DOI 10.31489/2022M4/107-116

UDC 519.624

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On bounded solutions of linear systems of differential equations with unbounded coefficients

This paper deals with a problem of finding a bounded solution of a system of nonhomogeneous linear differential equations with an unbounded matrix of coefficients on a finite interval. The right-hand side of the equation belongs to a space of continuous functions bounded with some weight; the weight function is chosen taking into account the behavior of the coefficient matrix. The problem is studied using a modified version of the parameterization method with non-uniform partitioning. Necessary and sufficient conditions of well-posedness of the problem are obtained in terms of a bilaterally infinite matrix of special structure.

Keywords: ordinary differential equation, singular boundary-value problem, well-posedness, parameterization method, bounded solution, linear system, unbounded coefficients.

In various branches of applied mathematics there arise problems leading to systems of ordinary differential equations involving singularities or defined on an infinite interval. Numerous works [1–12] have been studied the existence of bounded solutions of such problems. In [6], the boundedness condition for a solution at a singular point is replaced by an equivalent relation in a neighborhood of this point, namely, the equation of a stable initial manifold generated in the neighborhood of the singular point by the total set of bounded solutions of the system. In [8], the existence and approximation of a bounded (on the whole axis) solution of a linear ordinary differential equation are investigated by using the parameterization method. In this paper, we apply the parameterization method with non-uniform partition of the interval (0, T) to the linear differential equation

$$\frac{dx}{dt} = A(t)x + f(t), \quad x \in \mathbb{R}^n, \quad t \in (0,T), \tag{1}$$

where $A(t)$ and $f(t)$ are continuous on $(0,T)$, $\|A(t)\| = \max\limits_i \sum\limits_{j=1}^{n} |a_{i,j}(t)| = \alpha(t)$. We assume that the function $\alpha(t)$ is continuous on $(0,T)$ and satisfies the following conditions:

$$\int\limits_0^{T/2} \alpha(t)dt = \infty, \quad \lim_{t\to 0+0} \alpha(t) = \infty, \quad \int\limits_{T/2}^{T} \alpha(t)dt = \infty, \quad \lim_{t\to T-0} \alpha(t) = \infty.$$

We introduce the following spaces:
$\widetilde{C}((0,T),\mathbb{R}^n)$ is the space of functions $x:(0,T)\to\mathbb{R}^n$ that are continuous and bounded on $(0,T)$, equipped with the norm

$$\|x\|_1 = \sup_{t\in(0,T)} \|x(t)\|;$$

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$\widetilde{C}_{1/\alpha}((0,T),\mathbb{R}^n)$ is the space of functions $f:(0,T)\to\mathbb{R}^n$ that are continuous and bounded on $(0,T)$ with the weight $1/\alpha(t)$, equipped with the norm

$$\|f\|_\alpha = \sup_{t\in(0,T)} \|f(t)\alpha(t)\|;$$

m_n is the space bounded bilaterally infinite sequences $\lambda_s \in \mathbb{R}^n$ with the norm

$$\|\lambda\|_2 = \|(\ldots,\lambda_r,\lambda_{r+1},\ldots)'\|_2 = \sup_r \|\lambda_r\|, \quad r\in\mathbb{Z};$$

$L(m_n)$ is the space of all bounded linear operators mapping m_n into itself, equipped with the induced norm.

Let us choose a number $\theta>0$ and make the partition $(0,T)=\bigcup\limits_{r=-\infty}^{\infty}[t_{r-1},t_r)$ by the points t_r, $r\in\mathbb{Z}$, defined as follows: $t_0=T/2$, $\int\limits_{t_{r-1}}^{t_r}\alpha(t)dt=\theta$.

Let $\overline{h}(\theta)$ be the bilaterally infinite sequence of the partition step-sizes $h_r=t_r-t_{r-1}$, $r\in\mathbb{R}$. We denote by $x_r(t)$ the restriction of a function $x(t)\in\widetilde{C}((0,T),\mathbb{R}^n)$ to the r-th subinterval and introduce one more space $m_n(\overline{h})$ of bounded bilaterally infinite sequences of functions $x_r(t)$, $r\in\mathbb{Z}$, that are continuous and bounded on $[t_{r-1},t_r)$, equipped with the norm

$$\|x[t]\|_3 = \|(\ldots,x_r(t),x_{r+1}(t),\ldots)'\|_3 = \sup_r \sup_{t\in[t_{r-1},t_r)} \|x_r(t)\|.$$

Definition 1. We call Problem 1_α the problem of finding a bounded on $(0,T)$ solution of Eq. (1) with $f(t)\in\widetilde{C}_{1/\alpha}((0,T),\mathbb{R}^n)$.

The existence of a solution $x(t)\in\widetilde{C}((0,T),\mathbb{R}^n)$ of Problem 1_α is equivalent to the existence of a solution $x[t]\in m_n(\overline{h})$ of the multipont problem for the equations

$$\frac{dx_r}{dt} = A(t)x_r + f(t), \quad t\in[t_{r-1},t_r), \tag{2}$$

subject to the gluing conditions for $x(t)$ at the interior partition points:

$$\lim_{t\to t_r-0} x_r(t) = x_{r+1}(t_r), \quad r\in\mathbb{Z}. \tag{3}$$

Note that the derivative $\frac{dx_r}{dt}\big|_{t=t_{r-1}}$ in Eq. (2) is understood as the right-sided limit $\lim\limits_{t\to t_{r-1}+0}\frac{dx_r}{dt}$.

Indeed, let $\widehat{x}(t)$ be a solution of Problem 1_α. Let us show that the system of its restrictions to the partition subintervals, $\widehat{x}[t]=(\ldots,\widehat{x}_r(t),\widehat{x}_{r+1}(t),\ldots)'$, belongs to $m_n(\overline{h})$ and satisfies Eq. (2) and conditions (3).

Since $\widehat{x}(t)$ is a solution of Eq. (1), it is continuously differentiable on $(0,T)$. Hence $\widehat{x}_r(t)$ and $\frac{d\widehat{x}_r}{dt}$, $r\in\mathbb{Z}$, are continuous on $[t_{r-1},t_r)$. The boundedness of the function $\widehat{x}(t)$ on $(0,T)$ implies that the functions $\widehat{x}_r(t)$, $r\in\mathbb{Z}$, are bounded on $[t_{r-1},t_r)$, and $\widehat{x}[t]\in m_n(\overline{h})$.

The function system $\widehat{x}[t]$ satisfies Eq. (2) for all $t\in[t_{r-1},t_r)$, $r\in\mathbb{Z}$:

$$\frac{d\widehat{x}_r(t)}{dt} = \frac{d\widehat{x}(t)}{dt} = A(t)\widehat{x}(t)+f(t) = A(t)\widehat{x}_r(t)+f(t).$$

The continuity of $\widehat{x}(t)$ on $(0,T)$ implies the existence of the left-sided limits

$$\lim_{t\to t_r-0}\widehat{x}_r(t) = \lim_{t\to t_r-0}\widehat{x}(t) = \widehat{x}(t_r), \quad r\in\mathbb{Z},$$

that is, conditions (3) are satisfied:

$$\lim_{t\to t_r-0}\widehat{x}_r(t)=\widehat{x}(t_r)=\widehat{x}_{r+1}(t_r).$$

Let us now show that if $\widetilde{x}[t]=(\ldots,\widetilde{x}_r(t),\widetilde{x}_{r+1}(t),\ldots)'\in m_n(\overline{h})$ is a solution of problem (2),(3), then the function $\widetilde{x}(t)$, defined as $\widetilde{x}(t)=\widetilde{x}_r(t)$, $t\in[t_{r-1},t_r)$, $r\in\mathbb{Z}$, is a solution of Problem 1_α.

It follows from (3) that $\widetilde{x}(t)$ is continuous on $(0,T)$. Since the functions $\widetilde{x}_r(t)$, $r\in\mathbb{Z}$, satisfy Eq. (2) for all $t\in[t_{r-1},t_r)$, the function $\widetilde{x}(t)$ is continuously differentiable for all $x\in(0,T)$ except the points $t=t_r$, $r\in\mathbb{Z}$, and

$$\frac{d\widetilde{x}(t)}{dt}=\frac{d\widetilde{x}_r(t)}{dt}=A(t)\widetilde{x}_r(t)+f(t)=A(t)\widetilde{x}(t)+f(t),$$
$$t\in(0,T)\setminus\{t=t_r,r\in\mathbb{Z}\}.$$

The function $\widetilde{x}(t)$ has the right-hand derivative at the points $t=t_r$, $r\in\mathbb{Z}$. Let t_k be one of these points, and let us consider Eq. (1) on the intervals $[t_{k-1},t_k)$ and $[t_k,t_{k+1})$:

$$\frac{d\widetilde{x}(t)}{dt}=A(t)\widetilde{x}(t)+f(t),\quad [t_{k-1},t_k),\tag{4}$$

$$\frac{d\widetilde{x}(t)}{dt}=A(t)\widetilde{x}(t)+f(t),\quad [t_k,t_{k+1}).\tag{5}$$

From (4) and the continuity of $A(t)$, $f(t)$, and $\widetilde{x}(t)$ on $(0,T)$, we have

$$\lim_{t\to t_k-0}\frac{d\widetilde{x}}{dt}=A(t_k)\widetilde{x}(t_k)+f(t_k),$$

i.e., at $t=t_k$ there exists the left-hand derivative of $\widetilde{x}(t)$:

$$\dot{\widetilde{x}}(t_k-0)=A(t_k)\widetilde{x}(t_k)+f(t_k).$$

Taking into account (5) and the existence of $\dot{\widetilde{x}}(t_k+0)=A(t_k)\widetilde{x}(t_k)+f(t_k)$, we obtain that the continuous derivative of $\widetilde{x}$ exists at $t=t_k$, and Eq. (1) holds at this point.

Thus, the function $\widetilde{x}(t)$ is continuously differentiable on $(0,T)$ and satisfies Eq. (1) for all $x\in(0,T)$. It follows from $\widetilde{x}[t]\in m_n(\overline{h})$ that $\widetilde{x}(t)$ is a bounded solution of Eq. (1).

Let λ_r denote the values of $x_r(t)$ at $t=t_{r-1}$, $r\in\mathbb{Z}$. Setting $u_r(t)=x_r(t)-\lambda_r$ on each partition subinterval $[t_{r-1},t_r)$, we obtain the following boundary value problem with parameter:

$$\frac{du_r}{dt}=A(t)[u_r+\lambda_r]+f(t),\quad t\in[t_{r-1},t_r),\quad u_r(t_{r-1})=0,\tag{6}$$

$$\lim_{t\to t_r-0}u_r(t)+\lambda_r=\lambda_{r+1},\quad r\in\mathbb{Z}\tag{7}$$

$$(\lambda,u[t])\in m_n\times m_n(\overline{h}).\tag{8}$$

If a pair $(\lambda^*,u^*[t])\in m_n\times m_n(\overline{h})$ is a solution of problem (6)-(8), then the function $x^*(t)$, obtained by gluing the function systems $(\lambda_r^*+u_r^*[t])$, $r\in\mathbb{Z}$, belongs to the space $\widetilde{C}((0,T),\mathbb{R}^n)$ and satisfies Eq. (1) for all $t\in(0,T)$. Conversely, if $x(t)$ is a solution of Problem 1_α, then the pair $(\lambda,u[t])$ (with $\lambda=(\ldots,x_r(t_{r-1}),x_{r+1}(t_r),\ldots)$ and $u[t]=(\ldots,x_r(t)-x_r(t_{r-1}),x_{r+1}(t)-x_{r+1}(t_r),\ldots)$, where $x_r(t)$ are the restrictions of $x(t)$ to the $r-$th subintervals, $r\in\mathbb{Z}$) belongs to $m_n\times m_n(\overline{h})$ and satisfies Eq. (6) and conditions (7).

Since (6) is an initial-value problem with parameter, we obtain the integral representation of $u_r(t)$ for fixed parameter values λ_r:

$$u_r(t) = \int\limits_{t_{r-1}}^{t} A(\tau)[u_r(\tau) + \lambda_r]d\tau + \int\limits_{t_{r-1}}^{t} f(\tau)d\tau, \quad r \in \mathbb{Z}. \tag{9}$$

Replacing $u_r(\tau)$ with the right-hand side of (9) and repeating this procedure ν times ($\nu = 1, 2, \ldots$), we obtain

$$u_r(t) = D_{\nu,r}(t)\lambda_r + F_{\nu,r}(t) + G_{\nu,r}(u, t), \quad t \in [t_{r-1}, t_r), \tag{10}$$

where

$$D_{\nu,r}(t) = \sum_{j=0}^{\nu-1} \int\limits_{t_{r-1}}^{t} A(\tau_1) \ldots \int\limits_{t_{r-1}}^{\tau_j} A(\tau_{j+1}) d\tau_{j+1} \ldots d\tau_1,$$

$$F_{\nu,r}(t) = \int\limits_{t_{r-1}}^{t} f(\tau_1)d\tau_1 + \sum_{j=1}^{\nu-1} \int\limits_{t_{r-1}}^{t} A(\tau_1) \ldots \int\limits_{t_{r-1}}^{\tau_{j-1}} A(\tau_j) \int\limits_{t_{r-1}}^{\tau_j} f(\tau_{j+1}) d\tau_{j+1} d\tau_j \ldots d\tau_1,$$

$$G_{\nu,r}(u, t) = \int\limits_{t_{r-1}}^{t} A(\tau_1) \ldots \int\limits_{t_{r-1}}^{\tau_j} A(\tau_{j+1}) u_r(\tau_{j+1}) d\tau_{j+1} \ldots d\tau_1, \quad \tau_0 = t, \quad r \in \mathbb{Z}.$$

Now, substituting the values $\lim\limits_{t \to t_r - 0} u_r(t)$, $r \in \mathbb{Z}$, determined from (9), into equations (10), we obtain the bilaterally infinite system of algebraic equations in parameters λ_r:

$$[I + D_{\nu,r}(h_r)]\lambda_r - \lambda_{r+1} = -F_{\nu,r}(h_r) - G_{\nu,r}(u, h_r), \quad r \in \mathbb{Z}. \tag{11}$$

Here I is the identity matrix of order n.

Let us denote by $Q_{\nu,\overline{h}(\theta)}$ the bilaterally infinite block-banded matrix corresponding to the left-hand side of system (11). The only non-zero terms in each block row of $Q_{\nu,\overline{h}(\theta)}$ are $I + D_{\nu,r}(h_r)$ and $-I$. Hence, for any sequence $\overline{h}(\theta)$, the matrix $Q_{\nu,\overline{h}(\theta)}$ maps the space m_n into itself, and the following estimate holds:

$$\|Q_{\nu,\overline{h}(\theta)}\|_{L(m_n)} \leq 2 + \sum_{j=1}^{\nu} \frac{\theta^j}{j!}.$$

The matrix form of system (11) is

$$Q_{\nu,\overline{h}(\theta)}\lambda = -F_\nu(\overline{h}) - G_\nu(u, \overline{h}), \quad \lambda \in m_n,$$

where

$$F_\nu(\overline{h}) = (\ldots, F_{\nu,r}(h_r), F_{\nu,r+1}(h_{r+1}), \ldots)' \in m_n,$$

$$G_\nu(u, \overline{h}) = (\ldots, G_{\nu,r}(u, h_r), G_{\nu,r+1}(u, h_{r+1}), \ldots)' \in m_n$$

for all $u[t] \in m_n(\overline{h})$ and $\overline{h}(\theta)$.

Definition 2. Problem 1_α is well-posed if it has a unique solution $x(t) \in \widetilde{C}((0, T), \mathbb{R}^n)$ for any $f(t) \in \widetilde{C}_{1/\alpha}((0, T), \mathbb{R}^n)$, and $\|x\|_1 \leq K\|f\|_\alpha$, where K is a constant independent of $f(t)$.

Theorem 1. Let $Q_{\nu,\overline{h}(\theta)}$ have an inverse for some $\overline{h}(\theta)$ and ν $(\nu = 1, 2, \ldots)$, and let

$$\|Q^{-1}_{\nu,\overline{h}(\theta)}\|_{L(m_n)} \leq \gamma_\nu(\overline{h}), \tag{12}$$

$$q_\nu(\overline{h}) = \gamma_\nu(\overline{h}) \left(e^\theta - 1 - \theta - \ldots - \frac{\theta^\nu}{\nu!}\right) < 1. \tag{13}$$

Then Problem 1_α is well-posed and its solution satisfies the estimate

$$\|x^*\|_1 \leq e^\theta \left[\frac{\gamma_\nu(\overline{h}}{1 - q_\nu(\overline{h})} \frac{\theta^\nu}{\nu!}(\gamma_\nu(\overline{h})(e^\theta - 1)^2 + \theta e^\theta) + \gamma_\nu(\overline{h})(e^\theta - 1) + \theta\right] \|f\|_\alpha.$$

The proof of Theorem 1 follows the scheme of Theorem 1 in [7].

Let $x^*(t)$ be the solution of Problem 1_α. Then the pair $(\lambda^*, u^*[t])$ with components $\lambda^*_r = x^*_r(t_{r-1})$ and $u^*_r(t) = x^*(t) - x^*(t_{r-1})$, $t \in [t_{r-1}, t_r)$, $r \in \mathbb{Z}$, is the solution of problem (6)–(8). Moreover, there exist numbers δ_1 and δ_2 such that $\|\lambda^*\| \leq \delta_1$ and $\|u^*_r(t)\| \leq \delta_2$, $t \in [t_{r-1}, t_r)$, $r \in \mathbb{Z}$, and for any $\nu \in \mathbb{N}$ the following identities hold:

$$u^*_r(t) = D_{\nu,r}(t)\lambda^*_r + F_{\nu,r}(t) + G_{\nu,r}(u^*, t), \quad t \in [t_{r-1}, t_r), \quad r \in \mathbb{Z}, \tag{14}$$

$$Q_{\nu,\overline{h}(\theta)}\lambda^* = -F_\nu(\overline{h}) - G_\nu(u^*, \overline{h}). \tag{15}$$

It can be easily shown that $\|G_\nu(u^*, \overline{h})\|_2 \leq \frac{\theta^\nu}{\nu!}\|u^*[t]\|_3 \leq \frac{\theta^\nu}{\nu!}\delta_2$, and $D_{\nu,r}(t)$ and $F_{\nu,r}(t)$ converge uniformly to

$$D_{*,r}(t) = \sum_{j=0}^{\infty} \int\limits_{t_{r-1}}^{t} A(\tau_1) \ldots \int\limits_{t_{r-1}}^{\tau_j} A(\tau_{j+1}) d\tau_{j+1} \ldots d\tau_1,$$

and

$$F_{*,r}(t) = \int\limits_{t_{r-1}}^{t} f(\tau_1)d\tau_1 + \sum_{j=1}^{\infty} \int\limits_{t_{r-1}}^{t} A(\tau_1) \ldots \int\limits_{t_{r-1}}^{\tau_{j-1}} A(\tau_j) \int\limits_{t_{r-1}}^{\tau_j} f(\tau_{j+1}) d\tau_{j+1} d\tau_j \ldots d\tau_1,$$

respectively. Then, letting $\nu \to \infty$ in (14), (15), and dividing both sides of (15) by $\theta > 0$, we obtain

$$u^*_r(t) = D_{*,r}(t)\lambda^*_r + F_{*,r}(t), \quad t \in [t_{r-1}, t_r), \quad r \in \mathbb{Z}, \tag{16}$$

$$\frac{1}{\theta} Q_{*,\overline{h}(\theta)}\lambda^* = -F_*(A, f, \overline{h}(\theta)), \quad \lambda^* \in m_n. \tag{17}$$

Here $F_*(A, f, \overline{h}(\theta)) = \lim\limits_{\nu \to \infty} \frac{1}{\theta} F_\nu(\overline{h})$.

Thus, if $(\lambda^*, u^*[t])$ is a solution of problem (6)–(8), then the parameter $\lambda^* = (\ldots, \lambda^*_r, \lambda^*_{r+1}, \ldots)' \in m_n$ satisfies Eq.(17), and the solutions $u^*_r(t)$ of the Cauchy problems (6), corresponding to λ^*_r, $r \in \mathbb{Z}$, are of the form (16).

We now assume that $\widehat{\lambda} = (\ldots, \widehat{\lambda}_r, \widehat{\lambda}_{r+1}, \ldots)' \in m_n$ is a solution of the system

$$\frac{1}{\theta}[I + D_{*,r}(t_r)]\lambda_r - \frac{1}{\theta}\lambda_{r+1} = -\frac{1}{\theta} F_{*,r}(t_),$$

or

$$\frac{1}{\theta} Q_{*,\overline{h}(\theta)}\widehat{\lambda} = -F_*(A, f, \overline{h}(\theta)), \tag{18}$$

and $\widehat{u}[t] = (\ldots, \widehat{u}_r(t), \widehat{u}_{r+1}(t), \ldots)'$ is the system of solutions of the Cauchy problem (6) on $[t_{r-1}, t_r)$ with $\lambda_r = \widehat{\lambda}_r$, $r \in \mathbb{Z}$. Let us show that the pair $(\widehat{\lambda}, \widehat{u}[t])$ is the solution of problem (6)–(8). Since $\widehat{u}_r(t)$

is the solution of the Cauchy problem (6) with $\lambda_r = \widehat{\lambda}_r$, it follows from (16) and the unique solvability of the Cauchy problem (6) for fixed parameter values λ_r that

$$\widehat{u}_r(t) = D_{*,r}(t)\widehat{\lambda}_r + F_{*,r}(t), \quad t \in [t_{r-1}, t_r), \quad r \in \mathbb{Z}. \tag{19}$$

In view of (18), we have

$$\widehat{\lambda}_r + [D_{*,r}(t_r)\widehat{\lambda}_r + F_{*,r}(t_r)] = \widehat{\lambda}_{r+1}, \quad r \in \mathbb{Z}. \tag{20}$$

Then, by (19) the expressions in square brackets in (20) are equal to $\lim\limits_{t \to t_r - 0} \widehat{u}_r(t)$, $r \in \mathbb{Z}$, and the pair $(\widehat{\lambda}, \widehat{u}[t])$ satisfies (7) as well.

Theorem 2. Problem 1_α is well-posed iff, given an arbitrary $\nu \in \mathbb{N}$, there is a $\theta(\nu) > 0$ such that the matrix $Q_{\nu,\overline{h}(\theta)}$ has an inverse for all $\overline{h}(\theta) = (\ldots, h_r(\theta), h_{r+1}(\theta), \ldots)$ and the inequalities (12) and (13) hold.

Proof. The *sufficiency* of the conditions of Theorem 2 for the well-posedness of Problem 1_α follows from Theorem 1.

Necessity. Let us consider the equation

$$\frac{1}{\theta} Q_{*,\overline{h}(\theta)} \lambda = b, \quad \lambda, b \in m_n.$$

Obviously, the kernel of the matrix $\frac{1}{\theta} Q_{*,\overline{h}(\theta)}$ consists only of the zero vector of the space m_n. Suppose, contrary to this claim, that there is a $\overline{\lambda} \in m_n$ such that $\frac{1}{\theta} Q_{*,\overline{h}(\theta)} \overline{\lambda} = 0$, $\|\overline{\lambda}\| \neq 0$. Hence, as shown above, the pair $(\overline{\lambda}, \overline{u}[t])$, with $\overline{u}[t] = (\ldots, \overline{u}_r(t), \overline{u}_{r+1}(t), \ldots)'$ being the system of solutions of the Cauchy problems (6) with $\lambda_r = \overline{\lambda}_r$ on $[t_{r-1}, t_r)$, is the solution of problem (6)–(8) with $f(t) = 0$. The function $\overline{x}(t)$, obtained by gluing the function systems $(\overline{\lambda}_r + \overline{u}_r(t))$, $r \in \mathbb{Z}$, belongs to $\widetilde{C}((0,T), \mathbb{R}^n)$ and satisfies the equation $\frac{dx}{dt} = A(t)x$. But $\sup\limits_{t \in (0,T)} \|\overline{x}(t)\| \neq 0$, which contradicts the well-posedness of Problem 1_α. Thus, the matrix $Q_{*,\overline{h}(\theta)}$ has an inverse.

Let us fix $\varepsilon > 0$ and choose $\theta_0(\varepsilon) > 0$ satisfying the inequality

$$\frac{1}{\theta}(e^\theta - 1 - \theta) \leq \frac{\varepsilon/2}{2(1 + \varepsilon/4)(1 + \varepsilon/2)}. \tag{21}$$

Then, by Lemma in [12], for arbitrary $b_r \in \mathbb{R}^n$, $r \in \mathbb{Z}$, the functions $f_{b_r} \in C([t_{r-1}, t_r], \mathbb{R}^n)$ can be constructed such that

$$F_*(A, f_{b_r}) = b_r, \qquad \max_{t \in [t_{r-1}, t_r]} \|f_{b_r}(t)/\alpha(t)\| \leq (1 + \varepsilon/2)\|b_r\|.$$

Hence, the function $f_b(t)$ defined as $f_b(t) = f_{b_r}(t)$, $t \in [t_{r-1}, t_r]$, satisfies the relations

$$f_b(t) \in \widetilde{C}((0,T), \mathbb{R}^n), \quad \|f_b\|_\alpha \leq (1 + \varepsilon/2)\|b\|_2, \quad F_*(A, f_b, \overline{h}(\theta)) = b.$$

The well-posedness of Problem 1_α implies that Eq.(17) has a unique solution $\lambda_b \in m_n$ for any $f_b(t) \in \widetilde{C}_{1/\alpha}((0,T), \mathbb{R}^n)$, and

$$\|\lambda_b\|_2 = \sup_{r \in \mathbb{Z}} \|\lambda_{b_r}\| = \sup_{r \in \mathbb{Z}} \|x_b(t_{r-1})\| \leq \sup_{t \in (0,T)} \|x_b(t)\| \leq K\|f_b\|_\alpha \leq K(1 + \varepsilon/2)\|b\|_2.$$

Taking into account that $\|\lambda_b\|_2 = \|[\frac{1}{\theta} Q_{*,\overline{h}(\theta)}]^{-1} b\|_2$, the latter estimate yields

$$\frac{\|[\frac{1}{\theta} Q_{*,\overline{h}(\theta)}]^{-1} b\|_2}{\|b\|_2} \leq \left(1 + \frac{\varepsilon}{2}\right) K, \quad \forall b \in m_n.$$

This gives

$$\|[\frac{1}{\theta}Q_{*,\overline{h}(\theta)}]^{-1}\|_{L(m_n)} \le \Big(1+\frac{\varepsilon}{2}\Big)K, \quad \forall\theta\in(0,\theta_0].$$

Hence, choosing $\theta\in(0,\theta_0]$ such that

$$\frac{(1+\varepsilon/2)K}{\theta}\left(e^\theta-1-\theta-\ldots-\frac{\theta^\nu}{\nu!}\right)<\frac{\varepsilon}{2(1+\varepsilon)}$$

and taking into account

$$\left\|\frac{1}{\theta}Q_{*,\overline{h}(\theta)}-\frac{1}{\theta}Q_{\nu,\overline{h}(\theta)}\right\|_{L(m_n)}\le\frac{1}{\theta}\left(e^\theta-1-\theta-\ldots-\frac{\theta^\nu}{\nu!}\right),$$

by the theorem on small perturbations of boundedly invertible operators, we obtain that the matrix $Q_{\nu,\overline{h}(\theta)}$ has a bounded inverse satisfying the estimate

$$\|\left[\frac{1}{\theta}Q_{\nu,\overline{h}(\theta)}\right]^{-1}\|_{L(m_n)}\le(1+\varepsilon)K.$$

Finally, (17) yields

$$q_\nu(\overline{h}(\theta))=(1+\varepsilon)\frac{K}{\theta}\left(e^\theta-1-\theta-\ldots-\frac{\theta^\nu}{\nu!}\right)<\frac{\varepsilon}{2+\varepsilon}<1,$$

which completes the proof.

Theorem 3. Problem 1_α is well-posed iff, given an arbitrary $\nu\in\mathbb{N}$, there is a $\theta_0(\nu)$ such that the matrix $Q_{\nu,\overline{h}(\theta)}$ has an inverse for all sequences $\overline{h}(\theta)$, $\theta\in(0,\theta_0]$, and

$$\|\left[Q_{\nu,\overline{h}(\theta)}\right]^{-1}\|_{L(m_n)}\le\frac{\gamma}{\theta}, \tag{22}$$

where γ is a constant independent of $\overline{h}(\theta)$.

Moreover, if the well-posedness constant K is known, then for any $\varepsilon>0$ there exists $\overline{\theta}(\varepsilon,\nu)>0$ such that estimate (22) holds with constant $\gamma=(1+\varepsilon)K$ for all $\theta\in(0,\overline{\theta}(\varepsilon,\nu)]$. Conversely, if estimate (22) holds, then $K=\gamma$.

Proof. Necessity. Let Problem 1_α be well-posed with constant K. Given $\varepsilon>0$, we choose $\overline{\theta}(\varepsilon,\nu)\in(0,\theta_0(\varepsilon)]$, where $\theta_0(\varepsilon)$ satisfies condition (21). Then, as it was shown in Theorem 2, the matrix $Q_{\nu,\overline{h}(\theta)}$ is invertible for all $\theta\in(0,\overline{\theta}(\varepsilon,\nu)]$ and $\|\left[Q_{\nu,\overline{h}(\theta)}\right]^{-1}\|_{L(m_n)}\le\dfrac{(1+\varepsilon)K}{\theta}$, i.e. $\gamma=(1+\varepsilon)K$.

Sufficiency. Let estimate (22) hold. let us choose θ so that $q_\nu(\overline{h}(\theta))<1$. Then, by Theorem 1, Problem 1_α is well-posed and

$$\|x^*\|_1\le e^\theta\left[\frac{\gamma}{\theta}\cdot\frac{1}{1-q_\nu(\overline{h}(\theta))}\cdot\frac{\theta^\nu}{\nu!}\left(\frac{\gamma}{\theta}(e^\theta-1)^2+\theta e^\theta\right)+\frac{\gamma}{\theta}(e^\theta-1)+\theta\right]\|f\|_\alpha.$$

Letting $\theta\to0$, we obtain

$$\|x^*\|_1\le\gamma\|f\|_\alpha,$$

i.e. $K=\gamma$, which completes the proof.

Acknowledgments

This research is funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP09258829).

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Коэффициенттері шектелмеген дифференциалдық теңдеулер сызықты жүйелерінің шектелген шешімдері туралы

Мақалада шектелмеген коэффициенттер матрицасы бар біртекті емес сызықты дифференциалдық теңдеулер жүйесі үшін ақырлы интервалда шектелген шешімін табу есебі қарастырылған. Теңдеудің оң жағы үзіліссіз және қандай да бір салмақпен шектелген функциялар кеңістігіне жатады; салмақтық функция коэффициенттер матрицасының әрекетін ескере отырып таңдалды. Қарастырылып отырған есепті зерттеу үшін біркелкі емес бөліммен параметрлеу әдісінің модификациясы қолданылды. Арнайы құрылымды екі жақты шексіз матрицасы тұрғысынан зерттелген есептің дұрыс шешілімділігіне қажетті және жеткілікті шарттар алынған.

Кілт сөздер: жәй дифференциалдық теңдеулер, сингулярлы шеттік есеп, корректі шешілімділік, параметрлеу әдісі, шектелген шешім, сызықты жүйе, шектелмеген коэффициенттер.

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Об ограниченных решениях линейных систем дифференциальных уравнений с неограниченными коэффициентами

В статье рассмотрена задача нахождения ограниченного на конечном интервале решения системы неоднородных линейных дифференциальных уравнений с неограниченной матрицей коэффициентов. Правая часть уравнения принадлежит пространству непрерывных и ограниченных с некоторым весом функций; весовая функция выбирается с учетом поведения матрицы коэффициентов. Для исследования рассматриваемой задачи применена модификация метода параметризации с неравномерным разбиением. Получены необходимые и достаточные условия корректной разрешимости рассматриваемой задачи в терминах двусторонне-бесконечной матрицы специальной структуры.

Ключевые слова: обыкновенные дифференциальные уравнения, сингулярная краевая задача, корректная разрешимость, метод параметризации, ограниченное решение, линейная система, неограниченные коэффициенты.

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