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Generalized Hankel shifts and exact Jackson–Stechkin inequalities in L_2

In this paper, we have solved several extremal problems of the best mean-square approximation of functions f on the semiaxis with a power-law weight. In the Hilbert space L^2 with a power-law weight $t^{2\alpha+1}$ we obtain Jackson–Stechkin type inequalities between the value of the $E_\sigma(f)$-best approximation of a function $f(t)$ by partial Hankel integrals of an order not higher than σ over the Bessel functions of the first kind and the k-th order generalized modulus of smoothnes $\omega_k(B^r f, t)$, where B is a second–order differential operator.

Keywords: best approximation, generalized modulus of smoothness of m-th order, Hilbert space.

Introduction

At present, there is a number of meaningful papers [1–3] devoted to the theory of approximation of a function from $L_2[0, 2\pi]$. Let $\alpha > -\frac{1}{2}$. For $p = 2$ by L_{2,μ_α} we denote the space consisting of measurable functions f on $[0, \infty)$, for which the norm is finite

$$\|f\|_{2,\mu_\alpha} = \left(\int_0^\infty |f(x)|^2 d\mu_\alpha(x)\right)^{\frac{1}{p}},$$

where

$$d\mu_\alpha(x) = \frac{x^{2\alpha+1}}{2^\alpha \Gamma(\alpha+1)} dx.$$

Consider the Hankel transform defined for the function f:

$$h_\alpha(f)(\lambda) = \int_0^\infty x^{2\alpha+1}(x\lambda)^{-\alpha} J_\alpha(x\lambda) f(x) dx, \quad \lambda \in (0, \infty),$$

where $J_\alpha(z)$ is the Bessel function of the first kind of an order $\alpha \geq -\frac{1}{2}$, $\Gamma(x)$ is the gamma-function.

In particular, for $\alpha = \frac{1}{2}$ and $\alpha = -\frac{1}{2}$ the Hankel transforms turn into the sine transform and the cosine Fourier transform, respectively:

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\lambda x) dx,$$

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\lambda x) dx,$$

since the formulas $J_{\frac{1}{2}} = \sqrt{\frac{2}{\pi}} \sin x$ and $J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi}} \cos x$ hold.

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For a function $f \in L_{2,\mu_\alpha}$ the expansion into the Hankel integral [4], is valid:

$$\hat{H}_\alpha(f)(\lambda) = \int_0^\infty f(x) j_\alpha(\lambda x) d\mu_\alpha(x),$$

and

$$f(x) = \int_0^\infty \hat{H}_\alpha(f)(\lambda) j_\alpha(\lambda x) d\mu_\alpha(\lambda).$$

Let $T > 0$ and we denote by $S_T(f,x)$ the partial Hankel integral of a function $f \in L_{2,\mu_\alpha}$ i.e.

$$S_T(f,x) = \int_0^T \hat{H}_\alpha(f)(\lambda) j_\alpha(\lambda x) d\mu_\alpha(\lambda), \quad x \in (0,\infty).$$

For functions $f, g \in L_{2,\mu_\alpha}$, the generalized Plancherel's theorem holds [5]

$$(f,g) = (\hat{f}, \hat{g}),$$

where $(f,g) = \int_0^\infty f(x)\overline{g(x)} d\mu_\alpha$ is the scalar product of f and g.

In the space $L_{p,\alpha}$ consider the generalized shift operator of functions $f(x)$ [6]

$$(T^h f)(x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_0^\pi f(\sqrt{x^2+h^2-2xh\cos\varphi})(\sin\varphi)^{2\alpha} d\varphi.$$

For a function $f \in L_{2,\mu_\alpha}$, $\Delta_h^k f(x)$ finite differences of the k-th order with a step $h > 0$ are defined as follows (see [7]):

$$\Delta_h^1 f(x) = (I - T^h)(x), \Delta_h^k f(x) = (I - T^h)^k f(x), k > 1.$$

The value

$$\omega_k(f,\delta)_{2,\mu_\alpha} = \sup_{0 \le h \le \delta} \|\Delta_h^k f(x)\|_{2,\mu_\alpha} = \sup_{0 \le h \le \delta} \left\{ \int_0^\infty (1 - j_\alpha(\lambda h))^{2k} |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda) \right\}^{\frac{1}{2}} \tag{01}$$

will be called the generalized modulus of smoothness of the k-th order of a function$f \in L_{2,\mu_\alpha}$. We denote by $M(\nu,2,\alpha), \nu > 0$ the set of all functions $Q_\nu(x)$ satisfying the following conditions (see [7]):

1. $Q_\nu(x)$ is an even entire function of exponential type ν;
2. $Q_\nu(x)$ belongs to the class L_{2,μ_α}.

The best approximation of a function $f \in L_{2,\mu_\alpha}$ from the class $M(\sigma,2,\alpha), \nu > 0$ is defined as follows:

$$E_\sigma(f)_{2,\mu_\alpha} = \inf\{\|f - Q_\sigma\|_{2,\mu_\alpha} : Q_\sigma \in M(\sigma,2,\alpha)\} = \left\{ \int_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda) \right\}^{\frac{1}{2}}. \tag{02}$$

Let

$$B = B_t = \frac{d^2}{dt^2} + \frac{2\alpha+1}{t}\frac{d}{dt}$$

be a differential Bessel operator. We denote by $j_\alpha(\lambda t)$ the normalized Bessel function

$$j_\alpha(\lambda t) = \frac{2^\alpha \Gamma(\alpha+1) J_\alpha(\lambda t)}{(\lambda t)^\alpha}.$$

The function $j_\alpha(\sqrt{\lambda} t)$ is a solution to the problem

$$\frac{d^2 y}{dt^2} + \frac{2\alpha+1}{t}\frac{dy}{dt} + \lambda y = 0,$$

$$y(0) = 1, y'(0) = 0.$$

In [8], when solving problems of the theory approximations in the space L_{2,μ_α} associated with finding the exact constants in the Jackson–Stechkin inequality

$$E_\sigma(f) \leq \omega_r(f, \frac{\tau}{\sigma})$$

it is considered the following extreme characteristic:

$$K_{\sigma,r,m,\tau} = sup\left\{\frac{E_\sigma(f)}{\omega_r(f, \frac{\tau}{\sigma})} : f \in L_2(R^m)\right\}.$$

In this article, we want to get the exact constant in Jackson's inequality

$$E_\sigma(f) \leq K\sigma^{-2r}\omega_r(B^r f, \frac{\tau}{\sigma})$$

for the functions $f \in W^r_{2,\mu_\alpha}(B)$. For the goal, we introduce an extremal approximate characteristic of the following form

$$\Xi_{\sigma,r,m,p,s}(\varphi, h) = \sup_{f\in W^r_{2,\mu_\alpha}(B)} \frac{E_\sigma(f)}{\left(\int_0^h \omega^p_m(B^r f, t)\varphi(t)dt\right)^s}, \tag{03}$$

where $r, m \in \mathbb{N},\ 0 < p < 2,\ h > 0,\ \sigma > 0,\ \varphi(t) \geq 0$ is an arbitrary integrable, not equivalent to zero on the segment $[0, h]$, weight function and $W^r_{2,\mu_\alpha}(B), r = 1, 2, ...$ is a Sobolev space, constructed by the differential operator B, i.e.

$$W^r_{2,\mu_\alpha}(B) = \left\{f \in L_{2,\mu_\alpha} : B^j f \in L_{2,\mu_\alpha}, j = 1, 2, ..., r\right\}.$$

Note that values $\Xi_{\sigma,r,m,p,s}(\varphi, h)$ for different values of the parameters therein and specific weight functions were examined by Chernykh, Taykov, Yudin, Esmaganbetov, Ivanov, Babenko, Shalaev, Vakarchuk, Shabozov, Tukhliev and many others (see., e.g., [6-11] and the literature cited therein).

In the case of approximation of 2π-periodic function from L_2 by the subspace of trigonometric polynomials of an order $(n-1)$ in the metric L_2, similar problems were solved in [9] by Taikov, in [10] by M. Esmaganbetov, and in [11] by Sh.Shabozov and K. Tukhliev.

The extension of this question to the case of the best mean-square approximation by entire functions of exponential $\sigma > 0$ type in space L_2 with a power-law weight was carried out in [8] by A.G. Babenko and in [12] by D. V. Gorbachev, in [5] by V.I. Ivanov.

1 Auxiliary results

Lemma 1. Let $q_{\alpha+1,1}$ be the smallest positive zero of the function $j_{\alpha+1}(t)$. Let $\sigma > 0$ and $t \in (0, \frac{q_{\alpha+1,1}}{\sigma}],\ \alpha \geq -\frac{1}{2}$. Then

$$\sup_{0<h\leq t}(1 - j_\alpha(\sigma h)) = 1 - j_\alpha(\sigma t).$$

Proof of Lemma 1. Since

$$j'_\alpha(t) = -\frac{t}{2(\alpha+1)}j_{\alpha+1}(t), 0 \leq t \leq \infty$$

(see [5]), then from $j_{\alpha+1}(0) > 0$ and $j_{\alpha+1}(q_{\alpha+1,1}) = 0$ we obtain for all $t \in [0, q_{\alpha+1,1}]$ values $(1 - j_\alpha(t))' = \frac{t}{2(\alpha+1)}j_{\alpha+1}(t) > 0$. It follows that the function $1 - j_\alpha(t)$ increases on $[0, q_{\alpha+1,1}]$. Hence, for all $t \in (0, q_{\alpha+1,1}]$ we have

$$\sup_{0<h\leq t}(1 - j_\alpha(h)) = 1 - j_\alpha(t).$$

Therefore, for all $t \in (0, \frac{q_{\alpha+1,1}}{\sigma}]$ we get

$$\sup_{0<h\le t}(1 - j_\alpha(\sigma h)) = 1 - j_\alpha(\sigma t). \tag{1}$$

Lemma 1 is proved.

Lemma 2. Let $q_{\alpha+1,1}$ be the first positive zero of the function $j_{\alpha+1}(t),\ h \in (0, \frac{q_{\alpha+1,1}}{\sigma}],\ \alpha \ge -\frac{1}{2}$ and $\sigma > 0$. Let

$$\Psi(y) = y^{4r}\int_0^h (1 - j_\alpha(yt))^{2k}dt, \quad y \in G, \quad \text{where} \quad G = \{y : \sigma \le y < \infty\}.$$

Then

$$\min\{\Psi(y) : y \in G\} = \sigma^{4r}\int_0^h (1 - j_\alpha(\sigma t))^{2k}dt.$$

Proof of Lemma 2. Since $j'_\alpha(t) = -\frac{t}{2(\alpha+1)}j_{\alpha+1}(t),\ 0 \le x \le \infty$, then for $y \in G$ we have

$$\Psi'(y) = 4ry^{4r-1}\int_0^h (1 - j_\alpha(yt))^{2k}dt + y^{4r}\int_0^h \frac{\partial}{\partial y}\left((1 - j_\alpha(yt))^{2k}\right)dt. \tag{2}$$

Since it is not difficult to verify by direct verification that the equality is true

$$\frac{1}{y}\frac{\partial}{\partial t}\left((1 - j_\alpha(yt))^{2k}\right) = \frac{1}{t}\frac{\partial}{\partial y}\left((1 - j_\alpha(yt))^{2k}\right), \tag{3}$$

where t, y are non-zero, then from (2) by virtue of equality (3) we have

$$\Psi'(y) = y^{4r-1}\left[4r\int_0^h (1 - j_\alpha(yt))^{2k}dt + \int_0^h t\frac{\partial}{\partial t}\left((1 - j_\alpha(yt))^{2k}\right)dt\right]. \tag{4}$$

Applying the method of integration by parts to calculate the second integral in the right-hand side of (4), we conclude

$$\Psi'(y) = y^{4r-1}\left[(4r-1)\int_0^h (1 - j_\alpha(yt))^{2k}dt + h(1 - j_\alpha(yh))^{2k}\right]. \tag{5}$$

Since $|j_\alpha(u)| \le 1, \forall u \ge 0$ (see [8], formula (21)) and (1), then by virtue of (5), we have $\Psi'(y) > 0$ for all $y \ge \sigma$. Lemma is proved.

2 Main results

The main results of this work are the following theorems.

Theorem 1. For any function $f \in W^r_{2,\mu_\alpha}(B)$ for any $h > 0$, the following estimate holds:

$$E_\sigma(f)_{2,\mu_\alpha} \le \frac{\left(\int_0^h \omega_k^2(B^r f, t)_{2,\mu_\alpha}dt\right)^{\frac{1}{2}}}{\sigma^{2r}\left(\int_0^h (1 - j_\alpha(\sigma t))^{2k}dt\right)^{\frac{1}{2}}}.$$

Proof of Theorem 1. Let $f \in W^r_{2,\mu_\alpha}(B)$. Then from Parseval's equality, we have

$$\omega_k^2(B^r f, t)_{2,\mu_\alpha} \ge \int_\sigma^\infty (1 - j_\alpha(\lambda t))^{2k}\lambda^{4r}|\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda).$$

Integrating both sides of this inequality variable t over the range $t=0$ and $t=h$, we obtain

$$\int_0^h \omega_k^2(B^r f,t)_{2,\mu_\alpha}dt \geq \int_0^h \left(\int_\sigma^\infty (1-j_\alpha(\lambda t))^{2k}\lambda^{4r}|\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda)\right)dt =$$

$$= \int_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 \left(\int_0^h \lambda^{4r}(1-j_\alpha(\lambda t))^{2k}\right)dt d\mu_\alpha(\lambda). \quad (6)$$

From (6) by virtue of lemma 2, we have

$$\int_0^h \omega_k^2(B^r f,t)_{2,\mu_\alpha}dt \geq \sigma^{4r}\int_0^h (1-j_\alpha(\sigma t))^{2k}dt \int_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda).$$

It follows that

$$\int_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda) \leq \frac{\int_0^h \omega_k^2(B^r f,t)_{2,\mu_\alpha}dt}{\sigma^{4r}\int_0^h (1-j_\alpha(\sigma t))^{2k}dt}. \quad (7)$$

Further, given the following equality

$$\|f-S_\sigma(f,x)\|_{2,\mu_\alpha} = E_\sigma(f)_{2,\mu_\alpha}$$

in view of the inequality (7) we get

$$E_\sigma^2(f)_{2,\mu_\alpha} \leq \frac{\int_0^h \omega_k^2(B^r f,t)_{2,\mu_\alpha}dt}{\sigma^{4r}\int_0^h (1-j_\alpha(\sigma t))^{2k}dt}.$$

Theorem 1 is proved.

Theorem 2. For any function $f \in W_{2,\mu_\alpha}^r(B)$ for any $h>0$, the following estimate holds:

$$\sup_{f\in W_{2,\mu_\alpha}^r(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f,t)_{2,\mu_\alpha}dt\right)^{\frac{1}{2}}} = \frac{1}{\left(\int_0^h (1-j_\alpha(\sigma t))^{2k}dt\right)^{\frac{1}{2}}}. \quad (8)$$

Proof of Theorem 2. Let $f \in W_{2,\mu_\alpha}^r(B)$. Arguing in the same way as in Theorem 1, for $f \in W_{2,\mu_\alpha}^r(B)$ we have

$$\frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f,t)_{2,\mu_\alpha}dt\right)^{\frac{1}{2}}} \leq \frac{1}{\left(\int_0^h (1-j_\alpha(\sigma t))^{2k}dt\right)^{\frac{1}{2}}}.$$

Hence we get

$$\sup_{f\in W_{2,\mu_\alpha}^r(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f,t)_{2,\mu_\alpha}dt\right)^{\frac{1}{2}}} \leq \frac{1}{\left(\int_0^h (1-j_\alpha(\sigma t))^{2k}dt\right)^{\frac{1}{2}}}. \quad (9)$$

To obtain a lower estimate, we construct the function $f_\epsilon \in W_{2,\mu_\alpha}^r(B)$ so that it satisfies the inequality:

$$\sup_{f\in W_{2,\mu_\alpha}^r(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f,t)_{2,\mu_\alpha}dt\right)^{\frac{1}{2}}} \geq \frac{\sigma^{2r}E_\sigma(f_\epsilon)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f_\epsilon,t)_{2,\mu_\alpha}dt\right)^{\frac{1}{2}}}.$$

To do this, we use function $f_\epsilon \in W_{2,\mu_\alpha}^r(B)$ constructed by Babenko in [9] and such that

$$\hat{H}_\alpha(f_\epsilon)(\lambda) = \begin{cases} |\lambda|^{-\alpha-\frac{1}{2}} & \text{if} \quad \sigma<|\lambda|<\sigma+\epsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Relations (2) and using the properties of the Hankel transform (see [7]) imply the equality

$$E_\sigma^2(f_\epsilon)_{2,\mu_\alpha} = \int_\sigma^\infty |\hat{H}_\alpha(f_\epsilon)(\lambda)|^2 d\mu_\alpha(\lambda) = \int_\sigma^{\sigma+\epsilon} |\hat{H}_\alpha(f_\epsilon)(\lambda)|^2 d\mu_\alpha(\lambda) = \frac{\epsilon}{2^\alpha \Gamma(\alpha+1)}.$$

Therefore

$$E_\sigma(f_\epsilon)_{2,\mu_\alpha} = \sqrt{\frac{\epsilon}{2^\alpha \Gamma(\alpha+1)}}. \tag{10}$$

In virtue of the equality (01) and using the properties of the Hankel transform (see [7], [4])

$$\hat{H}_\alpha(B^r f_\epsilon)(\lambda) = \lambda^{2r} \hat{H}_\alpha(f_\epsilon)(\lambda)$$

we write:

$$\omega_k^2(B^r f_\epsilon, t)_{2,\mu_\alpha} = \int_\sigma^{\sigma+\epsilon} \lambda^{4r} |\hat{H}_\alpha(f_\epsilon)(\lambda)|^2 (1 - j_\alpha(\lambda t))^{2k} d\mu_\alpha(\lambda) \leq$$

$$\leq (\sigma+\epsilon)^{4r} (1 - j_\alpha((\sigma+\epsilon)t))^{2k} \frac{\epsilon}{2^\alpha \Gamma(\alpha+1)}. \tag{11}$$

Integrating both parts of the inequality (11), we have

$$\left\{ \int_0^h \omega_k^2(B^r f_\epsilon, t)_{2,\mu_\alpha} dt \right\}^{\frac{1}{2}} \leq (\sigma+\epsilon)^{2r} \sqrt{\frac{\epsilon}{2^\alpha \Gamma(\alpha+1)}} \left\{ \int_0^h (1 - j_\alpha((\sigma+\epsilon)t)^{2k} dt \right\}^{\frac{1}{2}}. \tag{12}$$

Using (10), (12) we write

$$\frac{\sigma^{2r} E_\sigma(f_\epsilon)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f_\epsilon, t)_{2,\mu_\alpha} dt \right)^{\frac{1}{p}}} \geq \frac{\sigma^{2r}}{(\sigma+\epsilon)^{2r} \{ \int_0^h (1 - j_\alpha((\sigma+\epsilon)t))^{2k} dt \}^{\frac{1}{p}}}. \tag{13}$$

Since $f_\epsilon \in W_{2,\mu_\alpha}^r(B)$, then from (13) and from left side of equality (8) we obtain

$$\sup_{f \in W_{2,\mu_\alpha}^r(B)} \frac{\sigma^{2r} E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f, t)_{2,\mu_\alpha} dt \right)^{\frac{1}{2}}} \geq \frac{\sigma^{2r}}{(\sigma+\epsilon)^{2r} \{ \int_0^h (1 - j_\alpha((\sigma+\epsilon)t))^{2k} dt \}^{\frac{1}{2}}}. \tag{14}$$

Obviously, the left side of inequality (14) does not depend on ϵ, and the expression located on its right side is the function of ϵ (with fixed values of other parameters). Since the left side of inequality (14) does not depend on ϵ, then calculating the supremum with respect to ϵ from its right side, we write

$$\sup_{f \in W_{2,\mu_\alpha}^r(B)} \frac{\sigma^{2r} E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f, t)_{2,\mu_\alpha} dt \right)^{\frac{1}{2}}} \geq \frac{\sigma^{2r} E_\sigma(f_\epsilon)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^2(B^r f_\epsilon, t)_{2,\mu_\alpha} dt \right)^{\frac{1}{2}}} = \frac{1}{\left(\int_0^h (1 - j_\alpha(\sigma t))^{2k} dt \right)^{\frac{1}{2}}}. \tag{15}$$

Comparing the upper estimate (9) and the lower estimate (15), we obtain the required equality. Theorem 2 is proved.

Theorem 3. Let $m, n \in \mathbb{N},\ r \in \mathbb{Z}_+,\ 0 < p \leq 2,\ h > 0,\ \alpha \geq -\frac{1}{2}$. Then the following estimate is valid

$$\sup_{f \in W_{2,\mu_\alpha}^r(B)} \frac{\sigma^{2r} E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt \right)^{\frac{1}{p}}} = \frac{1}{\{ \int_0^h (1 - j_\alpha(\sigma t))^{kp} dt \}^{\frac{1}{p}}}. \tag{16}$$

Proof of Theorem 3. Let $0 < p \leq 2$, then, arguing as in the previous theorem, we have

$$\omega_k^2(B^r f, t)_{2,\mu_\alpha} \geq \int_\sigma^\infty \lambda^{4r}(1 - j_\alpha(\lambda t))^{2k} |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda).$$

Raising both sides of this inequality by the power $p/2$, integrating the variable t over the range $t = 0$ and $t = h$ we obtain

$$\left(\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{p}} \geq \left\{\int_0^h \left(\int_\sigma^\infty \lambda^{4r}(1 - j_\alpha(\lambda t))^{2k} |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda)\right)^{\frac{p}{2}} dt\right\}^{\frac{1}{p}} = I.$$

Applying the inverse Minkowski inequality for $\frac{p}{2} \leq 1$, we have

$$I \geq \left\{\int_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 \left(\int_0^h \lambda^{2rp}(1 - j_\alpha(\lambda t))^{kp} dt\right)^{\frac{2}{p}} d\mu_\alpha(\lambda)\right\}^{\frac{1}{2}}. \tag{17}$$

Then from inequality (17) and in view of Lemma 2, we obtain

$$I \geq \sigma^{2r} \left\{\int_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda)\right\}^{\frac{1}{2}} \left(\int_0^h (1 - j_\alpha(\sigma t))^{kp} dt\right)^{\frac{1}{p}} =$$

$$= \sigma^{2r} E_\sigma(f)_{2,\mu_\alpha} \left(\int_0^h (1 - j_\alpha(\sigma t))^{kp} dt\right)^{\frac{1}{p}}. \tag{18}$$

So combining (17) and (18), we get

$$\left(\int_0^h \left(\omega_k^2(B^r f, t)_{2,\mu_\alpha}\right)^{\frac{p}{2}} dt\right)^{\frac{1}{p}} \geq \sigma^{2r} E_\sigma(f)_{2,\mu_\alpha} \left(\int_0^h (1 - j_\alpha(\sigma t))^{kp} dt\right)^{\frac{1}{p}}.$$

Hence it follows that for all $f \in W_{2,\mu_\alpha}^r(B)$ the inequality holds

$$\frac{\sigma^{2r} E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{p}}} \leq \frac{1}{\left(\int_0^h (1 - j_\alpha(\sigma t))^{kp} dt\right)^{\frac{1}{p}}}.$$

For all $f \in W_{2,\mu_\alpha}^r(B)$, we have

$$\sup_{f \in W_{2,\mu_\alpha}^r(B)} \frac{\sigma^{2r} E_\sigma(f)_{2,\mu_\alpha}}{\left\{\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt\right\}^{\frac{1}{p}}} \leq \frac{1}{\{\int_0^h (1 - j_\alpha(\sigma t))^{kp} dt\}^{\frac{1}{p}}}. \tag{19}$$

Thus, the upper estimate is proved.

To obtain a lower estimate, we construct a function $f_\epsilon \in W_{2,\mu_\alpha}^r(B)$ so that the inequality is fulfilled:

$$\sup_{f \in W_{2,\mu_\alpha}^r(B)} \frac{\sigma^{2r} E_\sigma(f)_{2,\mu\alpha}}{\left\{\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt\right\}^{\frac{1}{p}}} \geq \frac{\sigma^{2r} E_\sigma(f_\epsilon)_{2,\mu\alpha}}{\left\{\int_0^h \omega_k^p(B^r f_\epsilon, t)_{2,\mu_\alpha} dt\right\}^{\frac{1}{p}}}. \tag{20}$$

To do this, we use function $f_\epsilon \in W_{2,\mu_\alpha}^r(B)$ constructed by Babenko in [8] and such that

$$\hat{H}_\alpha(f_\epsilon)(\lambda) = \begin{cases} |\lambda|^{-\alpha-\frac{1}{2}} & \text{if} \quad \sigma < |\lambda| < \sigma + \epsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Raising both sides of the inequality (11) by the power $\frac{p}{2}$ and integrating the variable t over the range $t = 0$ to $t = h$, we have

$$\left\{\int_0^h \omega_k^p(B^r f_\epsilon, t)_{2,\mu_\alpha} dt\right\}^{\frac{1}{p}} \leq (\sigma+\epsilon)^{2r}\sqrt{\frac{\epsilon}{2^\alpha\Gamma(\alpha+1)}}\left\{\int_0^h (1-j_\alpha((\sigma+\epsilon)t)^{kp}dt\right\}^{\frac{1}{p}}. \tag{21}$$

Using (21), (10) we write

$$\frac{\sigma^{2r}E_\sigma(f_\epsilon)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^p(B^r f_\epsilon, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{p}}} \geq \frac{\sigma^{2r}}{(\sigma+\epsilon)^{2r}\{\int_0^h (1-j_\alpha((\sigma+\epsilon)t))^{kp}dt\}^{\frac{1}{p}}}. \tag{22}$$

In view of the fact that the function f_ϵ belongs to the class $W^r_{2,\mu_\alpha}(B)$ and from the right-hand side of equality (16) and by virtue of the inequality (22), (20) we obtain

$$\sup_{f\in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{p}}} \geq \frac{\sigma^{2r}}{(\sigma+\epsilon)^{2r}\{\int_0^h (1-j_\alpha((\sigma+\epsilon)t))^{kp}dt\}^{\frac{1}{p}}}. \tag{23}$$

Obviously, the left side of inequality (23) does not depend on ϵ, and the expression located on its right side is the function of ϵ (with fixed values of other parameters). Since the left side of inequality (23) does not depend on ϵ, then calculating the supremum with respect to ϵ from its right side, we write

$$\sup_{f\in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu\alpha}}{\left(\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha} dt\right)^{\frac{1}{p}}} \geq \frac{1}{\{\int_0^h (1-j_\alpha(\sigma t))^{kp}dt\}^{\frac{1}{p}}}. \tag{24}$$

Comparing the upper estimate (19) and the lower estimate (24), we obtain the required equality. The theorem 3 is proved.

Theorem 4. Let $m, n \in \mathbb{N},\ r \in \mathbb{Z}_+,\ 0 < p \leq 2,\ h > 0,\ \ \alpha \geq -\frac{1}{2}$ and $\varphi(t) \geq 0$ be a measurable function on the interval (0, h). Then the inequality

$$\left\{\gamma_{\sigma,r,m,p,\frac{1}{p}}(\varphi,h)\right\}^{-1} \leq \Xi_{\sigma,r,m,p,\frac{1}{p}}(\varphi,h) \leq \left\{\inf_{\sigma\leq\lambda<\infty}\gamma_{\lambda,r,m,p,\frac{1}{p}}(\varphi,h)\right\}^{-1}$$

holds, where

$$\gamma_{\lambda,r,m,p,\frac{1}{p}}(\varphi,h) = \left(\lambda^{2rp}\int_0^h (1-j_\alpha(\sigma t))^{kp}\varphi(t)dt\right)^{\frac{1}{p}}, \quad \lambda \geq \sigma.$$

Proof of Theorem 4. Let $0 < p \leq 2$ then, arguing as in the previous theorem, we have

$$\omega_k^2(B^r f, t)_{2,\mu_\alpha} \geq \int_\sigma^\infty \lambda^{4r}(1-j_\alpha(\lambda t))^{2k}|\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda).$$

Raising both sides of this inequality by the power $p/2$ and multiplying them by a function $\varphi(t)$ and integrating the variable t over the range $t = 0$ to $t = h$ we get

$$\left(\int_0^h \omega_k^p(B^r, t)_{2,\mu_\alpha}\varphi(t)dt\right)^{\frac{1}{p}} \geq \left\{\int_0^h \left(\int_\sigma^\infty \lambda^{4r}(1-j_\alpha(\lambda t))^{2k}|\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda)\right)^{\frac{p}{2}}\varphi(t)dt\right\}^{\frac{1}{p}} = I. \tag{25}$$

Applying the inverse Minkowski inequality for $\frac{p}{2} \le 1$ and by virtue of Lemma 2 we obtain

$$I \ge \left\{ \int_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda) \left(\int_0^h \lambda^{2rp}(1 - j_\alpha(\lambda t))^{kp}\varphi(t)dt \right)^{\frac{2}{p}} \right\}^{\frac{1}{2}} =$$

$$= \left\{ \int_\sigma^\infty |\hat{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda) \left\{ \gamma_{\lambda,r,k,p,\frac{1}{p}}(\varphi, h) \right\}^2 \right\}^{\frac{1}{2}} \ge$$

$$\ge E_\sigma(f)_{2,\mu_\alpha} \inf_{\sigma \le \lambda < \infty} \gamma_{\lambda,r,k,p,\frac{1}{p}}(\varphi, h). \tag{26}$$

So combining (25) and (26) we get

$$\left(\int_0^h \omega_k^p(B^r, t)_{2,\mu_\alpha}\varphi(t)dt \right)^{\frac{1}{p}} \ge E_\sigma(f)_{2,\mu_\alpha} \inf_{\sigma \le \lambda < \infty} \gamma_{\lambda,r,k,p,\frac{1}{p}}(\varphi, h).$$

Therefore, according the definition of quantity (03), by previous inequality we obtain an upper bound for the extremal characteristics $\Xi_{\sigma,r,k,p,\frac{1}{p}}(\varphi, h)$, namely

$$\Xi_{\sigma,r,k,p,\frac{1}{p}}(\varphi, h) = \sup_{f \in W_{2,\mu_\alpha}^r(B)} \frac{E_\sigma(f)_{2,\mu_\alpha}}{\left\{ \int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha}\varphi(t)dt \right\}^{\frac{1}{p}}} \le \frac{1}{\inf\limits_{\sigma \le \lambda < \infty} \gamma_{\lambda,r,k,p,\frac{1}{p}}(\varphi, h)}. \tag{27}$$

To obtain a lower estimate, we construct the function $f_\epsilon \in W_{2,\mu_\alpha}^r(B)$ so that the inequality would be fulfilled:

$$\Xi_{\sigma,r,k,p,\frac{1}{p}}(\varphi, h) = \sup_{f \in W_{2,\mu_\alpha}^r(B)} \frac{E_\sigma(f)_{2,\mu_\alpha}}{\left\{ \int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha}\varphi(t)dt \right\}^{\frac{1}{p}}} \ge \frac{E_\sigma(f_\epsilon)_{2,\mu_\alpha}}{\left\{ \int_0^h \omega_k^p(B^r f_\epsilon, t)_{2,\mu_\alpha}\varphi(t)dt \right\}^{\frac{1}{p}}}. \tag{28}$$

To do this, we use function $f_\epsilon \in W_{2,\mu_\alpha}^r(B)$ constructed by Babenko in [9] and such that

$$\hat{H}_\alpha(f_\epsilon)(\lambda) = \begin{cases} |\lambda|^{-\alpha-\frac{1}{2}} & \text{if} \quad \sigma < |\lambda| < \sigma + \epsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Raising both sides of the inequality (11) by the power $\frac{p}{2}$, multiplying them by the weight function $\varphi(t)$, and integrating the variable t over the range $t = 0$ to $t = h$, we have

$$\left\{ \int_0^h \omega_k^p(B^r f_\epsilon, t)_{2,\mu_\alpha}\varphi(t)dt \right\}^{\frac{1}{p}} \le (\sigma + \epsilon)^{2r} \sqrt{\frac{\epsilon}{2^\alpha \Gamma(\alpha + 1)}} \left\{ \int_0^h (1 - j_\alpha((\sigma + \epsilon)t)^{kp}\varphi(t)dt \right\}^{\frac{1}{p}}. \tag{29}$$

Using (29) and (10) we write

$$\frac{E_\sigma(f_\epsilon)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^p(B^r f_\epsilon, t)_{2,\mu_\alpha}\varphi(t)dt \right)^{\frac{1}{p}}} \ge \frac{1}{(\sigma + \epsilon)^{2r} \{ \int_0^h (1 - j_\alpha((\sigma + \epsilon)t))^{kp}\varphi(t)dt \}^{\frac{1}{p}}}. \tag{30}$$

In view of the fact that the function f_ϵ belongs to the class $W_{2,\mu_\alpha}^r(B)$, by virtue of inequality (30) and relation (03), (28) we obtain

$$\sup_{f \in W_{2,\mu_\alpha}^r(B)} \frac{E_\sigma(f)_{2,\mu_\alpha}}{\left(\int_0^h \omega_k^p(B^r f, t)_{2,\mu_\alpha}\varphi(t)dt \right)^{\frac{1}{p}}} \ge \frac{1}{(\sigma + \epsilon)^{2r} \{ \int_0^h (1 - j_\alpha((\sigma + \epsilon)t))^{kp}\varphi(t)dt \}^{\frac{1}{p}}}. \tag{31}$$

Obviously, the left side of the inequality (31) does not depend on ϵ, and the expression located on its right side is the function of ϵ (with fixed values of other parameters). Since the left side of inequality (31) does not depend on ϵ, then calculating the supremum with respect to ϵ from its right side, we write

$$\sup_{f\in W^r_{2,\mu_\alpha}(B)} \frac{\sigma^{2r}E_\sigma(f)_{2,\mu\alpha}}{\left(\int_0^h \omega_k^p(B^r f,t)_{2,\mu_\alpha}\varphi(t)dt\right)^{\frac{1}{p}}} \geq \frac{1}{\{\int_0^h (1-j_\alpha(\sigma t))^{kp}\varphi(t)dt\}^{\frac{1}{p}}}. \tag{32}$$

Comparing the upper estimate (27) and the lower estimate (32), we obtain the required equality. Theorem 4 is proved.

Let us find: what differential properties the weight function φ must possess in order that the following equality holds

$$\gamma_{\sigma,r,k,p,\frac{1}{p}}(\varphi,h) = \inf_{\sigma\leq\lambda<\infty}\gamma_{\lambda,r,k,p,\frac{1}{p}}(\varphi,h).$$

The following statement gives an answer to this question.

Theorem 5. Let $\varphi(t)$ be a non-negative continuously differentiable function on the interval $[0,h]$. If for some $p\in(0,2]$, $r\in\mathbb{N}$ any $t\in[0,h]$, $\alpha\geq-\frac{1}{2}$, φ satisfies the differential inequality

$$(2rp-1)\varphi(t) - t\varphi'(t) \geq 0,$$

then for all $\sigma\in(0,\infty)$ and $0<h\leq\frac{q_{\alpha+1,1}}{\sigma}$ we have

$$\inf\left\{\gamma_{\lambda,k,r,p,\frac{1}{p}}(\varphi,h) : \sigma\leq\lambda<\infty\right\} = \gamma_{\sigma,k,r,p,\frac{1}{p}}(\varphi,h)$$

and there is a relation

$$\Xi_{\sigma,k,r,p,\frac{1}{p}}(\varphi,h) = \left(\gamma_{\sigma,k,r,p,\frac{1}{p}}(\varphi,h)\right)^{-1}.$$

Proof of Theorem 5. Since

$$\gamma_{\lambda,r,k,p,\frac{1}{p}}(\varphi,h) = \left\{\lambda^{2rp}\int_0^h (1-j_\alpha(\lambda t))^{kp}\varphi(t)dt\right\}^{\frac{1}{p}}$$

it is sufficient to prove that under the above assumptions on $\varphi(t)$ and the function

$$\eta(y) = y^{2rp}\int_0^h (1-j_\alpha(yt))^{kp}\varphi(t)dt$$

is strictly increasing on the interval $G=\{y : y\geq\sigma\}$. Since

$$\eta'(y) = 2rpy^{2rp-1}\int_0^h (1-j_\alpha(yt))^{kp}\varphi(t)dt + y^{2rp}\int_0^h \frac{d}{dy}(1-j_\alpha(yt))^{kp}\varphi(t)dt, \tag{33}$$

then, using the obvious identity

$$\frac{d}{dy}(1-j_\alpha(yt))^{kp} = \frac{t}{y}\frac{d}{dt}(1-j_\alpha(yt))^{kp} \tag{34}$$

from (33) and taking into account (34) we have

$$\eta'(y) = 2rpy^{2rp-1}\int_0^h (1-j_\alpha(yt))^{kp}\varphi(t)dt + y^{2rp-1}\int_0^h \frac{d}{dt}(1-j_\alpha(yt))^{kp}(t\varphi(t))dt.$$

Applying the method of integration by parts when calculating the second integral, we come to the conclusion

$$\eta'(y) = y^{2rp-1}\left((1-j_\alpha(yh))^{kp}h\varphi(h) + \int_0^h (1-j_\alpha(yt))^{kp}[(2rp-1)\varphi(t)+t\varphi'(t)]dt\right). \quad (35)$$

Since $|j_\alpha(y)| \le 1$ for all $y \in [0,\infty)$, then by virtue of the

$$(2rp-1)\varphi(t) - t\varphi'(t) \ge 0,$$

taking into account the conditions $p \in (0,2], r \in \mathbb{N}$ from (35) we have $\eta'(y) \ge 0$, for $y \ge \sigma$. Whence follows $\inf\{\eta(y) : \sigma \le y < \infty\} = \eta(\sigma)$, which is equivalent to equality

$$\inf\left\{\gamma_{\lambda,k,r,p,\frac{1}{p}}(\varphi,h) : \sigma \le \lambda < \infty\right\} = \gamma_{\sigma,k,r,p,\frac{1}{p}}(\varphi,h).$$

Then by virtue of the double inequality from Theorem 4, we obtain the required equality. Theorem 5 is proved.

4 Approximation in $L^2(\mathbb{R}^m)$

The exact inequality and its various generalizations have been the subject of study for many specialists in the last 50 years. Some historical information on the Jackson–Stechkin inequalities in $L^2(\mathbb{R}^m)$ can be found in [5, 8, 13–18].

Let $L^2 = L^2(R^m)$ be the Hilbert space of complex functions on R^m with a scalar product and norm

$$(f,g) = \int_{R^m} f(x)g(x)dx, \quad \|f\| = \sqrt{(f,f)}.$$

The Fourier transform of the function $f \in L^2$ is defined by this formula

$$\hat{f}(y) = \frac{1}{(2\pi)^{m/2}}\int_{R^m} f(x)e^{-ix\cdot y}dx,$$

where $x \cdot y = \sum_{l=1}^m x_l \cdot y_l$ is the scalar product of vectors x and y of R^m.

The function f can be decomposed through its Fourier transform $\hat{f}$ as:

$$f(x) = \frac{1}{(2\pi)^{m/2}}\int_{R^m} f(y)e^{ix\cdot y}dy. \quad (36)$$

For the Fourier transform in L^2 space, the Plancherell formula applies

$$(f,g) = (\hat{f},\hat{g}), \quad f,g \in L^2.$$

Let us denote by W_σ the class of exponential spherical integer functions $\sigma > 0$ belonging to the space. The class W_σ of integer functions consists of integer functions $g \in L^2$ such that the support *supp* $\hat{g}$ of Fourier transform lies in a Euclidean ball $B_{\sigma^m} = \left\{x \in R^m : |x| = \sqrt{(x,x)} \le \sigma\right\}$ of a radius $\sigma > 0$ and with a center at the origin of the space R^m. The best approximation of the function f of L^2 by the class W_σ is

$$A_\sigma f = \inf\{\|f-g\| : g \in W_\sigma\}.$$

The spherical shift with a step h is the operator S_h acting according to the rule

$$S_h f(x) = \frac{1}{|\mathbb{S}^{m-1}|}\int_{\mathbb{S}^{m-1}} f(x+h\xi)d\xi,$$

where $\mathbb{S}^{m-1}$ is a unit Euclidean sphere in R^m, $|\mathbb{S}^{m-1}|$ is its surface area. Let I be an identical operator, k is a positive number. Following H.P. Rustamov's operator $(I-S_hf)^{\frac{k}{2}}$ (see [17]), will be called a difference operator of order k with step h and will be denoted by Δ_h^k:

$$\Delta_h^k=\sum_{l=0}^{\infty}(-1)^l\binom{\frac{k}{2}}{l}S_h^l,$$

and the k-order continuity module of the function $f\in L^2(S^{m-1})$ will be the function of the variable $\tau>0$:

$$\omega_k(f,\tau)=\sup\left\{\|\Delta_h^kf\|:0<h\le\tau\right\}.$$

Denote by $K_n(\tau,k,m),\tau>0,k\ge1,m=2,3,...$ the exact constant K the Jackson–Stechkin inequality in $L^2(S^{m-1})$

$$A_\sigma(f)\le K\omega_k(f,\tau),f\in L^2(S^{m-1}),$$

let's put

$$K_\sigma(\tau,k,m)=\sup\left\{\frac{A_\sigma(f)}{\omega_k(f,\tau)}:f\in L^2(S^{m-1})\right\}.$$

Using the Plancherell formula, it is easy to see that the value of the best approximation for the function $f\in L^2(S^{m-1})$ is expressed by

$$A_\sigma^2f=\int_{|y|>\sigma}|\hat f(y)|^2dy.$$

It is known ([19], [13; 176]) that the S_h spherical shift operator with step $h>0$ acts on the function $e_y(x)=e^{ix\cdot y}$ as follows:

$$S_he_y(x)=\frac{1}{|\mathbb{S}^{m-1}|}\int_{\mathbb{S}^{m-1}}e^{i(x+h\xi)\cdot y}d\xi=$$

$$=\frac{e^{ix\cdot y}}{|S^{m-1}|}\int_{S^{m-1}}e^{ih\xi\cdot y}d\xi=j_{\frac{m-2}{2}}(h|y|)e_y(x). \tag{37}$$

Applying k times to both parts of equality (36) the spherical shift operator and using relation (37) we have

$$S_h^kf(x)=\frac{1}{(2\pi)^{m/2}}\int_{\mathbb{S}^{m-1}}(j_{\frac{m-2}{2}}(h|y|))^k\hat f(y)e^{ix\cdot y}dy. \tag{38}$$

From the definition of the difference operator by virtue of (38) we obtain

$$\Delta_h^kf(x)=\int_{\mathbb{R}^m}(1-j_{\frac{m-2}{2}}(h|y|))^{\frac{k}{2}}\hat f(y)e^{ix\cdot y}dy. \tag{39}$$

Hence, by virtue of the Plancherell formula from (39) we have

$$\|\Delta_h^kf\|^2=\int_{\mathbb{R}^m}(1-j_{\frac{m-2}{2}}(h|y|))^k|\hat f(y)|^2dy.$$

5 The Jackson–Stechkin Theorem in $L^2(\mathbb{R}^m)$

Theorem 6. Let $k\ge1,\sigma>0$. Then for any function $f\in L^2(\mathbb{S}^{m-1})$ it holds:

$$A_\sigma(f)\le\frac{\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}}\omega_k^{\frac{2}{k}}(B^rf,t)j_{\frac{m-2}{2}}(\sigma t)t^{m-1}dt\right)^{\frac{k}{2}}}{\sigma^{2r}\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}}j_{\frac{m-2}{2}}(\sigma t)t^{m-1}dt\right)^{\frac{k}{2}}},$$

where $q_{\frac{m-2}{2},1}$ is the first positive zero of the function $j_{\frac{m-2}{2}}(t)$.

Proof of Theorem 6. For any function $f \in L^2(\mathbb{R}^m)$ and by the equality

$$A_\sigma(f) = \left\{ \int_{|y|>\sigma} |\hat{f}(y)|^2 dy \right\}^{\frac{1}{2}}$$

and applying the Hölder inequality we have

$$A_\sigma^2(f) - \int_\sigma^\infty |\hat{f}(y)|^2 j_{\frac{m-2}{2}}(t|y|)dy = \int_\sigma^\infty |\hat{f}(y)|^2 (1 - j_{\frac{m-2}{2}}(t|y|))dy =$$

$$= \int_\sigma^\infty |\hat{f}(y)|^{2-\frac{2}{k}} |\hat{f}(y)|^{\frac{2}{k}} (1 - j_{\frac{m-2}{2}}(t|y|))dy =$$

$$\leq A_\sigma^{2-\frac{2}{k}}(f) \left(\sigma^{-4r} \int_\sigma^\infty y^{4r} |\hat{f}(y)|^2 (1 - j_{\frac{m-2}{2}}(t|y|))^k dy \right)^{\frac{1}{k}}. \tag{40}$$

Since the equality holds

$$\omega_k^2(B^r f, t) = \int_0^\infty y^{4r} |\hat{f}(y)|^2 (1 - j_{\frac{m-2}{2}}(t|y|))^k dy$$

then from (40) we have

$$A_\sigma^2(f) - \int_\sigma^\infty |\hat{f}(y)|^2 j_{\frac{m-2}{2}}(t|y|)dy \leq A_\sigma^{2-\frac{2}{k}}(f) \sigma^{-\frac{4r}{k}} \omega_k^{\frac{2}{k}}(B^r f, t). \tag{41}$$

By multiplying both parts of the inequality (41) by the Babenko weight function (see [8]) $v(t) = t^{2\alpha+1} T_{\tau_{\alpha,1}} V(t),\ t \in R_+,\ \alpha > \frac{1}{2},\ \alpha = \frac{m-2}{2}$, where

$$V(t) = \begin{cases} j_{\frac{m-2}{2}}(\sigma t), & 0 < t < \frac{q_{\alpha,1}}{\sigma} \\ 0, & t > \frac{q_{\alpha,1}}{\sigma}, \end{cases}$$

$$T_h f(x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_0^\pi f(\sqrt{x^2 + h^2 - 2xh\cos\varphi})(sin\varphi)^{2\alpha} d\varphi$$

and integrating them over t to zero to $q_{\alpha,1} = q_{\frac{m-2}{2},1}$ we obtain

$$\int_0^{\frac{2q_{\alpha,1}}{\sigma}} A_\sigma^2(f) v(t)dt - \int_0^{\frac{2q_{\alpha,1}}{\sigma}} \int_\sigma^\infty |\hat{f}(y)|^2 j_{\frac{m-2}{2}}(t|y|)dy v(t)dt \leq$$

$$\leq \sigma^{-\frac{4r}{k}} \int_0^{\frac{2q_{\alpha,1}}{\sigma}} A_\sigma^{2-\frac{2}{k}}(f) \omega_k^{\frac{2}{k}}(B^r f, t) v(t)dt, \tag{42}$$

where $q_{\frac{m-2}{2},1}$ is the smallest root of the function $j_{\frac{m-2}{2}}(t)$. Since in [8] the inequality

$$\int_0^{\frac{2q_{\alpha,1}}{\sigma}} j_{\frac{m-2}{2}}(t|y|) v(t)dt < 0, \text{ for all } |y| > 1 \tag{43}$$

has been proved, so from (42) and (43), we obtain

$$A_\sigma^2(f) \int_0^{\frac{2q_{\alpha,1}}{\sigma}} v(t)dt \leq \sigma^{-\frac{4r}{k}} A_\sigma^{2-\frac{2}{k}}(f) \int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f, t) v(t)dt.$$

Then, applying the properties of the generalized shift operator $T_h f$ (see [6–8]) we have

$$A_\sigma^{\frac{2}{k}}(f) \le \frac{\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f, t) j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt}{\sigma^{\frac{4r}{k}} \int_0^{\frac{2q_{\alpha,1}}{\sigma}} j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt}.$$

It follows that

$$A_\sigma(f) \le \frac{\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f, t) j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}}{\sigma^{2r}\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}}.$$

Theorem 6 is proved.

Corollary 1. Let $k \in \mathbb{R}_+, k \ge 1,\ q_{\alpha,1} > 0, \sigma > 0, \alpha = \frac{m-2}{2}$. Then for any function $f \in L^2(\mathbb{R}^m)$ the inequality holds

$$A_\sigma(f) \le \sigma^{-2r} \omega_k(B^r f, \frac{2q_{\alpha,1}}{\sigma}),$$

where $q_{\alpha,1}$ is the smallest root of the function $j_\alpha(t)$.

Proof of Corollary 1. Let's first show that the functionality of the

$$J_k(f, q_{\alpha,1}) = \frac{\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f, t) j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}}{\sigma^{2r}\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}}$$

is smaller than $\omega_k(f, \frac{2q_{\alpha,1}}{\sigma})$. Indeed, it follows from the monotonicity of $\omega_k(f,t)$ that

$$J_k(f, \frac{2q_{\alpha,1}}{\sigma}) = \frac{\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f, t) j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}}{\sigma^{2r}\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}} \le \sigma^{-2r} \omega_k(B^r f, \frac{2q_{\alpha,1}}{\sigma}). \tag{44}$$

From Theorem 4 and by virtue of (44) we have

$$A_\sigma(f) \le \frac{\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} \omega_k^{\frac{2}{k}}(B^r f, t) j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}}{\sigma^{2r}\left(\int_0^{\frac{2q_{\alpha,1}}{\sigma}} j_{\frac{m-2}{2}}(\sigma t) t^{m-1} dt\right)^{\frac{k}{2}}} = J_k(f, \frac{2q_{\alpha,1}}{\sigma}) \le \sigma^{-2r} \omega_k(B^r f, \frac{2q_{\alpha,1}}{\sigma}).$$

Remark. Earlier in [5, 8, 12] similar results were obtained. The proof of Corollary 1 of Theorem 6 given here is new, i.e. it differs from the proofs of the theorems of A.G. Babenko [8], D.V. Gorbachev [12] and V.I. Ivanov [5]. The obtained result, which is a consequence of Theorem 6, coincides with the exact result of A.G. Babenko [8] at $k \ge 1$. In the works [20–22] , direct theorems of the theory of approximation were proved without refining the coefficients

References

1 Тихомиров В.М. Некоторые вопросы теории приближений / В.Н. Тихомиров. — М.: Моск.гос.ун-т, 1976. — 296 с.

2 Корнейчук Н.П. Точные константы в теории приближений / Н.П. Корнейчук. — М.: Наука, 1987. — 346 с.

3 Колмогоров Н.А. Избранные труды / Н.А. Колмогоров. — М.: Наука, 1987. — 346 с.

4 Trimeche K. Generalized Harmonic Analysis and Wavelet Packets: An Elementary Treatment of Theory and Applications / K. Trimeche. — London: CRC Press Taylor Francis Group, 2001. — 320 p.

5 Иванов В.И. О точности неравенства Джексона в пространствах L_p на полупрямой со степенным весом / В.И. Иванов // Мат. заметки. – 2015. – 98. — № 5. — С. 684–694.

6 Левитан В.М. Разложение по функциям Бесселя в ряды и интегралы Фурье / В.М. Левитан // Успехи математических наук. — 1951. — 6. — № 2(42). — С. 102–143.

7 Платонов С.С. Гармонический анализ Бесселя и аппроксимация функций на полуоси / С.С. Платонов // Изв. РАН. Сер. мат. — 2007. — 71.— № 5. — С. 149–196.

8 Бабенко А.Г. Точное неравенство Джексона–Стечкина в пространстве $L^2(\mathbb{R}^m)$ / А.Г. Бабенко // Тр. Ин-та математики и механики УрО РАН. — 1998. — 5. — С. 183–198.

9 Тайков Л.В. Неравенство, содержащее наилучшие приближения и модуль непрерывности функций из L_2 / Л.В. Тайков // Мат. заметки. — 1976. — 20. — № 3. — С. 433–438.

10 Есмаганбетов М.Г. Поперечники классов в L_2 и минимизация точных констант n-неравенств типа Джексона / М.Г. Есмаганбетов // Мат. заметки. — 1999. — 65. — № 6. — С. 816–820.

11 Шабозов М.Ш. Наилучшие полиномиальные приближения и поперечники некоторых классов из L_2 / М.Ш. Шабозов, К. Тухлиев // Мат. заметки. — 2013. — 94. — № 6 — С. 908–917.

12 Горбачев Д.В. Экстремальные задачи для целых функций экспоненциального сферического типа / Д.В. Горбачев // Мат. заметки. — 2000. — 68. — № 2. — С. 159–166.

13 Стейн И. Введение в гармонический анализ на евклидовых пространствах / И. Стейн, Г. Вейс. — М.: Мир, 1974. — 332 с.

14 Юдин В.А. Экстремальные свойства функций и дизайны на торе / В.А. Юдин // Матем. заметки. — 1997. — 61. — № 4. — С. 637–640.

15 Ватсон Г.Н. Теория бесселевых функций. — Ч. 1. / Г.Н. Ватсон. — М.: ИЛ, 1949. — 787 с.

16 Lizorkin P.I. A theorem concerning approximation on the sphere / P.I. Lizorkin, S.M. Nikol'skii // Anal. Math. — 1983. — 9. — P. 207–221.

17 Рустамов Х. П. О приближении функций на сфере / Х.П. Рустамов // Изв. РАН. Сер. мат. — 1993. — 57. — № 5. — С. 127–148.

18 Никольский С.М. Аппроксимация функций на сфере / С.М. Никольский, П.И. Лизоркин // Изв. РАН. Сер. мат. — 1987. — 51. — № 3. — С. 635–651.

19 Бейтмен Г. Высшие трансцендентные функции. Функции Бесселя, функции параболического цилиндра, ортогональные многочлены. — Т. II. / Г.А. Бейтмен, А. Эрдейи. — М.: Наука, 1974. — 295 с.

20 Абилов В.А. Точные оценки скорости сходимости рядов Фурье-Бесселя / В.А. Абилов, Ф.В. Абилова, М.К. Керимов // Журн. вычисл. мат. и мат. физ. — 2015. — 55. — № 6. — С. 917–927.

21 Абилов В.А. О некоторых оценках преобразования Фурье-Бесселя в пространстве L_2 / В.А. Абилов, Ф.В. Абилова, М.К. Керимов // Журн. вычисл. мат. и мат. физ. — 2013. — 53.— № 10. — С. 1062–1082.

22 Платонов С.С. Обобщенные сдвиги Бесселя и некоторые вопросы теории приближений функций в метрике L_2. I / С.С. Платонов // Труды ПГУ. Математика. — 2000.— 7.— С. 70–83.

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L_2 метрикасындағы дәл Джексон–Стечкин теңсіздіктері және жалпыланған Ганкелдің ығыстыруы

Жұмыста f функциясының ең жақсы орташа квадраттық жуықтауы бойынша, дәрежелі салмағы бар жарты осьте бірнеше экстремалды есептер шешілген. Гильберт кеңістігінде L_2 салмағы $t^{2\alpha+1}$ дәрежесі болатын, f функциясының Бессельдің бірінші текті функциялары бойынша құрылған σ-ретті дербес Ганкел интегралдарымен ең жақсы жуықтауы $E_\sigma(f)$ және k-ретті үздіксіздіктің жалпыланған модулі $\omega_k f(B^r)f,t)$ арасындағы Джексон–Стечкин типті теңсіздіктер алынған, мұндағы B-екінші ретті дифференциалдық оператор.

Кілт сөздер: ең жақсы жуықтау, үзіліссіздік модулі, m-ретті жалпыланған, тегістік модулі, гильберт кеңістігі.

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Обобщенные сдвиги Ганкеля и точные неравенства Джексона–Стечкина в L_2

В работе решено несколько экстремальных задач о наилучшем среднеквадратическом приближении функции f на полуоси о степенным весом. В гильбертовом пространстве L_2 со степенным весом $t^{2\alpha+1}$ получены неравенства типа Джексона–Стечкина между величиной $E_\sigma(f)$ — наилучшего приближения функции f частичными интегралами Ганкеля порядка не выше σ по функциям Бесселя первого рода и обобщенным модулем непрерывности k-го порядка $\omega_k f(B^r f,t)$, где B — дифференциальный оператор второго порядка.

Ключевые слова: наилучшее приближение, обобщенный модуль гладкости m-го порядка, гильбертово пространство.

References

1 Tihomirov, V.M. (1976). *Nekotorye voprosy teorii priblizhenii [Some questions of approximation theory]*. Moscow: Moskovskii gosudarstvennyi universitet [in Russian].

2 Korneychuk, N.P. (1987). *Tochnye konstanty v teorii priblizhenii [Exact constants in approximation theory]*. Moscow: Nauka [in Russian].

3 Kolmogorov, N.A. (1987). *Izbrannye trudy [Selected Works]*. Moscow: Nauka [in Russian].

4 Trimeche, K. (2001). *Generalized Harmonic Analysis and Wavelet Packets: An Elementary Treatment of Theory and Applications.* London: CRC Press Taylor Francis Group.

5 Ivanov, V.I. (2015). O tochnosti neravenstva Dzheksona v prostranstvakh L_p na polupriamoi so stepennym vesom [Sharpness of Jackson's inequality in the spaces L_p on the half-line with a power-law weight]. *Matematicheskie zametki — Math notes, 98*(5), 684–694 [in Russian].

6 Levitan, V.M. (1951). Razlozhenie po funktsiiam Besselia v riady i integraly Fure [Expansion in Bessel functions in Fourier series and integrals]. *Uspekhi matematicheskikh nauk — Advances in Mathematical Sciences, 6*, 2(42), 102–143 [in Russian].

7 Platonov, S.S. (2007). Garmonicheskii analiz Besselia i approksimatsiia funktsii na poluosi [Harmonic Bessel Analysis and Approximation of Functions on the Semiaxis]. *Izvestiia RAN. Seriia matematicheskaia — News of Russian Academy of Sciences. Series Mathematics, 71*(5), 149–196 [in Russian].

8 Babenko, A.G. (1998). Tochnoe neravenstvo Dzheksona–Stechkina v prostranstve $L^2(\mathbb{R}^m)$ [Sharp Jackson-Stechkin inequality in the space $L^2(\mathbb{R}^m)$]. *Trudy Instituta matematiki i mekhaniki Uralskogo otdeleniia Rossiiskoi akademii nauk — Proceedings of the Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences, 5*, 183–198 [in Russian].

9 Taykov, L.V. (1976). Neravenstvo, soderzhashchee nailuchshie priblizheniia i modul nepreryvnosti funktsii iz L^2 [Inequality containing the best approximations and the modulus of continuity of functions from L^2]. *Matematicheskie zametki — Math notes, 20*(3), 433–438 [in Russian].

10 Esmaganbetov, M.G. (1999). Poperechniki klassov v L_2 i minimizatsiia tochnykh konstant n neravenstv tipa Dzheksona [Widths of classes from L^2 and minimization of sharp constants in inequalities typ Jackson]. *Matematicheskie zametki — Math notes, 65*(6), 816–820 [in Russian].

11 Shabozov, M.Sh.,& Tukhliev, K. (2013). Nailuchshie polinomialnye priblizheniia i poperechniki nekotorykh klassov iz L^2 [Best polynomial approximations and widths of some functional classes in L^2]. *Matematicheskie zametki — Math notes, 94*(6), 908–917 [in Russian].

12 Gorbachev, D.V. (2000). Ekstremalnye zadachi dlia tselykh funktsii eksponentsialnogo sfericheskogo tipa [Extremal problems for entire functions of exponential spherical type]. *Matematicheskie zametki — Math notes, 68*(2), 159–166 [in Russian].

13 Stein, E., & Weiss, G. (1974). *Vvedenie v garmonicheskii analiz na evklidovykh prostranstvakh [Introduction to harmonic analysis on Euclidean spaces]*. Moscow: Mir [in Russian].

14 Yudin, V.A. (1997). Ekstremalnye svoistva funktsii i dizainy na tore [Extremal properties of functions and designs on a torus]. *Matematicheskie zametki — Math notes, 61*(4), 637–640 [in Russian].

15 Vatson, G.N. (1949). Teoriia besselevykh funktsii [Theory of Bessel functions]. Moscow: Inostrannaia literatura [in Russian].

16 Lizorkin, P.I., & Nikol'skii, S.M. (1983). A theorem concerning approximation on the sphere. *Anal. Math., 9*, 207–221.

17 Rustamov, Kh.P. (1993). O priblizhenii funktsii na sfere [On approximation of functions on the sphere]. *Izvestiia Rossiiskoi akademii nauk. Seriia matematicheskaia — News of Russian Academy of Sciences. Series Mathematics, 57*(5), 127–148 [in Russian].

18 Nikol'skii, S.M., & Lizorkin, P.I. (1987). Approksimatsiia funktsii na sfere [Approximation of functions on the sphere]. *Izvestiia Rossiiskoi akademii nauk. Seriia matematicheskaia — News of Russian Academy of Sciences. Series Mathematics, 51*(3), 635–651 [in Russian].

19 Bateman, G., & Erdeyi, A. (1974). *Vysshie transtsendentnye funktsii. Funktsii Besselia, funktsii parabolicheskogo tsilindra, ortogonalnye mnogochleny [Higher transcendental functions. Bessel functions, parabolic cylinder functions, orthogonal polynomials]*. Moscow: Nauka [in Russian].

20 Abilov, V.A., Abilova, F.V., & Kerimov, M.K. (2015). Tochnye otsenki skorosti skhodimosti riadov Fure-Besselia [Accurate Estimates of the Rate of Convergence of Fourier-Bessel Series]. *Zhurnal vychislitelnoi matematiki i matematicheskoi fiziki — Journal of Computational Mathematics and Mathematical Physics, 55*(6), 917–927 [in Russian].

21 Abilov, V.A., Abilova, F.V., & Kerimov, M.K. (2013). O nekotorykh otsenkakh preobrazovaniia

Fure-Besselia v prostranstve L_2 [On some estimates of the Fourier-Bessel transform in the space L_2]. *Zhurnal vychislitelnoi matematiki i matematicheskoi fiziki — Journal of Computational Mathematics and Mathematical Physics, 53*(10), 1062–1082 [in Russian].

22 Platonov, S.S. (2000). Obobshchennye sdvigi Besselia i nekotorye voprosy teorii priblizhenii funktsii v metrike L_2. I [Generalized Bessel shifts and some problems in the theory of approximation of functions in the metric L^2. I]. *Trudy Petrozavodskogo gosudarstvennogo universiteta. Matematika — Proceedings of Petrozavodsk State University. Mathematics, 7*, 70–83 [in Russian].