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Boundary control problem for the heat transfer equation associated with heating process of a rod

In this paper, we consider a boundary control problem for a parabolic equation in a segment. In the part of the domain's bound it is a given value of the solution and it is required to find controls to get the average value of the solution. The given control problem is reduced to a system of Volterra integral equations of the first kind. By the mathematical-physics methods it is proved that like this control functions exist over some domain, the necessary estimates were found and obtained.

Keywords: Heat conduction equation, system of integral equations, initial-boundary value problem, Laplace transform.

1 Introduction and statement of the Problem

Consider the following heat exchange process along the domain $\Omega = \{(x,t) : 0 < x < l, \quad t > 0\}$:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right), \quad (x,t) \in \Omega, \tag{1}$$

with boundary value conditions

$$u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t), \quad t > 0, \tag{2}$$

and an initial value condition

$$u(x,0) = 0, \quad 0 \le x \le l. \tag{3}$$

Assume that the function $k(x) \in C^1\big([0,l]\big)$ satisfies a condition

$$k(x) > 0, \quad 0 \le x \le l.$$

Let $M_j > 0$ be some given constants. We say that the functions $\mu_j(t)$ are an *admissible control* if this functions are differentiable on the half-line $t \ge 0$ and satisfies the following constraints

$$\mu_j(0) = 0, \quad |\mu_j(t)| \le M_j, \quad j = 1,2.$$

Consider the following eigenvalue problem

$$\frac{d}{dx}\left(k(x)\frac{dv_k(x)}{dx}\right) + \lambda_k\, v_k(x) = 0, \quad 0 < x < l, \tag{4}$$

with boundary value conditions

$$v_k(0) = v_k(l) = 0, \quad 0 \le x \le l. \tag{5}$$

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It is well-known that this problem is self-adjoint in $L_2(\Omega)$ and there exists a sequence of eigenvalues $\{\lambda_k\}$ so that

$$\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k \to \infty, \quad k \to \infty.$$

The corresponding eigenfunctions v_k form a complete orthonormal system $\{v_k(x)\}_{k\epsilon N}$ in $L_2(\Omega)$ and these function belong to $C(\bar{\Omega})$, where $\bar{\Omega} = \Omega \cup \partial\Omega$ (see, [1]).

Problem A. For the given functions $\theta_j(t)$ Problem A consists in looking for the admissible controls $\mu_j(t)$ such that the solution $u(x,t)$ of initial-boundary value problem (1)-(3) exists and for all $t > 0$ satisfies the equations

$$\int_0^l v_j(x)\, u(x,t)dx = \theta_j(t), \quad j = 1, 2. \tag{6}$$

We recall that the time-optimal control problem for partial differential equations of the parabolic type was first investigated in [2] and [3]. More recent results concerned with this problem were established in [4–13]. Detailed information on the optimal control problems for a distributed parameter systems is given in [14] and in monographs [15, 16] and [17].

General numerical optimization and optimal boundary control have been studied in a great number of publications such as [18]. The practical approaches to optimal control of the heat equation are described in publications like [19].

2 *System of integral equations*

Definition 1. By the solution of problem (1)–(3) we understand the function $u(x,t)$ represented in the form

$$u(x,t) = \mu_1(t) + \frac{x}{l}[\mu_2(t) - \mu_1(t)] - v(x,t), \tag{7}$$

where the function $v(x,t) \in C^{2,1}_{x,t}(\Omega) \cap C(\bar{\Omega})$, $v_x \in C(\bar{\Omega})$ is the solution to the problem:

$$v_t = \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial v}{\partial x}\right) + \mu_1'(t) + \frac{x}{l}\,[\mu_2'(t) - \mu_1'(t)] + \frac{k'(x)}{l}\,[\mu_1(t) - \mu_2(t)],$$

with the boundary value conditions

$$v(0,t) = 0, \quad v(l,t) = 0,$$

and the initial value condition

$$v(x,0) = 0, \quad 0 \leq x \leq l.$$

Set

$$a_k = \int_0^l v_k(x)dx, \quad b_k = \int_0^l \frac{x}{l}\, v_k(x)dx, \quad c_k = \int_0^l \frac{k'(x)}{l}\, v_k(x)dx. \tag{8}$$

Consequently,

$$v(x,t) = \sum_{k=1}^{\infty} v_k(x) \times$$

$$\times \int_0^t e^{-\lambda_k(t-s)} \big(a_k\, \mu_1'(s) + b_k\, [\mu_2'(s) - \mu_1'(s)] + c_k\, [\mu_1(s) - \mu_2(s)]\big)\, ds, \tag{9}$$

where a_k, b_k and c_k are defined by (8).

From (7) and (9), we get the solution of the problem (1)–(3) (see, [1]):

$$u(x,t) = \mu_1(t) + \frac{x}{l}\left[\mu_2(t) - \mu_1(t)\right] - \sum_{k=1}^{\infty} v_k(x) \times$$

$$\times \int\limits_0^t e^{-\lambda_k (t-s)} \big(a_k\, \mu_1'(s) + b_k \left[\mu_2'(s) - \mu_1'(s)\right] + c_k \left[\mu_1(s) - \mu_2(s)\right]\big)\, ds.$$

We know that the eigenvalues λ_k of boundary value problem (4), (5) satisfies the following inequalities

$$\lambda_k \geq 0, \quad k = 1, 2, ...$$

Indeed, since

$$\frac{d}{dx}\left(k(x)\frac{dv_k(x)}{dx}\right) + \lambda_k\, v_k(x) = 0, \quad 0 < x < l,$$

then we have

$$\lambda_k = -\int\limits_0^l \frac{d}{dx}\left(k(x)\frac{dv_k(x)}{dx}\right) v_k(x) dx = \int\limits_0^l k(x)\, |v_k'(x)|^2 dx \geq 0. \tag{10}$$

According to Jentsch's theorem $v_1(x) > 0$ (see, [20, 21]). Then, from $k(x) > 0$ and the estimate (10), we have

$$\lambda_1 > 0.$$

From condition (6) and the solution of the problem (1)–(3), we write

$$\theta_j(t) = \int\limits_0^l v_j(x)\, u(x,t) dx =$$

$$= \int\limits_0^l \left(\mu_1(t) + \frac{x}{l}\left[\mu_2(t) - \mu_1(t)\right]\right) v_j(x)\, dx - \sum_{k=1}^{\infty} \int\limits_0^l v_j(x)\, v_k(x)\, dx \times$$

$$\times \int\limits_0^t e^{-\lambda_k (t-s)} \big(a_k\, \mu_1'(s) + b_k \left[\mu_2'(s) - \mu_1'(s)\right] + c_k \left[\mu_1(s) - \mu_2(s)\right]\big)\, ds =$$

$$= \int\limits_0^l \left(\mu_1(t) + \frac{x}{l}\left[\mu_2(t) - \mu_1(t)\right]\right) v_j(x) dx -$$

$$- \int\limits_0^t e^{-\lambda_j (t-s)} \big(a_j\, \mu_1'(s) + b_j \left[\mu_2'(s) - \mu_1'(s)\right] + c_j \left[\mu_1(s) - \mu_2(s)\right]\big)\, ds =$$

$$= \int\limits_0^l \left(\mu_1(t) + \frac{x}{l}\left[\mu_2(t) - \mu_1(t)\right]\right) v_j(x) dx - a_j \mu_1(t) - b_j \left[\mu_2(t) - \mu_1(t)\right] +$$

$$+\int_0^t (a_j\,\lambda_j - b_j\,\lambda_j + c_j)\,e^{-\lambda_j(t-s)}\,\mu_1(s)ds + \int_0^t (b_j\,\lambda_j - c_j)\,e^{-\lambda_j(t-s)}\,\mu_2(s)ds. \tag{11}$$

Note that

$$\int_0^l \left(\mu_1(t) + \frac{x}{l}\,[\mu_2(t) - \mu_1(t)]\right) v_j(x)dx = a_j\mu_1(t) + b_j\,[\mu_2(t) - \mu_1(t)], \tag{12}$$

where a_j and b_j are defined by (8).

As a result, from (11) and (12), we obtain

$$\theta_j(t) = \int_0^t (a_j\,\lambda_j - b_j\,\lambda_j + c_j)\,e^{-\lambda_j(t-s)}\,\mu_1(s)ds+$$

$$+\int_0^t (b_j\,\lambda_j - c_j)\,e^{-\lambda_j(t-s)}\,\mu_2(s)ds.$$

Let

$$B_{1j}(t) = \alpha_j\,e^{-\lambda_j t}, \quad B_{2j}(t) = \beta_j\,e^{-\lambda_j t}, \quad j = 1, 2, \tag{13}$$

where

$$\alpha_j = a_j\,\lambda_j - b_j\,\lambda_j + c_j, \quad \beta_j = b_j\,\lambda_j - c_j. \tag{14}$$

Then we get a system of the main integral equations

$$\int_0^t B_{1j}(t-s)\,\mu_1(s)ds + \int_0^t B_{2j}(t-s)\,\mu_2(s)ds = \theta_j(t), \quad t > 0, \quad j = 1, 2. \tag{15}$$

Denote by $W(M_0)$ the set of functions $\theta \in W_2^2(-\infty, +\infty)$, $\theta(t) = 0$ for $t \leq 0$ which satisfy the condition

$$\|\theta\|_{W_2^2(R_+)} \leq M_0.$$

Theorem 1. There exists $M_0 > 0$ such that for any functions $\theta_j \in W(M_0)$ the solution $\mu_j(t)$ of system (15) exists and satisfies conditions

$$|\mu_j(t)| \leq M_j, \quad j = 1, 2.$$

3 Proof of the Theorem 1

To solve system (15), we use the Laplace transform method. We introduce the notation

$$\widetilde{\mu}_j(p) = \int_0^\infty e^{-pt}\,\mu_j(t)\,dt, \quad p = a + i\,\xi, \quad a > 0.$$

Then, we use the Laplace transform

$$\widetilde{\theta}_j(p) = \int_0^\infty e^{-pt}\,dt \int_0^t B_{1j}(t-s)\mu_1(s)ds + \int_0^\infty e^{-pt}\,dt \int_0^t B_{2j}(t-s)\mu_2(s)ds =$$

$$= \widetilde{B}_{1j}(p)\,\widetilde{\mu}_1(p) + \widetilde{B}_{2j}(p)\,\widetilde{\mu}_2(p). \tag{16}$$

According to (13), we get

$$\widetilde{B}_{1j}(p) = \int_0^\infty B_{1j}(t)e^{-pt}\,dt = \frac{\alpha_j}{p+\lambda_j}, \tag{17}$$

and

$$\widetilde{B}_{2j}(p) = \int_0^\infty B_{2j}(t)e^{-pt}\,dt = \frac{\beta_j}{p+\lambda_j}, \quad j = 1, 2, \tag{18}$$

where α_j, β_j are defined by (14).

Assume that the α_j, β_j $(j = 1, 2)$ satisfies the following condition

$$\alpha_1\,\beta_2 - \alpha_2\,\beta_1 \neq 0.$$

Consequently, from system (16) and (17), (18), we can obtain

$$\widetilde{\mu}_1(p) = \frac{\beta_1\,(\lambda_2 + p)}{\alpha_2\,\beta_1 - \alpha_1\,\beta_2}\,\widetilde{\theta}_2(p) - \frac{\beta_2\,(\lambda_1 + p)}{\alpha_2\,\beta_1 - \alpha_1\,\beta_2}\,\widetilde{\theta}_1(p), \tag{19}$$

and

$$\widetilde{\mu}_2(p) = \frac{\alpha_1\,(\lambda_2 + p)}{\alpha_1\,\beta_2 - \alpha_2\,\beta_1}\,\widetilde{\theta}_2(p) - \frac{\alpha_2\,(\lambda_1 + p)}{\alpha_1\,\beta_2 - \alpha_2\,\beta_1}\,\widetilde{\theta}_1(p). \tag{20}$$

Then, when $a \to 0$ from (19) and (20), we obtain the following equalities

$$\mu_1(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{\beta_1\,(\lambda_2 + i\xi)}{\alpha_2\,\beta_1 - \alpha_1\,\beta_2}\,\widetilde{\theta}_2(i\xi) - \frac{\beta_2\,(\lambda_1 + i\xi)}{\alpha_2\,\beta_1 - \alpha_1\,\beta_2}\,\widetilde{\theta}_1(i\xi)\right)e^{i\xi t}d\xi, \tag{21}$$

and

$$\mu_2(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{\alpha_1\,(\lambda_2 + i\xi)}{\alpha_1\,\beta_2 - \alpha_2\,\beta_1}\,\widetilde{\theta}_2(i\xi) - \frac{\alpha_2\,(\lambda_1 + i\xi)}{\alpha_1\,\beta_2 - \alpha_2\,\beta_1}\,\widetilde{\theta}_1(i\xi)\right)e^{i\xi t}d\xi. \tag{22}$$

Lemma 1. Let $\theta(t) \in W(M_0)$. Then for the image of the function $\theta(t)$ the following inequality

$$\int_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)|\sqrt{1+\xi^2}d\xi \leq C\,\|\theta\|_{W_2^2(R_+)}$$

is valid.

Proof. We calculate the Laplace transform of a function $\theta(t)$ as follows

$$\widetilde{\theta}(a+i\xi) = \int_0^\infty e^{-(a+i\xi)t}\theta(t)\,dt = -\theta(t)\,\frac{e^{-(a+i\xi)t}}{a+i\xi}\bigg|_{t=0}^{t=\infty} + \frac{1}{a+i\xi}\int_0^\infty e^{-(a+i\xi)t}\,\theta'(t)\,dt,$$

then we get

$$(a+i\xi)\,\widetilde{\theta}(a+i\xi) = \int_0^\infty e^{-(a+i\xi)t}\,\theta'(t)\,dt,$$

and for $a \to 0$ we have

$$i\xi\,\widetilde{\theta}(i\xi) = \int\limits_0^\infty e^{-i\xi t}\,\theta'(t)\,dt.$$

Also, we can write the following equality

$$(i\xi)^2\,\widetilde{\theta}(i\xi) = \int\limits_0^\infty e^{-i\xi t}\,\theta''(t)\,dt.$$

Then we have

$$\int\limits_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)|^2(1+\xi^2)^2 d\xi \le C_1\,\|\theta\|^2_{W_2^2(R_+)}. \tag{23}$$

Consequently, according to (23) we get the following estimate

$$\int\limits_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)|\sqrt{1+\xi^2}d\xi = \int\limits_{-\infty}^{+\infty} \frac{|\widetilde{\theta}(i\xi)|(1+\xi^2)}{\sqrt{1+\xi^2}} \le$$

$$\le \Big(\int\limits_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)|^2(1+\xi^2)^2 d\xi\Big)^{1/2}\Big(\int\limits_{-\infty}^{+\infty} \frac{1}{1+\xi^2}d\xi\Big)^{1/2} \le C\,\|\theta\|_{W_2^2(R_+)}.$$

Lemma 1 is proved.

Proof of Theorem 1. Note that

$$|\lambda_j + i\,\xi| = \sqrt{\lambda_j^2+\xi^2} \le (1+\lambda_j)\,\sqrt{1+\xi^2}.$$

According to (21), (22) and Lemma 1, we obtain the estimates

$$|\mu_1(t)| \le \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\Big|\frac{\beta_1}{\alpha_2\,\beta_1-\alpha_1\,\beta_2}\Big|\,|\lambda_2+i\xi|\,|\widetilde{\theta}_2(i\xi)|d\xi+$$

$$+\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\Big|\frac{\beta_2}{\alpha_2\,\beta_1-\alpha_1\,\beta_2}\Big|\,|\lambda_1+i\xi|\,|\widetilde{\theta}_1(i\xi)|d\xi \le$$

$$\le \frac{C_1\,(1+\lambda_2)}{2\pi}\int\limits_{-\infty}^{+\infty}\sqrt{1+\xi^2}\,|\widetilde{\theta}_2(i\xi)|\,d\xi + \frac{C_2\,(1+\lambda_1)}{2\pi}\int\limits_{-\infty}^{+\infty}\sqrt{1+\xi^2}\,|\widetilde{\theta}_1(i\xi)|d\xi \le$$

$$\le \frac{C_1\,C\,(1+\lambda_2)}{2\pi}\,\|\theta_2\|_{W_2^2(R_+)} + \frac{C_2\,C\,(1+\lambda_1)}{2\pi}\,\|\theta_1\|_{W_2^2(R_+)} \le$$

$$\le \frac{C_1\,C\,(1+\lambda_2)}{2\pi}\,M_0 + \frac{C_2\,C\,(1+\lambda_1)}{2\pi}\,M_0 = M_1,$$

and

$$|\mu_2(t)| \le \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\Big|\frac{\alpha_1}{\alpha_1\,\beta_2-\alpha_2\,\beta_1}\Big|\,|\lambda_2+i\xi|\,|\widetilde{\theta}_2(i\xi)|d\xi+$$

$$+\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left|\frac{\alpha_2}{\alpha_1\,\beta_2-\alpha_2\,\beta_1}\right||\lambda_1+i\xi|\,|\widetilde{\theta}_1(i\xi)|d\xi\le$$

$$\le\frac{C_3\,(1+\lambda_2)}{2\pi}\int\limits_{-\infty}^{+\infty}\sqrt{1+\xi^2}\,|\widetilde{\theta}_2(i\xi)|\,d\xi+\frac{C_4\,(1+\lambda_1)}{2\pi}\int\limits_{-\infty}^{+\infty}\sqrt{1+\xi^2}\,|\widetilde{\theta}_1(i\xi)|d\xi\le$$

$$\le\frac{C_3\,C\,(1+\lambda_2)}{2\pi}\,\|\theta_2\|_{W_2^2(R_+)}+\frac{C_4\,C\,(1+\lambda_1)}{2\pi}\,\|\theta_1\|_{W_2^2(R_+)}\le$$

$$\le\frac{C_3\,C\,(1+\lambda_2)}{2\pi}\,M_0+\frac{C_4\,C\,(1+\lambda_1)}{2\pi}\,M_0=M_2.$$

Theorem 1 is proved.

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Сырықты қыздыру процесіне байланысты жылуды бөлу теңдеуінің шекаралық мәнін бақылау есебі

Мақалада параболалық теңдеу үшін шекаралық бақылау есебі қарастырылған. Температураның мәні берілген аумақтың шекаралық бөлігінде берілген және температураның орташа мәнін алу үшін басқару элементтерін табу қажет. Берілген басқару есебі бірінші типті Вольтерра интегралдық теңдеулер жүйесіне келтірілді. Математиканың физикалық әдістерін қолдану арқылы белгілі бір салада ұқсас басқару функцияларының бар екендігі дәлелденді және қажетті бағалар алынды.

Кілт сөздер: жылуалмасу теңдеуі, интегралдық теңдеулер жүйесі, бастапқы-шекаралық есеп, Лаплас алмастыру.

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Задача граничного управления для уравнения теплопереноса, связанного с процессом нагрева стержня

В статье рассмотрена задача граничного управления для параболического уравнения на отрезке. В части границы данной области задано значение решения, и требуется найти управление, чтобы получить среднее значение решения. Данная задача управления сведена к системе интегральных уравнений Вольтерра первого рода. Методами математической физики доказано, что подобные функции управления существуют в некоторой области, найдены и получены необходимые оценки.

Ключевые слова: уравнение теплопроводности, система интегральных уравнений, начально-краевая задача, преобразование Лапласа.

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