcal{T})$, $\mathcal{T}_\varphi$ is finite or $\mathcal{T}_{\neg\varphi}$ is finite.

Recall that the *E-closure* for a family $\mathcal{T}$ of complete theories is characterized by the following proposition.

Proposition 1. [19] *Let $\mathcal{T}$ be a family of complete theories of the language Σ. Then $Cl_E(\mathcal{T}) = \mathcal{T}$ for a finite $\mathcal{T}$, and for an infinite $\mathcal{T}$, a theory T belongs to $Cl_E(\mathcal{T})$ if and only if T is a complete theory of the language Σ and $T \in \mathcal{T}$, or $T \neq \mathcal{T}$ and for any formula φ the set $\mathcal{T}_\varphi$ is infinite.*

We denote by $\overline{\mathcal{T}}$ the class of all complete theories of relational languages, by $\overline{\mathcal{T}}_{fin}$ the subclass of $\overline{\mathcal{T}}$ consisting of all theories with finite models, and by $\overline{\mathcal{T}}_{inf}$ the class $\overline{\mathcal{T}} \setminus \overline{\mathcal{T}}_{fin}$.

Proposition 2. [1] *For any theory T the following conditions are equivalent:*

(1) T is pseudofinite;

(2) T is $\overline{\mathcal{T}}_{fin}$-approximated;

(3) $T \in Cl_E(\overline{\mathcal{T}}_{fin}) \setminus \overline{\mathcal{T}}_{fin}$.

1.3 Preliminaries from ranks for families of theories

In [4], *rank* RS$(\cdot)$ is defined inductively for families of complete theories.

(1) The empty family $\mathcal{T}$ is assigned the rank $\mathrm{RS}(\mathcal{T}) = -1$.

(2) For finite nonempty families $\mathcal{T}$ set $\mathrm{RS}(\mathcal{T}) = 0$.

(3) For infinite families $\mathcal{T}$ we set $\mathrm{RS}(\mathcal{T}) \geq 1$.

(4) For the family $\mathcal{T}$ and the ordinal number we set $\alpha = \beta + 1$ $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if there are pairwise inconsistent $\Sigma(\mathcal{T})$ sets of φ_n, $n \in \omega$ such that $\mathrm{RS}(\mathcal{T}_{\varphi_n}) \geq \beta$, $n \in \omega$.

(5) If α is a limit ordinal, then $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \beta$ for each $\beta < \alpha$.

(6) Let $\mathrm{RS}(\mathcal{T}) = \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \alpha$ and $\mathrm{RS}(\mathcal{T}) \not\geq \alpha + 1$.

(7) If $\mathrm{RS}(\mathcal{T}) \geq \alpha$ for any α, we set $\mathrm{RS}(\mathcal{T}) = \infty$.

A family $\mathcal{T}$ is called *e-totally transcendental*, or *totally transcendental*, if $\mathrm{RS}(\mathcal{T})$ is an ordinal.

If $\mathcal{T}$ is e-totally transcendental, with $\mathrm{RS}(\mathcal{T}) = \alpha \geq 0$, we define the *degree* $\mathrm{ds}(\mathcal{T})$ of $\mathcal{T}$ as the maximal number of pairwise inconsistent sentences φ_i such that $\mathrm{RS}(\mathcal{T}_{\varphi_i}) = \alpha$.

Proposition 3. [4] *$\mathcal{T}$ is e-minimal $\Leftrightarrow$ $RS(\mathcal{T}) = 1$ and $ds(\mathcal{T}) = 1$*

Definition 4. [4] A family $\mathcal{T}$, with infinitely many accumulation points, is called *a-minimal* if for any sentence $\varphi \in \Sigma(T)$, $\mathcal{T}_\varphi$ or $\mathcal{T}_{\neg\varphi}$ has finitely many accumulation points.

Let α be an ordinal. A family $\mathcal{T}$ of rank α is called *α-minimal* if for any sentence $\varphi \in \Sigma(T)$, $\mathrm{RS}(\mathcal{T}_\varphi) < \alpha$ or $\mathrm{RS}(\mathcal{T}_{\neg\varphi}) < \alpha$.

Proposition 4. [4] (1) *A family $\mathcal{T}$ is* 0*-minimal $\Leftrightarrow$ $\mathcal{T}$ is a singleton.*

(2) *A family* $\mathcal{T}$ *is* 1*-minimal* $\Leftrightarrow$ $\mathcal{T}$ *is e-minimal.*
(3) *A family* $\mathcal{T}$ *is* 2*-minimal* $\Leftrightarrow$ $\mathcal{T}$ *is a-minimal.*
(4) *For any ordinal* α *a family* $\mathcal{T}$ *is* α*-minimal* $\Leftrightarrow$ $\mathrm{RS}(\mathcal{T}) = \alpha$ *and* $\mathrm{ds}(\mathcal{T}) = 1$.

2 *Ranks for families of cubic theories*

Consider a language Σ composed of $R^{(2)}$. Let $\mathcal{T}_{cub}$ be the family of all cubic theories of Σ. Let T be a cubic theory and $\mathcal{Q} \models T$. For a cubic theory T we consider the above invariants and the following possibilities:

2.1 *Family of cubic theories with a bounded number of* $Inv_T(n)$

If for each theory T from the subfamily $\mathcal{T} \subset \mathcal{T}_{cub}$ both diameters $d(T)$ and $Inv_T(n)$ are finite, and also $Inv_T(\infty) = 0$ or $Supp(T)$ is finite, the subfamily $\mathcal{T}$ is finite, so $RS(\mathcal{T}) = 0$, and the degree of $ds(\mathcal{T})$ is equal to the number of invariants. Let's illustrate how the grades of families differ.

Example 1. Now we consider a one-element family $\mathcal{T} = \{T_1\}$. If we consider n_0-cubes with invariant $Inv_{T_1}(n_0) = m$, then $RS(\mathcal{T}) = 0$, $ds(\mathcal{T}) = 1$. And if we work with n_0-cubes and n_1-cubes with $Inv_{T_1}(n_0) = m$ and $Inv_{T_1}(n_1) = l$ for $m \neq l$, then $ds(\mathcal{T}) = 2$. For a finite number k, if we are dealing with n_k-cubes with the set of invariants $\{Inv_{T_1}(n_0), \ldots, Inv_{T_1}(n_k)\}$, $n_i \neq n_j$, we still have $RS(\mathcal{T}) = 0$ and degree $ds(\mathcal{T}) = k + 1$.

Example 2. Let us deal with the finite family $\mathcal{T} \subset \mathcal{T}_{cub}$ consisting of theories $T_1, \ldots, T_n$. If the number of m_i-cubes in each theory T_i is equal to k, in other words, each theory has the same number of m_i-cubes, that is, $Inv_{T_i}(m_i) = k$ with $Inv_{T_i}(m_i) \neq Inv_{T_j}(m_j)$, $i \neq j$, then $RS(\mathcal{T}) = 0$, $ds(\mathcal{T}) = n$, since $\mathcal{T}$ is represented as a disjoint union of finite subfamilies $\mathcal{T}_{\varphi_i} = \{T_i \in \mathcal{T} | \varphi_i \in T_i$ is a sentence describing m_i-cubes $\}$.

In the examples above, one can notice that the degree of the family depends on the number of invariants. If for the theories considered in Example 2 we add the conditions that each theory has the same number of invariants, let, for example, s, then $ds(\mathcal{T}) = n \cdot s$. And if for different $s_1, \ldots, s_n$, in each theory T_i there are s_i invariants, then $ds(\mathcal{T}) = \sum_{i=1}^{n} s_i$.

For a family $\mathcal{T} \subset \mathcal{T}_{cub}$ such that $Inv_T(\infty) = 0$ and $Supp(T)$ is finite for every theory $T \in \mathcal{T}$, the degree varies in a similar way.

Let us now consider infinite subfamilies $\mathcal{T} \subset \mathcal{T}_{cub}$ of all cubic theories with a bounded number of $Inv_T(n) = \infty$ and $Inv_T(\infty) = 0$ for every $T \in \mathcal{T}$. In this case, $Supp(T)$ is infinite and the rank of the family increases, and for the degree of the family, we consider the number of accumulation points.

For natural numbers $n, m \in \omega$, with $n \neq m$, we denote by $\mathcal{T}_n$ the family of cubic theories from $\mathcal{T}_{cub}$ with one arbitrary value $Inv_T(n)$, where $T \in \mathcal{T}_n$ and $Inv_T(m) = 0$.

Proposition 5. Each subfamily $\mathcal{T}_n$ *of* $\mathcal{T}_{cub}$ *is e-minimal.*

Proof. By Proposition 3, it suffices to prove that $RS(\mathcal{T}_n) = 1$ and $ds(\mathcal{T}_n) = 1$. The family $\mathcal{T}_n$ consists of theories $T_1, \ldots, T_s$ with $Inv_{T_i}(n) = k_i$, $k_i > 0$ $1 \leq i \leq s$ and the only theory T_∞ with $Inv_{T_\infty}(n) = \infty$. The theory T_∞ is the only accumulation point for $\mathcal{T}_n$, and the number of accumulation points is equal to the degree of the family. We get $RS(\mathcal{T}_n) = 1$ and $ds(\mathcal{T}_n) = 1$, which implies an e-minimality of $\mathcal{T}_n$.

Example 3. We are dealing with cubes of different sizes n_0 and n_1. Then we get a countable number of options $(Inv_T(n_0), Inv_T(n_1))$. Thus there is a countable set of theories with n_0-cubes and n_1-cubes forming the family $\mathcal{T}'$. Here every family with an infinite $Inv_T(n_0)$ or $Inv_T(n_1)$ has $RS = 1$, and the only accumulation point with $Inv_T(n_0) = Inv_T(n_1) = \infty$, has infinitely many n_0 cubes, infinitely many n_1 cubes and $RS(\mathcal{T}') = 2$. Thus for the given family $\mathcal{T}'$ $RS(\mathcal{T}') = 2$ and $ds(\mathcal{T}') = 1$. Hence the family is a-minimal.

Example 4. If there exists a countable number of n_i-cubes, $i \in \omega$, with countable $(Inv_T(n_0), Inv_T(n_1), \ldots, Inv_T(n_i))$ one can construct an α-minimal family $\mathcal{T}$ consisting of a countable number of e-minimal subfamilies $\mathcal{T}_i$, $i \in \omega$. According to the definition of α-minimality, the family $\mathcal{T}$ has $RS(\mathcal{T}) = \alpha$, $ds(\mathcal{T}) = 1$ and is represented as $\mathcal{T}_{\bigwedge_{i\in\omega}\varphi_i}$.

So by increasing the number of $Inv_T(n)$ invariants and the dimension of the cubes, one can unlimited increase rank to any natural number. If the set $Inv_T(n)$ is countable, then the family $\mathcal{T} \subset \mathcal{T}_{cub}$ of cubic theories is e-totally transcendental and can contain e-minimal, a-minimal, α-minimal subfamilies.

Realizations of e-minimal, a-minimal, α-minimal subfamilies of the family $\mathcal{T}_{cub}$ of all cubic theories allow one to construct a subfamily $\mathcal{T}$ with a given countable rank and degree. According to the definition of α-minimality, a family of $\mathcal{T}$ cubic theories with $RS(\mathcal{T}) = \alpha$ and $ds(\mathcal{T}) = n$ can be represented as a disjoint union of subfamilies $\mathcal{T}_{Inv_T(k_0)}, \ldots, \mathcal{T}_{Inv_T(k_{n-1})}$, somewhat differently $Inv_T(k_0), \ldots, Inv_T(k_{n-1})$, so every $\mathcal{T}_{Inv_T(k_i)}$ is α-minimal.

2.2 Family of cubic theories with an unbounded number of $Inv_T(n)$

The next result shows that the family $\mathcal{T}_{cub}$ of all cubic theories is not e-totally transcendental.

Theorem 1. $RS(\mathcal{T}_{cub}) = \infty$.

Proof. Repeating the arguments of [1; Proposition 4.4] and [8; Proposition 2.5] we can construct a 2-tree of sentences φ, φ_0, φ_1, φ_{01}, ... indicating an infinite rank.

3 Approximations of cubic theories

The following theorem shows that any cubic theory is approximated by theories of finite cubic structures.

Theorem 2. Any cubic theory T with an infinite model is pseudofinite.

Proof. Let $\mathcal{Q}$ be an infinite model of a cubic theory T. Since for finite k and n, $Inv_T(n) = k$ and $Inv_T(\infty) = 0$, the cubic model $\mathcal{Q}$ is finite and consists of a finite number of finite connected components (n-cubes), we will consider only the following cases:

Case 1. If $Inv_T(n) = \infty$ and $Inv_T(\infty) = 0$ (that is, ∞-support is a singleton), then $\mathcal{Q}$ consists of an infinite number of connected components of finite diameters. The $\mathcal{Q}$ model is approximated by the disjoint union $\bigsqcup_{i\in\omega} \mathcal{Q}_i$ of models $\mathcal{Q}_i$, $i \in \omega$ which the connected components are n-cubes. Each such n-cubes are pairwise isomorphic that implies the pseudofiniteness of T.

Case 2. If for finite k and $n \in \omega$, $Inv_T(n) = k$ and $Inv_T(\infty) = 1$, then the theory T has models $\mathcal{Q} = \mathcal{Q}_0 \bigsqcup \mathcal{Q}_1$, where $\mathcal{Q}_0$ is a finite cubic model consisting of $m \leq k$ connected components (n-cubes) of finite diameters, $\mathcal{Q}_1$ is an infinite cubic model consisting of $k - m$ connected components of infinite diameters. Since the components of the model $\mathcal{Q}_0$ do not affect the pseudofiniteness, $\mathcal{Q}_1$ is approximated by increasing the dimension, as well as the diameters of the connected components. Let $\mathcal{Q}'_n$ be a finite model with $k - m$ connected components which are n-cubes. Using $\mathcal{Q}'_i = \mathcal{Q}'_2 \cup \mathcal{Q}'_{i-1}, i > 2$ in the limit, we obtain the desired model $\mathcal{Q}_1$. The set of theories $\{Th(\mathcal{Q}'_i) | i \in \omega\}$ approximate the theory $Th(\mathcal{Q}_1)$ and theories $\{Th(\mathcal{Q}_0 \bigsqcup \mathcal{Q}'_i) | i \in \omega\}$ approximate the T theory.

We can also grow connected components to get a pseudofinite model $\mathcal{Q}'$ with $Inv_T(n) = \infty$ and $Inv_T(\infty) = 1$, having components of both finite and infinite diameters.

Case 3. Let $Inv_T(n) = \infty$ and $Inv_T(\infty) = 1$. Let the cubic model $\mathcal{Q}$ have only an infinite number of connected components of infinite diameters. For the cubic model $\mathcal{Q}$, it is true that $\mathcal{Q} = \bigsqcup_{i\in\omega} \mathcal{Q}'_i$, where $\mathcal{Q}'_i = \mathcal{Q}'_2 \cup \mathcal{Q}'_{i-1}, i > 2$. That is, first we take the finite model and increase the diameters of the connected components, we get a model with a finite number of connected components, each of which is infinite-dimensional cubes, then, increasing the number of the connected components, we get the desired model $\mathcal{Q}$.

4 Futher direction

Recently, various methods similar to the "transfer principle" have been rapidly developing, where one property of the structure or pieces of this structure is satisfied in all infinite structures or in another algebraic structure. Such methods include smoothly approximable structures, holographic structures, almost sure theory, and pseudofinite structures approximable by finite structures. Pseudofinite structures in an explicit form after J. Ax were not studied for a long time. Until the 1990s, only a few results on this topic were obtained, and the very first result is the result of B.I. Zilber [20] asserting that ω-categorical theory is not finitely axiomatizable. At the time, the property of being pseudofinite was not considered particularly important or interesting, but the proof is based on pseudofiniteness.

One of the first results in the theory of classification of pseudofinite structures is the famous theorem of G. Cherlin, L. Harrington and A. Lachlan [21], which generalizes Zilber's theorem to the class of ω-stable ω-categorical structures, stating that totally categorical theories (and in more generally, ω-categorical ω-stable theories) are pseudofinite. They also proved that such structures are smoothly approximated by finite structures.

Definition 5. [22] Let L be a countable language and let $\mathcal{M}$ be a countable and ω-categorical L-structure. L-structure $\mathcal{M}$ (or $Th(\mathcal{M})$) is said to be *smoothly approximable* if there is an ascending chain of finite substructures $A_0 \subseteq A_1 \subseteq \ldots \subseteq \mathcal{M}$ such that $\bigcup_{i\in\omega} A_i = \mathcal{M}$ and for every i, and for every $\bar{a}, \bar{b} \in A_i$ if $tp_{\mathcal{M}}(\bar{a}) = tp_{\mathcal{M}}(\bar{b})$, then there is an automorphism σ of M such that $\sigma(\bar{a}) = \bar{b}$ and $\sigma(A_i) = A_i$, or equivalently, if it is the union of an ω-chain of finite homogeneous substructures; or equivalently, if any sentence in $Th(\mathcal{M})$ is true of some finite homogeneous substructure of $\mathcal{M}$.

A. Lachlan introduced the concept of *smoothly approximable* structures to change the direction of analysis from finite to infinite, that is, to classify large finite structures that appear to be *smooth approximations* to an infinite limit.

Smoothly approximated structures were first examined in generality in [22], subsequently in [23]. The model theory of smoothly approximable structures has been developed very much further by G. Cherlin and E. Hrushovski [18]. The class of smoothly approximable structures is a class of ω-categorical supersimple structures of finite rank which properly contains the class of ω-categorical ω-stable structures (so in particular the totally categorical structures).

Recall [24,25] that a countable model $\mathcal{Q}$ of a theory T is called a *limit model* if $\mathcal{Q}$ is represented as the union of a countable elementary chain of models of the theory T that are prime over tuples, and the model $\mathcal{Q}$ itself is not prime over any tuple. A theory T is called *l-categorical* if T has a unique (up to isomorphism) limit model.

Homogeneity and l-categoricity, as well as the Morley rank for a fixed cubic theory, are studied in [9,10].

Proposition 6. Any model $\mathcal{Q}$ of the l-categorical cubic theory T is smoothly approximable by finite cubic structures.

Proof. The limit model $\mathcal{Q}$ of l-categorical cubic theories T is represented as an ascending chain of finite prime substructures $\mathcal{Q}'_0 \subseteq \mathcal{Q}'_1 \subseteq \ldots \subseteq \mathcal{Q}$ such that $\mathcal{Q} = \bigcup_{i\in\omega} \mathcal{Q}'_i$ and there is an automorphism σ of $\mathcal{Q}$ such that $\sigma(\mathcal{Q}'_i) = \mathcal{Q}'_i$.

Conclusions

In the paper the ranks and degrees for families of cubic theories are described. Several examples of families of finite rank cubic theories are given. It is proved that any cubic theory with an infinite model is pseudofinite.

Acknowledgments

The study was partially supported by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (AP19674850 (Section 3), AP19677451 (Section 2 and 4)).

References

1 Sudoplatov S.V. Approximations of theories / S.V. Sudoplatov // Siberian Electronic Mathematical Reports. — 2020. — 17. — P. 715–725. https://doi.org/10.33048/semi.2020.17.049

2 Morozov A.S. Minimal Predicates for Δ-Definability / A.S. Morozov, D.A. Tussupov // Algebra & Logic. — 2020. — 59. — No. 4. — P. 328–340.

3 Morozov A.S. On the existential interpretability of structures / A.S. Morozov, A.Zh. Satekbaeva, D.A. Tussupov // Siberian Electronic Mathematical Reports. — 2014. — 11. — P. 557–566.

4 Sudoplatov S.V. Ranks for families of theories and their spectra / S.V. Sudoplatov // Lobachevskii J. Math. — 2021. — 42. — P. 2959–2968 https://doi.org/10.1134/S1995080221120313

5 Markhabatov N.D. Topologies, ranks and closures for families of theories. I / N.D. Markhabatov, S.V. Sudoplatov // Algebra and Logic. — 2020. — 59. — No. 6. — P. 437–455. https://doi.org/10.-1007/s10469-021-09620-4

6 Markhabatov N.D. Topologies, ranks and closures for families of theories. II / N.D. Markhabatov, S.V. Sudoplatov // Algebra and Logic. — 2021. — 60. — No. 1. — P. 38–52. https://doi.org/10.10-07/s10469-021-09626-y

7 Markhabatov N.D. Ranks for families of permutation theories / N.D. Markhabatov // The Bulletin of Irkutsk State University. Series Mathematics. — 2019. — 28. — P. 85–94. https://doi.org /10.26516/1997-7670.2019.28.85

8 Markhabatov N.D. Ranks for families of all theories of given languages / N.D. Markhabatov, S.V. Sudoplatov // Eurasian Mathematical Journal. — 2021. — 12. — No. 2. — C. 52–58. https://doi.org/10.32523/2077-9879-2021-12-2-52-58

9 Sudoplatov S.V. Models of Cubic Theories / S.V. Sudoplatov // Bulletin of the Section of Logic. — 2014. — 43(1-2). — P. 19–34.

10 Судоплатов С.В. Полигонометрии групп: моногр. / С.В. Судоплатов. — 2-е изд. — Новосибирск: Изд-во Новосиб. гос. техн. ун-та, 2013. — 302 с.

11 Судоплатов С.В. Дискретная математика / С.В. Судоплатов, Е.В. Овчинникова. — М.: Юрайт, 2023. —279 с.

12 Ax J. Diophantine Problems Over Local Fields I / J. Ax, S. Kochen // American Journal of Mathematics. — 1965. — 87. — No. 3. — P. 605–630. https://doi.org/10.2307/2373065

13 Ax J. The Elementary Theory of Finite Fields / J. Ax // Annals of Mathematics. — 1968. — 88. — No. 2. — P. 239–271. https://doi.org/10.2307/1970573

14 Ершов Ю.Л. О полях с разрешимой теорией / Ю.Л. Ершов // Докл. АН СССР. — 1967. — 174. — № 1. — С. 19,20.

15 Ax J. Solving diophantine problems modulo every prime / J. Ax // Ann. Math. — 1967. — 85. — No. 2. — P. 161–183. https://doi.org/10.2307/1970438

16 Hrushovski E. Finite Structures with Few Types / E. Hrushovski // In: Sauer N.W., Woodrow R.E., Sands B. (eds) Finite and Infinite Combinatorics in Sets and Logic. NATO ASI Series (Series C: Mathematical and Physical Sciences). — 1993. — Vol. 411. — Dordrecht: Springer. https://doi.org/10.1007/978-94-011-2080-7

17 Cherlin G. Large finite structures with few types / G. Cherlin // In: Hart B.T., Lachlan A.H., Valeriote M.A. (eds) Algebraic Model Theory. NATO ASI Series. — Vol. 496. — Dordrecht: Springer. — 1997. https://doi.org/10.1007/978-94-015-8923-9_3

18 Cherlin G. Finite Structures with Few Types / G. Cherlin, E. Hrushovski // Annals of Mathematics Studies. — Princeton University Press, Princeton. — 2003. — Vol. 152.

19 Sudoplatov S.V. Closures and generating sets related to combinations of structures / S.V. Sudoplatov // The Bulletin of Irkutsk State University. Series Mathematics. — 2016. — 16. — P. 131–144.

20 Зильбер Б. К проблеме конечной аксиоматизируемости теорий, категоричных во всех бесконечных мощностях / Б. Зильбер /под ред. Б. Байжанова. Исследования по теоретическому программированию. — Алма-Ата, 1981. — С. 69–75.

21 Cherlin G. $\aleph_0$-categorical, $\aleph_0$-stable structures / G. Cherlin, L. Harrington, A.H. Lachlan // Ann. Pure Appl. Logic. — 1985. — 28. — P. 103–135. https://doi.org/10.1016/0168-0072(85)90023-5

22 Kantor W.M. $\aleph_0$-categorical structures smoothly approximated by finite substructures / W.M. Kantor, M.W. Liebeck, H.D. Macpherson // Proc. London Math. Soc. — 1989. — 59. — P. 439–463. https://doi.org/10.1112/plms/s3-59.3.439

23 Macpherson D. Homogeneous and Smoothly Approximated Structures / H.D. Macpherson // In: Hart B.T., Lachlan A.H., Valeriote M.A. (eds) Algebraic Model Theory. NATO ASI Series. — 1997. — Vol. 496. — Dordrecht: Springer. https://doi.org/10.1007/978-94-015-8923-9_7

24 Sudoplatov S.V. Complete Theories with Finitely Many Countable Models I / S.V. Sudoplatov // Algebra and Logic. — 2004. — 43. — P.62–69 https://doi.org/10.1023/B:ALLO.0000015131.4121-8.f4

25 Sudoplatov S.V. Complete theories with finitely many countable models II / S.V. Sudoplatov // Algebra and Logic. — 2006. — 45. — P. 180–200 https://doi.org/10.1007/s10469-006-0016-5

Н.Д. Мархабатов

Л.Н. Гумилев атындағы Еуразия ұлттық университеті, Астана, Қазақстан;
Қазақ-Британ техникалық университеті, Алматы, Қазақстан

Кубтық теориялардың үйірлері үшін рангілер мен аппроксимациялар

Жұмыста кубтық теориялар үйірлерінің рангтық сипаттамалары, сонымен қатар псевдоақырлы және тегіс аппроксимациялау сияқты кубтық теориялардың жаңа қасиеттері зерттелген. Кубтық теориялар үйіріндегі кез келген теория ақырлы құрылым теориясы болып табылатыны немесе ақырлы құрылымдардың теорияларымен аппроксимацияланатыны дәлелденді. Псевдоақырлылық немесе тегіс аппроксимациялану қасиеті күрделі шексіз құрылымдардың орнына ақырлы объектілерді зерттеуге немесе керісінше қарапайым құрылымдардан күрделі құрылымдарды тудыруға мүмкіндік береді.

Кілт сөздер: теориялар аппроксимациялары, куб, кубтық құрылым, кубтық теория, псевдоақырлы теория, тегіс аппроксимацияланатын құрылым.

Н.Д. Мархабатов

Евразийский национальный университет имени Л.Н. Гумилева, Астана, Казахстан;
Казахстанско-Британский технический университет, Алматы, Казахстан

Ранги и аппроксимация для семейств кубических теорий

В работе изучены ранговые характеристики семейств кубических теорий, а также новые свойства кубических теорий, такие как псевдоконечность и гладкая аппроксимируемость. Доказано, что в семействе кубических теорий любая теория является теорией конечной структуры или аппроксимируется теориями конечных структур. Свойство псевдоконечности или гладкой аппроксимируемости позволяет исследовать конечные объекты вместо сложных бесконечных или, наоборот, из простых структур производить более сложные.

Ключевые слова: аппроксимация теории, куб, кубическая структура, кубическая теория, псевдоконечная теория, гладко аппроксимируемая структура.

References

1 Sudoplatov, S.V. (2020). Approximations of theories. *Siberian Electronic Mathematical Reports, 17*, 715–725. https://doi.org/10.33048/semi.2020.17.049

2 Morozov, A.S., & Tussupov, D.A. (2020). Minimal Predicates for Δ-Definability. *Algebra and Logic, 59*(4), 328–340. https://doi.org/10.1007/s10469-020-09605-9

3 Morozov, A.S., Satekbayeva, A.Z., & Tusupov, D.A. (2014). On the existential interpretability of structures. *Siberian Electronic Mathematical Reports, 11*, 557–566.

4 Sudoplatov, S.V. (2021). Ranks for families of theories and their spectra. *Lobachevskii Journal of Mathematics, 42*, 2959–2968 https://doi.org/10.1134/S1995080221120313.

5 Markhabatov, N.D., & Sudoplatov, S.V. (2021). Topologies, Ranks, and Closures for Families of Theories. I. *Algebra and Logic, 59*, 437–455. https://doi.org/10.1007/s10469-021-09620-4

6 Markhabatov, N.D., & Sudoplatov, S.V. (2021). Topologies, Ranks, and Closures for Families of Theories. II. *Algebra and Logic, 60*, 38–52. https://doi.org/10.1007/s10469-021-09626-y

7 Markhabatov, N.D. (2019). Ranks for Families of Permutation Theories. *The Bulletin of Irkutsk State University. Series Mathematics, 28*, 85–94. https://doi.org/10.26516/1997-7670.2019.28.85

8 Markhabatov, N.D., & Sudoplatov, S.V. (2021). Ranks for families of all theories of given languages. *Eurasian Mathematical Journal, 12*(2), 52–58. https://doi.org/10.32523/2077-9879-2021-12-2-52-58

9 Sudoplatov, S.V. (2014). Models of Cubic Theories. *Bulletin of the Section of Logic, 43*(1-2), 19–34.

10 Sudoplatov, S.V. (2013). *Poligonometrii grupp: monografiia [Group polygonometries].* Novosibirsk: Izdatelstvo Novosibirskogo gosudarstvennogo techniheskogo universiteta [in Russian].

11 Sudoplatov, S.V., & Ovchinnikova, E.V. (2023) *Diskretnaia matematika [Discrete mathematics].* Moscow: Yurait [in Russian].

12 Ax, J., & Kochen, S. (1965). Diophantine Problems Over Local Fields I. *American Journal of Mathematics, 87*(3), 605–630. https://doi.org/10.2307/2373065

13 Ax, J. (1968). The Elementary Theory of Finite Fields. *Annals of Mathematics, 88*(2), 239–271. https://doi.org/10.2307/1970573

14 Ershov, Yu.L. (1967). O poliakh s razreshimoi teoriei [Fields with a solvable theory]. *Doklady Akademii nauk SSSR — Report Academy of Sciences of the USSR, 174* (1), 19,20 [in Russian].

15 Ax, J. (1967) Solving diophantine problems modulo every prime. *Ann. Math., 85*(2), 161–183. https://doi.org/10.2307/1970438

16 Hrushovski, E. (1993). Finite Structures with Few Types. *In: Sauer N.W., Woodrow R.E., Sands B. (eds) Finite and Infinite Combinatorics in Sets and Logic. NATO ASI Series (Series C: Mathematical and Physical Sciences), Vol. 411.* Dordrecht: Springer. https://doi.org/10.1007/978-94-011-2080-7

17 Cherlin, G. (1997). Large finite structures with few types. *In: Hart, B.T., Lachlan, A.H., Valeriote, M.A. (eds) Algebraic Model Theory. NATO ASI Series, Vol. 496.* Dordrecht: Springer. https://doi.org/10.1007/978-94-015-8923-9_3

18 Cherlin, G., & Hrushovski, E. (2003). Finite Structures with Few Types. *Annals of Mathematics Studies, Vol. 152.* Princeton: Princeton University Press.

19 Sudoplatov, S.V. (2016). Closures and generating sets related to combinations of structures. *The Bulletin of Irkutsk State University. Series Mathematics, 16*, 131–144.

20 Zilber, B. (1981). K probleme konechnoi aksiomatiziruemosti teorii, kategorichnykh vo vsekh beskonechnykh moshchnostiakh [On the problem of finite axiomatizability for theories categorical in all infinite powers]. *B. Baizhanov* (Ed.) *Issledovaniia po teoreticheskomu programmirovaniiu — Investigations in Theoretical Programming.* Alma-Ata, 69–75 [in Russian].

21 Cherlin, G. (1985). $\aleph_0$-categorical, $\aleph_0$-stable structures. *Ann. Pure Appl. Logic., 28*, 103–135. https://doi.org/10.1016/0168-0072(85)90023-5

22 Kantor, W.M., Liebeck, M.W., & Macpherson, H.D. (1989). $\aleph_0$-categorical structures smoothly approximated by finite substructures. *Proceedings of the London Mathematical Society, 59*, 439–463. https://doi.org/10.1112/plms/s3-59.3.439

23 Macpherson, D. (1997). Homogeneous and Smoothly Approximated Structures. *In: Hart, B.T., Lachlan, A.H., Valeriote, M.A. (eds) Algebraic Model Theory. NATO ASI Series, Vol. 496.* Dordrecht: Springer. https://doi.org/10.1007/978-94-015-8923-9_7

24 Sudoplatov, S.V. (2004). Complete Theories with Finitely Many Countable Models. I. *Algebra and Logic, 43*, 62–69. https://doi.org/10.1023/B:ALLO.0000015131.41218.f4

25 Sudoplatov, S.V. (2006). Complete theories with finitely many countable models. II. *Algebra and Logic, 45*, 180–200. https://doi.org/10.1007/s10469-006-0016-5