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On weighted integrability of the sum of series with monotone coefficients with respect to multiplicative systems

In this paper, we consider the questions about the weighted integrability of the sum of series with respect to multiplicative systems with monotone coefficients. Conditions are obtained for weight functions that ensure that the sum of such series belongs to the weighted Lebesgue space. The main theorems are proved without the condition that the generator sequence is bounded; in particular, it can be unbounded. In the case of boundedness of the generator sequence, the proved theorems imply an analogue of the well-known Hardy-Littlewood theorem on trigonometric series with monotone coefficients.

Keywords: multiplicative systems, decomposition, weighted integrability, sum of series, generator sequence, monotone coefficients, Hardy-Littlewood theorem, Lebesgue space.

Introduction

In the theory of trigonometric series, the Hardy-Littlewood theorem on series with monotone coefficients is known [1, 2]: in order to the series $\sum_{n=0}^{\infty} a_n \cos nx$, where $a_n \downarrow 0$ at $n \to \infty$, was the Fourier series of some function $f(x) \in L_p[0, 2\pi]$, $1 < p < \infty$, is necessary and sufficient to $\sum_{n=1}^{\infty} a_n^p n^{p-2} < \infty$.

An analogue of this theorem for the Walsh system was proved by Moricz F. [3], for multiplicative systems with bounded generating sequences p $(1 \leq \sup_n p_n < c)$ was proved by Timan M.F., Tukhliev K. [4].

The weighted integrability of the trigonometric series' sum with generalized monotone coefficients was studied in the works of Tikhonov S.Yu., Dyachenko M.I. [5, 6] and others. Weighted integrability for the sum of series with respect to multiplicative systems is considered in the works of Volosivecs S.S., Fadeev R.N. [7, 8], Bokayev N.A., Mukanov Zh.B. [9].

In this paper, we consider weight functions with other conditions.

1 Notation and Preliminaries

In this paper we consider series with monotone coefficients on multiplicative systems. We investigate the problem: under what conditions imposed on the weight function and the coefficients of the series, the sum of this series will belong to the space L_p with weight. Let us give a definition of the multiplicative systems.

Definition 1. Let $\{p_k\}_{k=1}^{\infty}$ is a sequence of natural numbers $p_k \geq 2$, $k \in \mathbb{N}$, $\sup_k p_k = N < \infty$. By definition let us put

$$m_0 = 1,\ m_n = p_1 p_2 \cdots p_n,\ n \in \mathbb{N}.$$

Then every point $x \in [0, 1)$ has a decomposition

$$x = \sum_{k=1}^{\infty} \frac{x_k}{m_k}, \quad x_k \in \mathbb{Z} \cap [0, p_k), \tag{1}$$

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where $\mathbb{Z}$ is the set of integers. Decomposition (1) is uniquely defined if for $x = n/m_k$ take a decomposition with a finite number of nonzero x_k. If $n \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$ is represented as

$$n = \sum_{j=1}^{\infty} \alpha_j m_{j-1}, \quad \alpha_j \in \mathbb{Z} \cap [0, p_j),$$

then for the numbers $x \in [0, 1)$ we put by definition

$$\psi_n(x) = exp\left(2\pi i \sum_{j=1}^{\infty} \frac{\alpha_j x_j}{p_j}\right), \quad n \in \mathbb{Z}_+.$$

It is known that the system $\{\psi_n\}_{n=0}^{\infty}$, called the Price system, is an orthonormalized system that is complete in $L^1(0, 1)$ (see [10] or [11]). If all $p_k = 2$, then $\{\psi_n\}_{n=0}^{\infty}$ coincides with Walsh system in the Paley numbering.

Let $L^p(G)$, $G := [0, 1)$, $1 \leq p < \infty$, be a Lebesgue space with a norm

$$\|f\|_p = \left(\int_G |f(x)|^p dx\right)^{\frac{1}{p}}, \quad \|f\|_\infty = ess \sup_{x \in G} |f(x)|.$$

Definition 2. Let $\varphi(x)$ be a non-negative measurable function on $[1, \infty)$. We say that $\varphi(x)$ satisfies condition B_1, if for all $x \geq 1$

$$\int_x^{\infty} \frac{\varphi(t)}{t^2} dt \leq C \frac{\varphi(x)}{x},$$

where C is a positive number independent of x.

For example, the function $\varphi(t) = t^\alpha$ $(\alpha < 1)$ satisfies condition B_1.

To prove the main results, we need the following auxiliary assertions.

Lemma A. (Potapov M.K. [12]). If $a_n,\ b_n \geq 0$ $(n = 1,\ 2,\ \dots)$, $1 \leq p < \infty$ and $\sum_{m=n}^{\infty} b_m = \gamma_n b_n$, then

$$\sum_{m=1}^{\infty} b_m \left(\sum_{n=1}^{m} a_n\right)^p \leq C_p \sum_{m=1}^{\infty} b_m \left(a_m \gamma_m\right)^p.$$

Lemma B. (Simonyan A.S. [13, 14]) Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $f(x) \in L[0, 1]$, $f(x) \geq 0$ and function $[\varphi(x)]^{-p'}$ satisfies condition B_1,

$$F(x) = \int_0^x f(t) dt.$$

Then

$$\int_0^1 \varphi^p\left(\frac{1}{x}\right)\left(\frac{F(x)}{x}\right)^p dx \leq C_p \int_0^1 \varphi^p\left(\frac{1}{x}\right) f^p(x)\, dx.$$

By

$$D_n(x) = \sum_{k=0}^{n-1} \psi_k(x), \quad n = 1, 2, ...,$$

denote the Dirichlet kernel of the system $\{\psi_n(x)\}$.

Lemma C. (see [10] or [11]) For any $k \in \mathbb{N}$ and $x \in [0,1)$ the Dirichlet kernels have the following properties:

$$D_{m_k} = \begin{cases} m_k, & \text{if } x \in \left[0, \frac{1}{m_k}\right), \\ 0, & \text{if } x \notin \left[0, \frac{1}{m_k}\right). \end{cases} \tag{2}$$

The Dirichlet kernel $D_n(x)$ satisfies the estimate

$$\frac{q(x)}{2} \leq \sup_{j \leq p_{n+1}} |D_{j_{m_k}}(x)| \leq 2q(x), \tag{3}$$

where $q(x)$ is the function introduced in [15]:

$$q(x) = \frac{m_n(x)}{\sin \frac{\pi l(x)}{p_{n(x)+1}}}, \quad x \in [0,1], \tag{4}$$

where $n(x)$ is the number of the last zero in the initial series of the decomposition of the element $x \in [0,1]$, $l(x)$ is the value of the first nonzero coordinate of this decomposition.

Lemma 1. Let $S_n(x) = \sum_{k=0}^{n-1} a_k \psi_k(x)$, $(n = 0,\ 1,\ 2, \ldots)$ $a_k \downarrow 0$ at $k \to \infty$. Then for any $x \in \left[\frac{1}{m_{\nu+1}},\ \frac{1}{m_\nu}\right]$,

$$|S_n(x)| \leq \sum_{k=0}^{m_\nu - 1} a_k + a_{m_\nu} \cdot m_{\nu+1}.$$

Proof of Lemma 1. Let $x \in \left[\frac{1}{m_{\nu+1}},\ \frac{1}{m_\nu}\right]$, $0 \leq \nu < \infty$. Considering, $|\psi_k(x)| = 1$, we have

$$|S_n(x)| \leq \sum_{k=0}^{\nu-1} a_k + \sum_{k=\nu}^{n-1} a_k \psi_k(x).$$

Applying the Abel transformation for the second sum, by inequality (3) we have

$$\left|\sum_{k=m_\nu}^{n-1} a_k \psi_k(x)\right| = \left|\sum_{k=m_\nu}^{n-2} \Delta a_k D_{k+1}(x) + a_{n-1} D_n(x) - a_{m_\nu} D_{m_\nu}(x)\right| \leq$$

$$\leq q(x) \left|\sum_{k=m_\nu}^{n-2} \Delta a_k + a_{n-1} + a_{m_\nu}\right| \leq 2 a_{m_\nu} q(x),$$

but for the function $q(x)$ at $x \in \left[\frac{1}{m_{\nu+1}},\ \frac{1}{m_\nu}\right]$, an estimate

$$q(x) \leq \frac{m_{\nu+1}}{2}$$

holds (see (4)).

Consequently,

$$|S_n(x)| \leq \sum_{k=0}^{m_\nu - 1} a_k + a_{m_\nu} \cdot m_{\nu+1}.$$

Lemma 1 is proved.

2 Main Results

The following theorems about integrability with weight of the series' sum with monotone coefficients are valid.

Theorem 1. Let $1 < p < \infty,\ \frac{1}{p} + \frac{1}{p'} = 1$

$$f(x) = \textstyle\sum_{k=0}^{\infty} a_k \psi_k(x), \quad a_k \downarrow 0 \text{ at } k \to \infty$$

and let $\varphi(x)$ is a non-negative measurable function on $[1,\ \infty)$. Then

1^0. If $\varphi\left(\frac{1}{x}\right) \in L_p(0,\ 1)$ and

$$\sum_{\mathrm{n}=1}^{\infty} \left(\sum_{k=0}^{m_n - 1} a_k + a_{m_n} m_{n+1} \right)^p \int_{1/m_{n+1}}^{1/m_n} \varphi^p\left(\frac{1}{x}\right) dx < \infty, \tag{5}$$

then $\varphi\left(\frac{1}{x}\right) f(x) \in L_p(0,\ 1)$.

2^0. If the function $\varphi^{-p'}(x)$ satisfies the condition B_1 and $\varphi\left(\frac{1}{x}\right) f(x) \in L_p(0,\ 1)$, $\sup\limits_n p_n = K < \infty$, then

$$\sum_{n=1}^{\infty} \left(\sum_{k=0}^{m_{n+1}-1} a_k \right)^p \int_{1/m_{n+1}}^{1/m_n} \varphi^p\left(\frac{1}{x}\right) dx < \infty.$$

Theorem 2. Let $1 < p < \infty,\ \frac{1}{p} + \frac{1}{p'} = 1$

$$f(x) = \textstyle\sum_{k=0}^{\infty} a_k \psi_k(x), \quad a_k \downarrow 0 \text{ at } k \to \infty$$

and let $\varphi(x)$ be a non-negative measurable function on $[1,\ \infty)$. Then

1^0. If function $\varphi^p(x)$ satisfies condition B_1 and

$$\sum_{n=0}^{\infty} a_{m_n}^p \cdot m_{n+1}^p \int_{1/m_{n+1}}^{1/m_n} \varphi^p\left(\frac{1}{x}\right) dx < \infty, \tag{6}$$

then $\varphi\left(\frac{1}{x}\right) f(x) \in L_p(0,\ 1)$.

2^0. If $\varphi^{-p'}(x)$ satisfies condition B_1 and $\varphi\left(\frac{1}{x}\right) f(x) \in L_p(0,\ 1)$, then

$$\sum_{n=0}^{\infty} a_{m_{n+1}}^p \cdot m_{n+1}^p \int_{1/m_{n+1}}^{1/m_n} \varphi^p\left(\frac{1}{x}\right) dx < \infty. \tag{7}$$

In the case $\sup p_n = k < \infty$ theorem 1 is equivalent to the following theorem:

Theorem 3. Let $1 < p < \infty,\ \frac{1}{p} + \frac{1}{p'} = 1$,

$$f(x) = \textstyle\sum_{k=0}^{\infty} a_k \psi_k(x), \quad a_k \downarrow 0 \text{ at } k \to \infty$$

and let $\varphi(x)$ be a non-negative measurable function on $[1,\ \infty)$, $\sup\limits_n p_n = N < \infty$. Then

1^0. If the function $\varphi^p(x)$ satisfies condition B_1 and

$$\sum_{n=1}^{\infty} a_n^p \cdot n^p \int_n^{n+1} \frac{\varphi^p(x)}{x^2} dx < \infty,$$

then $\varphi\left(\frac{1}{x}\right) f(x) \in L_p(0,\ 1)$.

2^0. If $\varphi^{-p'}(x)$ it satisfies the condition B_1 and $\varphi\left(\frac{1}{x}\right) f(x) \in L_p(0,\ 1)$, then it takes place (6).

From this theorem in the case of the Walsh system follow the corresponding results of A.S. Simonyan [13].

Remark. If the weight function $\varphi(x)$ has the form $\varphi(x) = x^{\alpha}$, then in this case $\varphi^p(x)$ and $\varphi^{-p'}(x)$ satisfy condition B_1 at $-\frac{1}{p'} < \alpha < \frac{1}{p}$ and condition (7) has the form

$$\sum_{n=1}^{\infty} a_n^p \cdot n^{p(\alpha+1)-1} < \infty.$$

Proof of Theorem 1. 1^0. By Lemma 1 and condition (5) we have

$$\int_0^1 \varphi^p\left(\frac{1}{x}\right) |f(x)|^p\, dx = \sum_{n=0}^{\infty} \int_{1/m_{n+1}}^{1/m_n} \varphi^p\left(\frac{1}{x}\right) |f(x)|^p\, dx \le$$

$$\le \sum_{n=0}^{\infty} \left(\sum_{k=0}^{m_n-1} a_k + a_{m_n} \cdot m_{n+1} \right)^p \int_{1/m_{n+1}}^{1/m_n} \varphi^p\left(\frac{1}{x}\right) dx < \infty,$$

that is

$$f(x)\ \varphi\left(\frac{1}{x}\right) \in L_p[0,\ 1].$$

2^0. Let

$$\varphi\left(\frac{1}{x}\right) f(x) \in L_p(0,\ 1) \quad \textit{and} \quad \varphi^{-p'}\left(\frac{1}{x}\right) \in L(0,\ 1).$$

By Gelder's inequality

$$\int_0^1 |f(x)|\ dx \le \left(\int_0^1 \varphi^p\left(\frac{1}{x}\right) |f(x)|^p\, dx \right)^{1/p} \left(\int_0^1 \varphi^{-p'}\left(\frac{1}{x}\right) dx \right)^{1/p'} < \infty.$$

Consequently, $f(x) \in L(0,1)$ and $a_k = a_k(f)$.

Let $F(x) = \int_0^x |f(t)|\, dt$. By (2) from Lemma C we get

$$\sum_{k=0}^{m_n-1} a_k(f) = \sum_{k=0}^{m_n-1} \int_0^1 f(x) \overline{\psi_x(x)} dx = \int_0^1 f(x) \sum_{k=0}^{m_n-1} \overline{\psi_k(x)} dx =$$

$$= \int_0^1 f(x) D_{m_n}(x)\, dx = m_n \int_0^{1/m_n} f(x)\, dx \le m_n F\left(\frac{1}{m_n}\right),$$

where $F(x) = \int_0^x |f(x)|\, dx$.

From here, using the monotonicity of the sequence $\{a_k\}$ and Lemma B, we obtain

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{m_{n+1}-1} a_k \right)^p \int_{1/m_{n=1}}^{1/m_n} \varphi^p\left(\frac{1}{x}\right) dx \le \sum_{n=0}^{\infty} \left[m_{n+1} F\left(\frac{1}{m_{n+1}}\right) \right]^p \int_{1/m_{n+1}}^{1/m_n} \varphi^p\left(\frac{1}{x}\right) dx \le$$

$$\le C_p \sum_{n=0}^{\infty} \int_{1/m_{n+1}}^{1/m_n} \varphi^p\left(\frac{1}{x}\right) \left(\frac{F(x)}{x}\right)^p dx \le C_p \int_0^1 \varphi^p\left(\frac{1}{x}\right) |f(x)|^p\, dx < \infty.$$

Theorem 1 is proved.

Proof of Theorem 2. 1^0. We denote

$$b_n = \int_{m_n}^{m_{n+1}} \frac{\varphi^p(t)}{t^2} dt.$$

Then
$$\sum_{n=\nu}^{\infty} b_n = \left(\int_{m_\nu}^{\infty} \frac{\varphi^p(t)}{t^2}dt\right)\left(\int_{m_\nu}^{m_{\nu+1}} \frac{\varphi^p(t)}{t^2}dt\right)^{-1} b_\nu = \gamma_\nu \cdot b_\nu,$$
where
$$\gamma_\nu = \left(\int_{m_\nu}^{\infty} \frac{\varphi^p(t)}{t^2}dt\right)\cdot\left(\int_{m_\nu}^{m_{\nu+1}} \frac{\varphi^p(t)}{t^2}dt\right)^{-1}.$$

The function $\varphi^p(x)$ satisfies the condition B_1, therefore
$$\gamma_\nu = 1 + \left(\int_{m_\nu}^{m_{\nu+1}} \frac{\varphi^p(t)}{t^2}dt\right)^{-1}\cdot\int_{m_\nu}^{m_{\nu+1}}\left(\int_x^{\infty} \frac{\varphi^p(t)}{t^2}dt\right)dx \le$$
$$\le 1 + C_1\left(\int_{m_\nu}^{m_{\nu+1}} \frac{\varphi^p(t)}{t^2}dt\right)^{-1}\cdot\int_{m_\nu}^{m_{\nu+1}} \frac{\varphi^p(x)}{x}dx \le C_2.$$
Using the Lemma A we have
$$\sum_{n=1}^{\infty} b_n\left(\sum_{k=0}^{m_n-1} a_k\right)^p = \sum_{n=1}^{\infty} b_n\left[\sum_{k=0}^{n-1}\left(\sum_{j=m_k}^{m_{k+1}-1} a_j\right)\right]^p \le C\sum_{n=1}^{\infty} a_{m_n}^p\cdot m_n^p\cdot b_n;$$
consequently,
$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{m_n-1} a_k + a_{m_n}\cdot m_{n+1}\right)^p\cdot b_n \le C\sum_{n=0}^{\infty} a_{m_n}^p\cdot m_{n+1}^p\cdot b_n < \infty.$$
Hence, on the basis of the first point of Theorem 1 follows the first point of Theorem 2.

2^0. Due to the monotonicity of the sequence a_n
$$\sum_{n=1}^{\infty}\left(\sum_{k=0}^{m_{n+1}-1} a_n\right)^p\int_{m_n}^{m_{n+1}} \frac{\varphi^p(x)}{x^2}dx \ge \sum_{n=1}^{\infty} a_{m_{n+1}}\cdot m_{n+1}^p\int_{m_n}^{m_{n+1}} \frac{\varphi^p(x)}{x^2}dx.$$
Therefore, the statement of Theorem 2 follows from Theorem 1.

Theorem 2 is proved.

Proof of Theorem 3. Sufficiency. By the monotonicity of the sequence $\{a_n\}$ and by the condition $\sup p_n = N < \infty$, we have
$$\sum_{n=1}^{\infty} a_n^p n^p\int_n^{n+1} \frac{\varphi^p(x)}{x^2}dx = \sum_{n=0}^{\infty}\sum_{k=m_n}^{m_{n+1}-1} a_k^p k^p\int_k^{k+1} \frac{\varphi^p(x)}{x^2}dx \ge$$
$$\ge \sum_{n=0}^{\infty} a_{m_{n+1}}^p\cdot m_n^p\sum_{k=m_n}^{m_{n+1}-1}\int_k^{k+1} \frac{\varphi^p(x)}{x^2}dx \ge C_p\sum_{n=0}^{\infty} a_{m_{n+1}}^p\cdot m_{n+2}^p\int_{m_n}^{m_{n+1}} \frac{\varphi^p(x)}{x^2}dx \ge$$
$$\ge C_p\sum_{n=1}^{\infty} a_{m_n}^p\cdot m_{n+1}^p\int_{1/m_{n+1}}^{1/m_n}\varphi^p\left(\frac{1}{t}\right)dx.$$
Therefore, from the condition of point 1^0 of Theorem 3 it follows the condition of point 1^0 of the Theorem 2. Therefore $f(x)\,\varphi\left(\frac{1}{x}\right) \in L_p(0,\ 1)$.

On the other hand, also due to monotonicity of the sequence $\{a_n\}$ and boundedness of the sequence $\{p_n\}$, we have
$$\sum_{n=1}^{\infty} a_n^p n^p\int_n^{n+1} \frac{\varphi^p(x)}{x^2}dx = \sum_{n=0}^{\infty}\sum_{k=m_n}^{m_{n+1}-1} a_k^p k^p\int_k^{k+1} \frac{\varphi^p(x)}{x^2}dx \le$$

$$\leq \sum_{n=0}^{\infty} a_{m_n}^p \cdot m_{n+1}^p \sum_{k=m_n}^{m_{n+1}-1} \int_k^{k+1} \frac{\varphi^p(x)}{x^2} dx = C_p \sum_{n=0}^{\infty} a_{m_{n+1}}^p \cdot m_{n+1}^p \int_{1/m_{n+1}}^{1/m_n} \varphi^p\left(\frac{1}{t}\right) dt.$$

Therefore, from the point 2^0 of Theorem 3 follows the condition of point 2^0 of Theorem 2. From this follows the necessity of the Theorem 3.

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Коэффициенттері монотонды мультипликативтік жүйелер бойынша қатарлардың қосындысының салмақты интегралдануы туралы

Жұмыста коэффициенттері монотонды мультипликативті жүйелер бойынша құрылған қатарлар қосындысының салмақты интегралдануы туралы сұрақтар қарастырылған. Осындай қатарлардың қосындысы салмақты Лебег кеңістігінде жататынын қамтамасыз ететін салмақты функцияларға шарттар алынған. Негізгі теоремалар жасаушы тізбегіне шенелгендік шарт қойылмағанда дәлелденеді; атап айтқанда, ол шенелмеген болуы мүмкін. Жасаушы тізбегі шенелгендігі жағдайында дәлелденген теоремалар монотонды коэффициенттері бар тригонометриялық қатарлар бойынша белгілі Харди-Литлвуд теоремасының аналогын білдіреді.

Кілт сөздер: мультипликативті жүйелер, жіктеу, қатарлар қосындысы, салмақты интегралдану, жасаушы тізбек, монотонды коэффициенттер, Харди-Литлвуд теоремасы, Лебег кеңістігі.

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Об интегрируемости с весом суммы рядов с монотонными коэффициентами по мультипликативным системам

В работе рассмотрены вопросы о весовой интегрируемости суммы рядов по мультипликативным системам с монотонными коэффициентами. Получены условия на весовые функции, обеспечивающие принадлежность суммы таких рядов весовому пространству Лебега. Основные теоремы доказаны без условия ограниченности образующей последовательности, в частности, она может быть неограниченной. В случае ограниченности образующей последовательности из доказанных теорем следует аналог известной теоремы Харди–Литлвуда о тригонометрических рядах с монотонными коэффициентами.

Ключевые слова: мультипликативные системы, разложение, весовая интегрируемость, сумма рядов, образующая последовательность, монотонные коэффициенты, теорема Харди-Литлвуда, пространство Лебега.

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