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On a mixed problem for Hilfer type differential equation of higher order

The study considers the solvability of a mixed problem for a Hilfer type partial differential equation of the even order with initial value conditions and small positive parameters in mixed derivatives in three-dimensional domain. It studies the solution to this fractional differential equation of higher order in the class of regular functions. The case, when the order of fractional operator is $1 < \alpha < 2$, is examined. During this study the authors use the Fourier series method and obtain a countable system of ordinary differential equations. The initial value problem is integrated as an ordinary differential equation and the integrated constants find by the aid of given initial value conditions. Using the Cauchy–Schwarz inequality and the Bessel inequality, it is proved the absolute and uniform convergence of the obtained Fourier series. The stability of the solution to the mixed problem on the given functions is studied.

Keywords: fractional order, Hilfer operator, mixed problem, Fourier series, initial value conditions, unique solvability.

Introduction

The theory of the mixed problems is one of the most important directions of the modern theory of differential equations. A large number of works are devoted to the study of the mixed problems for differential and integro-differential equations (see, for example, [1–12]). Many problems of gas dynamics, theory of elasticity, plates, and shells are described by higher-order partial differential equations.

Fractional calculus plays an important role for the mathematical modeling in many applied problems. In [13], it is considered problems of continuum and statistical mechanics. The work [14] studies the mathematical problems of the Ebola epidemic model. The studies [15] and [16] investigate the fractional model for the dynamics of tuberculosis infection and novel coronavirus (nCoV-2019), respectively. The construction of various models of theoretical physics by the aid of fractional calculus is described in [17, Vol. 4, 5], [18], [19]. Some applications of fractional calculus in solving applied problems are given in [17, Vol. 6–8], [20]. In [21], the solvability of an initial value problem for Hilfer type fractional differential equation with nonlinear maxima is studied. In [22], by analytical method, the unique solvability of boundary value problem for weak nonlinear partial differential equations of mixed type with fractional Hilfer operator is studied. In [23], the solvability of nonlocal problem for a mixed type fourth-order differential equation with Hilfer fractional operator is examined. In [24], it is considered an inverse problem for a mixed type integro-differential equation with fractional order Gerasimov-Caputo operators. The research works [25–34] obtained the results on the direction of applications of fractional derivatives to solve partial differential equations.

Let $\big(t_0; T\big) \subset \mathbb{R}^+ \equiv [0; \infty)$ be an interval on the set of positive real numbers, where $0 \le t_0 < T < \infty$. The Riemann–Liouville $0 < \alpha$-order fractional integral of a function $\eta(t)$ is defined as follows:

$$I^{\alpha}_{t_0+}\eta(t) = \frac{1}{\Gamma(\alpha)} \int\limits_{t_0}^{t} (t-s)^{\alpha-1}\eta(s)ds, \quad \alpha > 0, \quad t \in \big(t_0; T\big),$$

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where $\Gamma(\alpha)$ is the Gamma function.

Let $n-1<\alpha\leq n,\ n\in\mathbb{N}$. The Riemann–Liouville α-order fractional derivative of a function $\eta(t)$ is defined as follows:

$$D_{t_0+}^{\alpha}\eta(t)=\frac{d^n}{dt^n}I_{t_0+}^{n-\alpha}\eta(t),\quad t\in\big(t_0;\,T\big).$$

The Hilfer fractional derivatives of α-order ($n-1<\alpha\leq n,\ n\in\mathbb{N}$) and β-type ($0\leq\beta\leq 1$), are defined by the following composition of three operators:

$$D_{t_0+}^{\alpha,\,\beta}\eta(t)=I_{t_0+}^{\beta(n-\alpha)}\frac{d^n}{dt^n}I_{t_0+}^{(1-\beta)(n-\alpha)}\eta(t),\quad t\in\big(t_0;\,T\big).$$

Let $\gamma=\alpha+\beta n-\alpha\beta$. It is easy to see that $\alpha\leq\gamma\leq n$. Then it is convenient to use another designation for the operator $D^{\alpha,\gamma}\eta(t)=D_{t_0+}^{\alpha,\beta}\eta(t)$. Hilfer operator is generalization of the Riemann–Liouville operator and was introduced by R. Hilfer based on fractional time evolutions that arise during the transition from the microscopic scale to the macroscopic time scale (see [17]).

In this paper, for the case $1<\alpha<2$ we study the regular solvability of mixed value problem for a Hilfer type partial differential equation of higher even order with positive small parameters. The stability of the solution from the given functions is proved.

In three-dimensional domain $\Omega=\{(t,\,x,\,y)\,|\,t_0<t<T,\ 0<x,\,y<l\}$ a higher order partial differential equation of the following form is considered

$$D_{\varepsilon_1,\,\varepsilon_2}^{\alpha,\,\gamma}\left[U\right]=a\left(t\right)b\left(x,\,y\right)\tag{1}$$

with initial value conditions

$$\lim_{t\to+t_0}J_{t_0+}^{2-\gamma}U(t,\,x,\,y)=\varphi_1(x,\,y),\quad \lim_{t\to+t_0}\frac{d}{dt}J_{t_0+}^{2-\gamma}U(t,\,x,\,y)=\varphi_2(x,\,y),\tag{2}$$

where T and l are given positive real numbers, $0\leq t_0<T$,

$$D_{\varepsilon_1,\,\varepsilon_2}^{\alpha,\,\gamma}\left[U\right]=\left[D^{\alpha,\,\gamma}-D^{\alpha,\,\gamma}\left(\varepsilon_1\left(\frac{\partial^{2k}}{\partial x^{2k}}+\frac{\partial^{2k}}{\partial y^{2k}}\right)-\varepsilon_2\left(\frac{\partial^{4k}}{\partial x^{4k}}+\frac{\partial^{4k}}{\partial y^{4k}}\right)\right)-\right.$$

$$\left.-\,\omega\left(\left(\frac{\partial^{2k}}{\partial x^{2k}}+\frac{\partial^{2k}}{\partial y^{2k}}\right)-\left(\frac{\partial^{4k}}{\partial x^{4k}}+\frac{\partial^{4k}}{\partial y^{4k}}\right)\right)\right]U\left(t,\,x,\,y\right),$$

ω is positive parameter, ε_1 and ε_2 are positive small parameters, $1<\alpha<\gamma\leq 2,\ k$ is given positive integer, $a\left(t\right)\in C\left(\Omega_T\right),\ \Omega_T\equiv\left[t_0;\,T\right],\ \Omega_l\equiv[0;\,l],\ b\left(x,\,y\right)\in C\left(\Omega_l^2\right)$ is known function, $\varphi_1(x,\,y)$ and $\varphi_2(x,\,y)$ are given continuous functions, $\Omega_l^2\equiv\Omega_l\times\Omega_l$. We assume that for given functions the following boundary conditions hold

$$\varphi_i(0,\,y)=\varphi_i(l,\,y)=\varphi_i(x,\,0)=\varphi_i\left(x,\,l\right)=0,\ \ i=1,\,2,$$

$$b\left(0,\,y\right)=b\left(l,\,y\right)=b\left(x,\,0\right)=b\left(x,\,l\right)=0.$$

Problem Statement. We find the function $U\left(t,\,x,\,y\right)$, which satisfies differential equation (1), initial value conditions (2), zero boundary value conditions for $t_0\leq t\leq T$

$$U\left(t,\,0,\,y\right)=U\left(t,\,l,\,y\right)=U\left(t,\,x,\,0\right)=U\left(t,\,x,\,l\right)=$$

$$=\frac{\partial^2}{\partial x^2}U\left(t,\,0,\,y\right)=\frac{\partial^2}{\partial x^2}U\left(t,\,l,\,y\right)=\frac{\partial^2}{\partial x^2}U\left(t,\,x,\,0\right)=\frac{\partial^2}{\partial x^2}U\left(t,\,x,\,l\right)=$$

$$=\frac{\partial^2}{\partial y^2}U\left(t,\,0,\,y\right)=\frac{\partial^2}{\partial y^2}U\left(t,\,l,\,y\right)=\frac{\partial^2}{\partial y^2}U\left(t,\,x,\,0\right)=\frac{\partial^2}{\partial y^2}U\left(t,\,x,\,l\right)=\ldots=$$

$$= \frac{\partial^{4k-2}}{\partial x^{4k-2}} U(t, 0, y) = \frac{\partial^{4k-2}}{\partial x^{4k-2}} U(t, l, y) = \frac{\partial^{4k-2}}{\partial x^{4k-2}} U(t, x, 0) = \frac{\partial^{4k-2}}{\partial x^{4k-2}} U(t, x, l) =$$

$$= \frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t, 0, y) = \frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t, l, y) = \frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t, x, 0) = \frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t, x, l) = 0, \quad (3)$$

class of functions

$$\left[\begin{array}{l} (t-t_0)^{2-\gamma} U(t, x, y) \in C(\overline{\Omega}), \\ D^{\alpha, \gamma} U(t, x, y) \in C^{4k, 4k}_{x, y}(\Omega) \cap C^{4k+0}_{x, y}(\Omega) \cap C^{0+4k}_{x, y}(\Omega), \end{array} \right. \quad (4)$$

where $C^{4k+0}_{x, y}(\Omega)$ is the class of continuous functions $\frac{\partial^{4k} U(t, x, y)}{\partial x^{4k}}$ on Ω, while $C^{0+4k}_{x, y}(\Omega)$ is the class of continuous functions $\frac{\partial^{4k} U(t, x, y)}{\partial y^{4k}}$ on Ω, $\frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t, x, l)$ we understand as $\left.\frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t, x, y)\right|_{y=l}$, $\overline{\Omega} = \{(t, x, y) \,|\, t_0 \le t \le T, \ 0 \le x, y \le l\}$.

1 Transform of fractional differential equation

Lemma. The solution to the ordinary fractional differential equation

$$D^{\alpha, \gamma} v(t) + \omega v(t) = f(t, v(t)) \quad (5)$$

with initial value condition

$$\lim_{t \to +t_0} J^{2-\gamma}_{t_0+} v(t) = v_0, \quad \lim_{t \to +t_0} \frac{d}{dt} J^{2-\gamma}_{t_0+} v(t) = v_1, \quad (6)$$

is represented as follows

$$v(t) = v_0 (t-t_0)^{\gamma-2} E_{\alpha, \gamma-1}(\omega (t-t_0)^{\alpha}) + v_1 (t-t_0)^{\gamma-1} E_{\alpha, \gamma}(\omega (t-t_0)^{\alpha}) +$$

$$+ \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha, \alpha}(-\omega (t-s)^{\alpha}) f(s, v(s)) \, ds, \quad (7)$$

where $E_{\alpha, \gamma}(z)$ is the Mittag–Leffler function and has the form [17, vol. 1, 269–295]

$$E_{\alpha, \gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \gamma)}, \quad z, \alpha, \gamma \in (0; \infty),$$

$f(t, v) \in C(\Omega_1)$, $0 < \omega$ is real parameter, $v_0, v_1 = \text{const}$, $\Omega_1 \equiv [t_0; T] \times X$, $0 \le t_0$, $X \subset \mathbb{R} \equiv (-\infty; \infty)$, X is closed set,

$$D^{\alpha, \gamma} = J^{\gamma-\alpha}_{t_0+} \frac{d^2}{dt^2} J^{2-\gamma}_{t_0+}, \ 1 < \alpha < \gamma \le 2, \ \gamma = \alpha + 2\beta - \alpha\beta.$$

Proof. We rewrite the differential equation (5) in the form

$$J^{\gamma-\alpha}_{t_0+} D^{\gamma}_{t_0+} v(t) = -\omega v(t) + f(t, v).$$

Applying the operator $J^{\alpha}_{t_0+}$ to both sides of this equation (5), taking into account the linearity of this operator and the formula [35]

$$J^{\delta}_{t_0+} D^{\delta}_{t_0+} v(t) = v(t) - \sum_{k=0}^{n-1} \frac{(t-t_0)^{\delta+k-n}}{\Gamma(\delta+k+1-n)} \lim_{t \to t_0+} \frac{d^k}{dt^k} J^{n-\delta}_{t \to t_0+} v(t), \quad \delta \in (n-1; n],$$

we obtain

$$v\,(t) = -\omega J^{\alpha}_{t_0+} v\,(t) \frac{v_0}{\Gamma(\gamma-1)}\,(t-t_0)^{\gamma-2} + \frac{v_1}{\Gamma(\gamma)}\,(t-t_0)^{\gamma-1} + J^{\alpha}_{t_0+} f(t,\,v\,(t)). \tag{8}$$

Using the lemma from [26], we represent the solution to equation (8) in the form

$$v\,(t) = \frac{v_0}{\Gamma(\gamma-1)}\,(t-t_0)^{\gamma-2} + \frac{v_1}{\Gamma(\gamma)}\,(t-t_0)^{\gamma-1} + J^{\alpha}_{t_0+} f(t,\,v\,(t)) -$$

$$-\omega \int\limits_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\,\alpha}\left(-\omega\ (t-s)^{\alpha}\right) \times$$

$$\times \left[\frac{v_0}{\Gamma(\gamma-1)}\,(s-t_0)^{\gamma-2} + \frac{v_1}{\Gamma(\gamma)}\,(s-t_0)^{\gamma-1} + J^{\alpha}_{t_0+} f(s,\,v\,(s))\right] ds. \tag{9}$$

We rewrite the presentation (9) as the sum of two expressions:

$$I_1(t) = \frac{v_0}{\Gamma(\gamma-1)} \left[(t-t_0)^{\gamma-2} - \omega \int\limits_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\,\alpha}\left(-\omega\ (s-t_0)^{\alpha}\right)(s-t_0)^{\gamma-2}\,d\,s\right] +$$

$$+\frac{v_1}{\Gamma(\gamma)} \left[(t-t_0)^{\gamma-1} - \omega \int\limits_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\omega\ (t-s)^{\alpha}\right)(s-t_0)^{\gamma-1}\,ds\right], \tag{10}$$

$$I_2\,(t) = J^{\alpha}_{t_0+} f\,(t,\,v\,(t)) - \omega \int\limits_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\,\alpha}\left(-\omega\cdot(t-s)^{\alpha}\right) J^{\alpha}_{t_0+} f\,(s,\,v\,(s))\,d\,s. \tag{11}$$

We apply the following presentations [17, vol. 1, 269–295]

$$E_{\alpha,\,\mu}(z) = \frac{1}{\Gamma(\mu)} + z\cdot E_{\alpha,\,\mu+\alpha}(t), \quad \alpha>0, \quad \mu>0, \tag{12}$$

$$\frac{1}{\Gamma(\nu)} \int\limits_{0}^{z} (z-t)^{\nu-1} E_{\alpha,\,\beta}\left(\lambda\,t^{\alpha}\right) t^{\beta-1} d\,t = z^{\beta+\nu-1}\cdot E_{\alpha,\,\beta+\nu}\left(\lambda\,z^{\alpha}\right), \quad \nu>0,\ \beta>0. \tag{13}$$

Then for the integral (10) we obtain the presentation

$$I_1(t) = v_0\,(t-t_0)^{\gamma-2}\,E_{\alpha,\,\gamma-1}\left(-\omega\ (t-t_0)^{\alpha}\right) + v_1\ (t-t_0)^{\gamma-1}\,E_{\alpha,\,\gamma}\left(-\omega\ (t-t_0)^{\alpha}\right). \tag{14}$$

The integral in (11) is easily transformed to the form

$$\int\limits_{t_0}^{t} (t-\xi)^{\alpha-1}\,E_{\alpha,\,\alpha}\left(-\omega\cdot(t-\xi)^{\alpha}\right)\ J^{\alpha}_{t_0+} f\,(\xi,\,v\,(\xi))\,d\,\xi =$$

$$= \frac{1}{\Gamma\,(\alpha)} \int\limits_{t_0}^{t} (t-\xi)^{\alpha-1} E_{\alpha,\,\alpha}\left(-\omega\cdot(t-\xi)^{\alpha}\right)\,d\,\xi \int\limits_{t_0}^{\xi} (\xi-s)^{\alpha-1} f\,(s,\,v\,(s))\,d\,s =$$

$$= \frac{1}{\Gamma(\alpha)} \int\limits_{t_0}^{t} f(s, v(s))\, ds \int\limits_{s}^{t} (t-\xi)^{\alpha-1} (\xi - s)^{\alpha-1} E_{\alpha,\alpha}\left(-\omega \cdot (t-\xi)^{\alpha}\right) d\xi. \tag{15}$$

Taking into account the (13) the second integral in the last equality of (15) can be written as

$$\int\limits_{s}^{t} (t-\xi)^{\alpha-1} (\xi - s)^{\alpha-1} E_{\alpha,\alpha}\left(-\omega \cdot (t-\xi)^{\alpha}\right) d\xi = \Gamma(\alpha)\, (t-\xi)^{2\alpha-1} E_{\alpha,2\alpha}\left(-\omega \cdot (t-\xi)^{\alpha}\right).$$

Then, taking into account (12), we represent (11) in the following form

$$I_2(t) = \int\limits_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\omega \cdot (t-s)^{\alpha}\right) f(s, v(s))\, ds. \tag{16}$$

Substituting (14) and (16) into the sum $v(t) = I_1(t) + I_2(t)$, we obtain (7). The Lemma is proved.

2 Expansion of the solution into Fourier series

Nontrivial solutions to the problem are sought as a Fourier series

$$U(t, x, y) = \sum_{n,m=1}^{\infty} u_{n,m}(t)\, \vartheta_{n,m}(x, y), \tag{17}$$

where

$$u_{n,m}(t) = \int\limits_0^l \int\limits_0^l U(t, x, y)\, \vartheta_{n,m}(x, y)\, dx\, dy, \tag{18}$$

$$\vartheta_{n,m}(x, y) = \frac{2}{l} \sin \frac{\pi n}{l} x \sin \frac{\pi m}{l} y, \ \ n, m = 1, 2, \ldots$$

We also suppose that the following function is expanded to Fourier series

$$b(x, y) = \sum_{n,m=1}^{\infty} b_{n,m}\, \vartheta_{n,m}(x, y), \tag{19}$$

where

$$b_{n,m} = \int\limits_0^l \int\limits_0^l b(x, y)\, \vartheta_{n,m}(x, y)\, dx\, dy. \tag{20}$$

Substituting Fourier series (17) and (19) into partial differential equation (1), we obtain the countable system of ordinary fractional differential equations of order: $1 < \alpha, \gamma < 2$

$$D^{\alpha,\gamma} u_{n,m}(t) + \lambda_{n,m}^{2k}(\varepsilon_1, \varepsilon_2)\, \omega\, u_{n,m}(t) = \frac{a(t)\, b_{n,m}}{1 + \mu_{n,m}^{2k}\left(\varepsilon_1 + \varepsilon_2 \mu_{n,m}^{2k}\right)}, \tag{21}$$

where

$$\lambda_{n,m}^{2k}(\varepsilon_1, \varepsilon_2) = \frac{\mu_{n,m}^{2k}\left(1 + \mu_{n,m}^{2k}\right)}{1 + \mu_{n,m}^{2k}\left(\varepsilon_1 + \varepsilon_2 \mu_{n,m}^{2k}\right)}, \ \ \mu_{n,m}^{k} = \left(\frac{\pi}{l}\right)^k \sqrt{n^{2k} + m^{2k}}.$$

According to the Lemma, the general solution to countable system of differential equations (21) has the form

$$u_{n,m}(t) = C_{1\,n,m}\,(t-t_0)^{\gamma-2}\,E_{\alpha,\,\gamma-1}\left(-\lambda_{n,m}^{2k}\,(\varepsilon_1,\,\varepsilon_2)\;\omega\;(t-t_0)^{\alpha}\right)+$$

$$+C_{2\,n,m}\,(t-t_0)^{\gamma-1}\,E_{\alpha,\,\gamma}\left(-\lambda_{n,m}^{2k}\,(\varepsilon_1,\,\varepsilon_2)\;\omega\;(t-t_0)^{\alpha}\right)+b_{n,m}h_{n,m}(t), \tag{22}$$

where

$$h_{n,m}(t) = \frac{1}{1+\mu_{n,m}^{2k}\left(\varepsilon_1+\varepsilon_2\mu_{n,m}^{2k}\right)}\int\limits_{t_0}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{n,m}^{2k}\,(\varepsilon_1,\,\varepsilon_2)\;\omega\,(t-s)^{\alpha}\right)\,a\,(s)\,d\,s,$$

$C_{1\,n,m}$ and $C_{2\,n,m}$ are arbitrary constants.

By Fourier coefficients (18), the initial conditions (2) we rewrite in the forms

$$\lim_{t\to+t_0}J_{t_0+}^{2-\gamma}u_{n,m}(t) = \int\limits_0^l\int\limits_0^l\lim_{t\to+t_0}J_{t_0+}^{2-\gamma}U\;(t,\,x,\,y)\;\vartheta_{n,m}(x,\,y)\,d\,x\,d\,y=$$

$$=\int\limits_0^l\int\limits_0^l\varphi_1(x,\,y)\,\vartheta_{n,m}(x,\,y)\,d\,x\,d\,y=\varphi_{1\,n,m}, \tag{23}$$

$$\lim_{t\to+t_0}\frac{d}{dt}J_{t_0+}^{2-\gamma}u_{n,m}(t) = \int\limits_0^l\int\limits_0^l\lim_{t\to+t_0}\frac{d}{dt}J_{t_0+}^{2-\gamma}U\,(t,\,x,\,y)\;\vartheta_{n,m}(x,\,y)\,d\,x\,d\,y=$$

$$=\int\limits_0^l\int\limits_0^l\varphi_2(x,\,y)\,\vartheta_{n,m}(x,\,y)\,d\,x\,d\,y=\varphi_{2\,n,m}\,. \tag{24}$$

To find the unknown coefficients $C_{1\,n,m}$ and $C_{2\,n,m}$ in (22), we use conditions (23) and (24). Then from (22) we have

$$u_{n,m}(t) = \varphi_{1\,n,m}\,(t-t_0)^{\gamma-2}\,E_{\alpha,\,\gamma-1}\left(-\lambda_{n,m}^{2k}\,(\varepsilon_1,\,\varepsilon_2)\;\omega\;(t-t_0)^{\alpha}\right)+$$

$$+\varphi_{2\,n,m}\,(t-t_0)^{\gamma-1}\,E_{\alpha,\,\gamma}\left(-\lambda_{n,m}^{2k}\,(\varepsilon_1,\,\varepsilon_2)\;\omega\;(t-t_0)^{\alpha}\right)+$$

$$+\frac{b_{n,m}}{1+\mu_{n,m}^{2k}\left(\varepsilon_1+\varepsilon_2\mu_{n,m}^{2k}\right)}\int\limits_{t_0}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{n,m}^{2k}\,(\varepsilon_1,\,\varepsilon_2)\;\omega\,(t-s)^{\alpha}\right)\,a\,(s)\,d\,s. \tag{25}$$

Substituting the presentation of the Fourier coefficients (25) of main unknown function into Fourier series (6), we obtain

$$U\,(t,\,x,\,y)=\sum_{n,\,m=1}^{\infty}\vartheta_{n\,,m}(x,\,y)\left[\varphi_{1\,n,m}\,(t-t_0)^{\gamma-2}\,E_{\alpha,\,\gamma-1}\left(-\lambda_{n,m}^{2k}\,(\varepsilon_1,\,\varepsilon_2)\;\omega\;(t-t_0)^{\alpha}\right)+\right.$$

$$+\varphi_{2\,n,m}\,(t-t_0)^{\gamma-1}\,E_{\alpha,\,\gamma}\left(-\lambda_{n,m}^{2k}\,(\varepsilon_1,\,\varepsilon_2)\;\omega\;(t-t_0)^{\alpha}\right)+$$

$$\left.+\frac{b_{n,m}}{1+\mu_{n,m}^{2k}\left(\varepsilon_1+\varepsilon_2\mu_{n,m}^{2k}\right)}\int\limits_{t_0}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{n,m}^{2k}\,(\varepsilon_1,\,\varepsilon_2)\;\omega\,(t-s)^{\alpha}\right)\,a\,(s)\,d\,s\right]. \tag{26}$$

This Fourier series (26) is a formal solution to the initial value problem (1)–(4).

3 Convergence of the Fourier series (26)

We prove absolute and uniform convergence of the Fourier series (26). We need to use the concepts of the following Banach spaces. Hilbert coordinate space ℓ_2 of number sequences $\{\varphi_{n,m}\}_{n,m=1}^{\infty}$ with norm

$$\|\varphi\|_{\ell_2} = \sqrt{\sum_{n,m=1}^{\infty} |\varphi_{n,m}|^2} < \infty.$$

The space $L_2(\Omega_l^2)$ of square-summable functions on the domain $\Omega_l^2 = \Omega_l \times \Omega_l$ with norm

$$\|\vartheta(x, y)\|_{L_2(\Omega_l^2)} = \sqrt{\int_0^l \int_0^l |\vartheta(x, y)|^2 \, dx\, dy} < \infty.$$

Conditions of smoothness. Let for functions

$$\varphi_i(x, y)\ (i = 1, 2),\ b(x, y) \in C^{4k}(\Omega_l^2)$$

there exist piecewise continuous $4k+1$ order derivatives. Then by integrating in parts the functions (20), (23) and (24) $4k+1$ times over every variable x, y, we obtain the following relations

$$|\varphi_{i\,n,m}| = \left(\frac{l}{\pi}\right)^{8k+2} \frac{\left|\varphi_{i\,n,m}^{(8k+2)}\right|}{n^{4k+1} m^{4k+1}}, \quad |b_{n,m}| = \left(\frac{l}{\pi}\right)^{8k+2} \frac{\left|b_{n,m}^{(8k+2)}\right|}{n^{4k+1} m^{4k+1}}, \tag{27}$$

$$\left\|\varphi_{i\,n,m}^{(8k+2)}\right\|_{\ell_2} \le \frac{2}{l} \left\| \frac{\partial^{8k+2}\varphi_i(x, y)}{\partial x^{4k+1}\, \partial y^{4k+1}} \right\|_{L_2(\Omega_l^2)}, \tag{28}$$

$$\left\|b_{n,m}^{(8k+2)}\right\|_{\ell_2} \le \frac{2}{l} \left\| \frac{\partial^{8k+2} b(x, y)}{\partial x^{4k+1} \partial y^{4k+1}} \right\|_{L_2(\Omega_l^2)}, \tag{29}$$

where

$$\varphi_{i\,n,m}^{(8k+2)} = \int_0^l \int_0^l \frac{\partial^{8k+2}\varphi_i(x, y)}{\partial x^{4k+1}\, \partial y^{4k+1}}\, \vartheta_{n,m}(x, y)\, dx\, dy, \quad i = 1, 2,$$

$$b_{n,m}^{(8k+2)} = \int_0^l \int_0^l \frac{\partial^{8k+2} b(x, y)}{\partial x^{4k+1}\, \partial y^{4k+1}}\, \vartheta_{n,m}(x, y)\, dx\, dy.$$

To obtain estimates for solution, we use the properties of the Mittag–Leffler function [36]. Let $\alpha \in (0; 2)$ and $\gamma \in \mathbb{R}$. If $\arg z = \pi$, then there takes place the following estimate

$$|E_{\alpha,\gamma}(z)| \le \frac{M_1}{1 + |z|},$$

where $0 < M_1 = \text{const}$ does not depend from z.

Therefore, it is easy to see that there exists constant M_2 such that

$$\max_{t_0 \le t \le T} \left| E_{\alpha,\gamma-1}\left(-\lambda_{n,m}^{2k}(\varepsilon_1, \varepsilon_2)\, \omega\, (t - t_0)^{\alpha}\right) \right| \le M_2 < \infty, \tag{30}$$

$$\max_{t_0 \le t \le T} \left| E_{\alpha,\gamma}\left(-\lambda_{n,m}^{2k}(\varepsilon_1, \varepsilon_2)\, \omega\, (t - t_0)^{\alpha}\right) \right| \le M_2 < \infty, \tag{31}$$

$$\max_{t_0\le t\le T}\left|\int_{t_0}^{t}(t-s)^{\alpha-1}(t-t_0)^{2-\gamma}E_{\alpha,\alpha}\left(-\lambda_{n,m}^{2k}(\varepsilon_1,\varepsilon_2)\,\omega\,(t-s)^{\alpha}\right)a(s)\,ds\right|\le M_2<\infty. \tag{32}$$

Theorem 1. Suppose that the conditions of smoothness and estimates (27)–(29) are fulfilled. Then Fourier series (26) convergence is absolute and uniform.

Proof. We apply the formulas (27)–(29) and estimates (30)–(32) to estimate the series (26). Using the Cauchy–Schwartz inequality for series (26), we get the estimate

$$\left|(t-t_0)^{2-\gamma}U(t,x,y)\right|\le M_2\sum_{n,m=1}^{\infty}|\vartheta_{n,m}(x,y)|\cdot[|\varphi_{1\,n,m}|+|\varphi_{2\,n,m}|+|b_{n,m}|]\le$$

$$\le\frac{2}{l}M_2\left[\sum_{n,m=1}^{\infty}|\varphi_{1\,n,m}|+\sum_{n,m=1}^{\infty}|\varphi_{2\,n,m}|+\sum_{n,m=1}^{\infty}|b_{n,m}|\right]\le$$

$$\le\frac{2}{l}\left(\frac{l}{\pi}\right)^{8k+2}M_2\left[\sum_{n,m=1}^{\infty}\frac{\left|\varphi_{1\,n,m}^{(8k+2)}\right|}{n^{4k+1}m^{4k+1}}+\sum_{n,m=1}^{\infty}\frac{\left|\varphi_{2\,n,m}^{(8k+2)}\right|}{n^{4k+1}m^{4k+1}}+\sum_{n,m=1}^{\infty}\frac{\left|b_{n,m}^{(8k+2)}\right|}{n^{4k+1}m^{4k+1}}\right]\le$$

$$\le\frac{2}{l}\left(\frac{l}{\pi}\right)^{8k+2}M_2M_3\left[\left\|\varphi_{1\,n,m}^{(8k+2)}\right\|_{\ell_2}+\left\|\varphi_{2\,n,m}^{(8k+2)}\right\|_{\ell_2}+\left\|b_{n,m}^{(8k+2)}\right\|_{\ell_2}\right]\le$$

$$\le\gamma_1\left[\sum_{i=1}^{2}\left\|\frac{\partial^{8k+2}\varphi_i(x,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}+\left\|\frac{\partial^{8k+2}b(x,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}\right]<\infty, \tag{33}$$

where

$$\gamma_1=M_2M_3\left(\frac{2}{l}\right)^2\left(\frac{l}{\pi}\right)^{8k+2},\quad M_3=\sqrt{\sum_{n,m=1}^{\infty}\frac{1}{n^{8k+2}m^{8k+2}}}<\infty.$$

From the estimate (33) the absolute and uniform convergence of Fourier series (26) implies. The Theorem 1 is proved.

4 Uniqueness of the solution

To establish the uniqueness of the function $U(t,x,y)$ we suppose that there are two functions U_1 and U_2 that satisfy the given conditions (1)–(4). Then their difference $U=U_1-U_2$ is a solution to differential equation (1), satisfying conditions (2)–(4) with zero functions $\varphi_1(x,y)=\varphi_2(x,y)=0$. By virtue of relations (23) and (24) we have that $\varphi_{1\,n,m}=\varphi_{2\,n,m}=0$. Hence, we that obtain from formulas (18) and (26) in the domain Ω follows the zero identity

$$\int_0^l\int_0^l(t-t_0)^{2-\gamma}U(t,x,y)\,\vartheta_{n,m}(x,y)\,dx\,dy\equiv0.$$

Hence, by virtue of the completeness of the systems of eigenfunctions $\left\{\sqrt{\frac{2}{l}}\sin\frac{\pi n}{l}x\right\}$, $\left\{\sqrt{\frac{2}{l}}\sin\frac{\pi m}{l}y\right\}$ in $L_2\left(\Omega_l^2\right)$ we deduce that $U(t,x,y)\equiv0$ for all $x\in\Omega_l^2\equiv[0,l]^2$ and $t\in\Omega_T\equiv[0;T]$.

Since $(t-t_0)^{2-\gamma}U(t,x,y)\in C\left(\overline{\Omega}\right)$, then $(t-t_0)^{2-\gamma}U(t,x,y)\equiv0$ in the domain $\overline{\Omega}$. Therefore, the solution to the initial value problem (1)–(4) is unique in the domain $\overline{\Omega}$.

5 Term-by-term differentiation possibility

Theorem 2. Let the conditions of the Theorem 1 be fulfilled. Then term-by-term differentiation of the series (26) is possible.

Proof. The function (26) we differentiate the required number of times

$$\frac{\partial^{4k}}{\partial x^{4k}}(t-t_0)^{2-\gamma}U(t,x,y)=\sum_{n,m=1}^{\infty}\left(\frac{\pi n}{l}\right)^{4k}\vartheta_{n,m}(x,y)\,(t-t_0)^{2-\gamma}u_{n,m}(t), \tag{34}$$

$$\frac{\partial^{4k}}{\partial y^{4k}}(t-t_0)^{2-\gamma}U(t,x,y)=\sum_{n,m=1}^{\infty}\left(\frac{\pi m}{l}\right)^{4k}\vartheta_{n,m}(x,y)\,(t-t_0)^{2-\gamma}u_{n,m}(t), \tag{35}$$

where $u_{n,m}(t)$ is defined from the presentation (25).

The expansion of the following functions into Fourier series are defined in a similar way

$$(t-t_0)^{2-\gamma}D^{\alpha,\gamma}U(t,x,y),\quad \frac{\partial^{4k}}{\partial x^{4k}}(t-t_0)^{2-\gamma}D^{\alpha,\gamma}U(t,x,y),\quad \frac{\partial^{4k}}{\partial y^{4k}}(t-t_0)^{2-\gamma}D^{\alpha,\gamma}U(t,x,y).$$

We show the convergence of series (34) and (35). Analogously to the case of estimate (33), applying the Cauchy–Schwarz inequality, we obtain

$$\left|\frac{\partial^{4k}}{\partial x^{4k}}(t-t_0)^{2-\gamma}U(t,x,y)\right|\le\sum_{n,m=1}^{\infty}\left(\frac{\pi n}{l}\right)^{4k}\left|(t-t_0)^{2-\gamma}u_{n,m}(t)\right|\cdot|\vartheta_{n,m}(x,y)|\le$$

$$\le\frac{2}{l}\left(\frac{\pi}{l}\right)^{4k}M_2\left[\sum_{n,m=1}^{\infty}n^{4k}|\varphi_{1n,m}|+\sum_{n,m=1}^{\infty}n^{4k}|\varphi_{2n,m}|+\sum_{n,m=1}^{\infty}n^{4k}|b_{n,m}|\right]\le$$

$$\le\frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2}M_2\left[\sum_{n,m=1}^{\infty}\frac{\left|\varphi_{1n,m}^{(8k+2)}\right|}{n\,m^{4k+1}}+\sum_{n,m=1}^{\infty}\frac{\left|\varphi_{2n,m}^{(8k+2)}\right|}{n\,m^{4k+1}}+\sum_{n,m=1}^{\infty}\frac{\left|b_{n,m}^{(8k+2)}\right|}{n\,m^{4k+1}}\right]\le$$

$$\le\frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2}M_2M_4\left[\left\|\varphi_{1n,m}^{(8k+2)}\right\|_{\ell_2}+\left\|\varphi_{2n,m}^{(8k+2)}\right\|_{\ell_2}+\left\|b_{n,m}^{(8k+2)}\right\|_{\ell_2}\right]\le$$

$$\le\gamma_2\left[\sum_{i=1}^{2}\left\|\frac{\partial^{8k+2}\varphi_i(x,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}+\left\|\frac{\partial^{8k+2}b(x,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}\right]<\infty, \tag{36}$$

where $\gamma_2=\left(\frac{2}{l}\right)^2\left(\frac{l}{\pi}\right)^{4k+2}M_2M_4$, $M_4=\sqrt{\sum\limits_{n,m=1}^{\infty}\frac{1}{n\,m^{8k+2}}}<\infty$;

$$\left|\frac{\partial^{4k}}{\partial y^{4k}}(t-t_0)^{2-\gamma}U(t,x,y)\right|\le\sum_{n,m=1}^{\infty}\left(\frac{\pi m}{l}\right)^{4k}\left|(t-t_0)^{2-\gamma}u_{n,m}(t)\right|\cdot|\vartheta_{n,m}(x,y)|\le$$

$$\le\frac{2}{l}\left(\frac{\pi}{l}\right)^{4k}M_2\left[\sum_{n,m=1}^{\infty}m^{4k}|\varphi_{1n,m}|+\sum_{n,m=1}^{\infty}m^{4k}|\varphi_{2n,m}|+\sum_{n,m=1}^{\infty}m^{4k}|b_{n,m}|\right]\le$$

$$\le\frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2}M_2\left[\sum_{n,m=1}^{\infty}\frac{\left|\varphi_{1n,m}^{(8k+2)}\right|}{n^{4k+1}m}+\sum_{n,m=1}^{\infty}\frac{\left|\varphi_{2n,m}^{(8k+2)}\right|}{n^{4k+1}m}+\sum_{n,m=1}^{\infty}\frac{\left|b_{n,m}^{(8k+2)}\right|}{n^{4k+1}m}\right]\le$$

$$\leq \frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2} M_2 M_5 \left[\left\| \varphi_{1\,n,m}^{(8k+2)} \right\|_{\ell_2} + \left\| \varphi_{2\,n,m}^{(8k+2)} \right\|_{\ell_2} + \left\| b_{n,m}^{(8k+2)} \right\|_{\ell_2}\right] \leq$$

$$\leq \gamma_3 \left[\sum_{i=1}^{2} \left\| \frac{\partial^{8k+2}\varphi_i(x,\,y)}{\partial\, x^{4k+1}\partial\, y^{4k+1}} \right\|_{L_2(\Omega_l^2)} + \left\| \frac{\partial^{8k+2} b\,(x,\,y)}{\partial\, x^{4k+1}\partial\, y^{4k+1}} \right\|_{L_2(\Omega_l^2)}\right] < \infty, \tag{37}$$

where $\gamma_3 = \left(\frac{2}{l}\right)^2 \left(\frac{l}{\pi}\right)^{4k+2} M_2 M_5$, $M_5 = \sqrt{\sum\limits_{n,\,m=1}^{\infty} \frac{1}{n^{8k+2m}}} < \infty$.

The convergence of Fourier series for functions

$$(t-t_0)^{2-\gamma}\, D^{\alpha,\gamma} U\,(t,\,x,\,y), \quad \frac{\partial^{4k}}{\partial\, x^{4k}}\,(t-t_0)^{2-\gamma}\, D^{\alpha,\gamma} U\,(t,\,x,\,y), \quad \frac{\partial^{4k}}{\partial\, y^{4k}}\,(t-t_0)^{2-\gamma}\, D^{\alpha,\gamma} U\,(t,\,x,\,y)$$

is easy to prove, and the necessary estimates are obtained similarly to the cases of estimates (33), (36), and (37). Therefore, the function $U\,(t,\,x,\,y)$ belongs to the class of functions (4). Theorem 2 is proved.

6 Stability of the solution $U(t,x,y)$ with respect to given functions

Theorem 3. Suppose that all the conditions of Theorem 2 are fulfilled. Then, the function $U\,(t,\,x,\,y)$ as a solution to the problem (1)–(4) is stable with respect to given functions $\varphi_1(x,\,y)$, $\varphi_2(x,\,y)$.

Proof. We show that the solution to the differential equation (1) $U\,(t,\,x,\,y)$ is stable with respect to a given functions $\varphi_1(x,\,y)$, $\varphi_2(x,\,y)$. Let $U_1(t,\,x,\,y)$ and $U_2(t,\,x,\,y)$ be two different solutions to the initial value problem (1)–(4), corresponding to two different values of the functions $\varphi_{11}(x,\,y)$, $\varphi_{12}(x,\,y)$ and $\varphi_{21}(x,\,y)$, $\varphi_{22}(x,\,y)$, respectively.

We put that $|\,\varphi_{11\,n,m} - \varphi_{12\,n,m}\,| + |\,\varphi_{21\,n,m} - \varphi_{22\,n,m}\,| < \delta_{n,m}$, where $0 < \delta_{n,m}$ is sufficiently small positive quantity and the series $\sum\limits_{n,\,m=1}^{\infty} |\,\delta_{n,m}\,|$ is convergent. Then, considering this, by virtue of the conditions of the theorem, from the Fourier series (26) it is easy to obtain that

$$\left|\, t^{2-\gamma}\left[U_1(t,\,x,\,y) - U_2(t,\,x,\,y)\right] \right| \leq$$

$$\leq \frac{2}{l}\sigma_3 \sum_{n,\,m=1}^{\infty} \left[|\,\varphi_{11\,n,m} - \varphi_{12\,n,m}\,| + |\,\varphi_{21\,n,m} - \varphi_{22\,n,m}\,|\right] < \frac{2}{l}\sigma_3 \sum_{n,\,m=1}^{\infty} |\,\delta_{n,m}\,| < \infty.$$

If we put $\varepsilon = \frac{2}{l}\sigma_3 \sum\limits_{n,\,m=1}^{\infty} |\,\delta_{n,m}\,| < \infty$, then from last estimate we finally obtain assertions about the stability of the solution to the differential equation (1) with respect to a given functions $\varphi_1(x,\,y)$, $\varphi_2(x,\,y)$ in (2). The theorem 3 is proved.

Similarly, it is proved that there holds the following theorem.

Theorem 4. Suppose that all the conditions of Theorem 2 are fulfilled. Then, the function $U\,(t,\,x,\,y)$ as a solution to the problem (1)–(4) is stable with respect to given function $b\,(x,\,y)$ in the right-hand side of the differential equation (1).

Conclusions

In three-dimensional domain, the solvability of a mixed problem for a Hilfer type partial differential equation (1) of the higher even order with initial value conditions (2) and small positive parameters in mixed derivatives is considered. Suppose that the conditions of smoothness are fulfilled. Then the solution to this fractional differential equation of higher order for $1 < \alpha < \gamma \leq 2$ is studied in the class of regular functions. The Fourier series method is used and a countable system of ordinary differential equations is obtained (21). The initial value problem is integrated as an ordinary differential equation. We obtained the presentation for unknown function $U(t, x, y)$. Using the Cauchy–Schwarz inequality and the Bessel inequality, we proved the absolute and uniform convergence of the obtained Fourier series (26) for function $U(t, x, y)$ and its derivatives. It is proved that solution to the problem (1)–(4) $U(t, x, y)$ is stable with respect to given functions $b(x, y)$ and $\varphi_i(x, y),\ \ i = 1, 2$.

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References

1 Асанова А.Т. Об однозначной разрешимости начально-краевой задачи для системы дифференциальных уравнений в частных производных третьего порядка / А.Т. Асанова // Дифференц. уравнения. — 2021. — 57. — № 1. — С. 125–129.

2 Асанова А.Т. О решении начально-краевой задачи для системы дифференциальных уравнений в частных производных третьего порядка / А.Т. Асанова // Изв. вузов. Математика. — 2019. — № 4. — С. 15–26.

3 Assanova A.T. An initial-boundary value problem for a higher-order partial differential equation / A.T. Assanova, A.D. Abildayeva, A.P. Sabalakhova // News of the National Academy of Sciences of the Republic of Kazakhstan. Series Physico-Mathematical. — 2020. — 302. — No. 2. — P. 133–141.

4 Assanova A.T. Initial-boundary-value problem for an integro-differential equation of the third order / A.T. Assanova, G.K. Vasilina, A.E. Imanchiev // Journal of Mathematical Sciences. — 2021. — 253. — No. 2. — P. 181–203.

5 Assanova A.T. A nonlocal problem for loaded partial differential equations of fourth order / A.T. Assanova, A.E. Imanchiyev, Zh.M. Kadirbayeva // Bulletin of the Karaganda University. Mathematics series. — 2020. — 97. — No. 1. — P. 6–16.

6 Барановская С.Н. Задача Коши и вторая смешанная задача для параболических уравнений с потенциалом Дирака, сосредоточенным в конечном числе заданных точек / С.Н. Барановская, Н.И. Юрчук // Дифференц. уравнения. — 2019. — 55. — № 3. — С. 356–360.

7 Ильин В.А. Единственность обобщенных решений смешанных задач для волнового уравнения с нелокальными граничными условиями // В.А. Ильин // Дифференц. уравнения. — 2008. — 44. — № 5. — С. 672–680.

8 Ильин В.А. Об эквивалентности двух определений обобщенного из класса L_p решения смешанной задачи для волнового уравнения / В.А. Ильин, А.А. Кулешов // Тр. МИАН. — 2014. — 284. — С. 163–168.

9 Ильин В.А. Смешанные задачи, описывающие придольные колебания стержня, состоящего из двух участков, имеющих разные плотности и разные упругости, но одинаковые импедансы / В.А. Ильин, П.В. Луференко // Докл. РАН. — 2009. — 428. — № 1. — С. 12–15.

10 Дженалиев М.Т. К обобщенной разрешимости нагруженного волнового уравнения / М.Т. Дженалиев // Дифференц. уравнения. — 1994. — 30. — № 4. — С. 723, 724.

11 Юлдашев Т.К. Нелокальная смешанная задача для интегро-дифференциального уравнения типа Буссинеска с вырожденным ядром / Т.К. Юлдашев // Укр. мат. журн. — 2016. — 68. — № 8. — С. 1115–1131.

12 Юлдашев Т.К. Смешанная задача для псевдопараболического интегро-дифференциального уравнения с вырожденным ядром / Т.К. Юлдашев // Дифференц. уравнения. — 2017. — 53. — № 1. — С. 101–110.

13 Mainardi F. Fractional calculus: some basic problems in continuum and statistical mechanics / F. Mainardi // In: Carpinteri, A., Mainardi, F. (eds.) Fractals and Fractional Calculus in Continuum Mechanics. Springer, Wien, 1997.

14 Area I. On a fractional order Ebola epidemic model / I. Area, H. Batarfi, J. Losada, J.J. Nieto, W. Shammakh, A. Torres // Advances in Difference Equations. — 2015. — 2015. — № 1. — ID 278. — P. 1–12.

15 Hussain A. Existence of solution and stability for the fractional order novel coronavirus (nCoV-2019) model / A. Hussain, D. Baleanu, M. Adeel // Advances in Difference Equations. — 2020. — 384.

16 Ullah S. A fractional model for the dynamics of tuberculosis infection using Caputo–Fabrizio derivative / S. Ullah, M.A. Khan, M. Farooq, Z. Hammouch, D. Baleanu // Discrete and Continuous Dynamical Systems. Ser. S. — 2020. — 13. — № 3. — P. 975–993.

17 Handbook of Fractional Calculus with Applications. Vols. 1-8. Tenreiro Machado J.A. (ed.). Berlin, Boston: Walter de Gruyter GmbH, 2019.

18 Kumar D. Editorial: fractional calculus and its applications in physics / D. Kumar, D. Baleanu // Frontiers Physics. — 2019. — 7. — № 6.

19 Sun H. A review on variable-order fractional differential equations: mathematical foundations, physical models, numerical methods and applications / H. Sun, A. Chang, Y. Zhang, W. Chen // Fractional Calculus and Applied Analysis. — 2019. — 22. — № 1. — P. 27–59.

20 Patnaik S. Applications of variable-order fractional operators: a review / S. Patnaik, J.P. Hollkamp, F. Semperlotti // Proceedings of the Royal Society, A. — 2020. — 476. — No. 2234. — P. 1–32.

21 Yuldashev T.K. On solvability of an initial value problem for Hilfer type fractional differential equation with nonlinear Maxima / T.K. Yuldashev, B.J. Kadirkulov // Daghestan Electronic Mathematical Reports. — 2020. — 14. — P. 48–65.

22 Yuldashev T.K. Boundary value problem for weak nonlinear partial differential equations of mixed type with fractional Hilfer operator / T.K. Yuldashev, B.J. Kadirkulov // Axioms. — 2020. — 9. — No. 2. — ID 68. — P. 1–19.

23 Yuldashev T.K. Nonlocal problem for a mixed type fourth-order differential equation with Hilfer fractional operator / T.K. Yuldashev, B.J. Kadirkulov // Ural Mathematical Journal. — 2020. — 6. — No. 1. — P. 153–167.

24 Yuldashev T.K. Inverse problem for a mixed type integro-differential equation with fractional order Caputo operators and spectral parameters / T.K. Yuldashev, E.T. Karimov // Axioms. — 2020. — 9. — 4. — ID 121. — P. 1–19.

25 Abdullaev O.Kh. Non-local problems with integral gluing condition for loaded mixed type equations involving the Caputo fractional derivative / O.Kh. Abdullaev, K.B. Sadarangani // Electronic Journal of Differential Equations. — 2016. — 2016. — No. 164. — 1–10 p.

26 Бердышев А.С. Об одной нелокальной задаче для параболического уравнения четвертого порядка с дробным оператором Джрбашяна-Нерсесяна / А.С. Бердышев, Б.Ж. Кадиркулов // Дифференц. уравнения. — 2016. — 52. — No. 1. — С. 123–128.

27 Islomov B.I. On a boundary value problem for a parabolic-hyperbolic equation with fractional order Caputo operator in rectangular domain / B.I. Islomov, U.Sh. Ubaydullayev // Lobachevskii Journal of Mathematics. — 2020. — 41. — No. 9. — P. 1801–1810.

28 Karimov E.T. Frankl-type problem for a mixed type equation with the Caputo fractional derivative / E.T. Karimov // Lobachevskii Journal of Mathematics. — 2020. — 41. — No. 9. — P. 1829–1836.

29 Karimov E. Nonlocal initial problem for second order time-fractional and space-singular equation / E. Karimov, M. Mamchuev, M. Ruzhansky // Hokkaido Mathematical Journal. 2020. — 49. — No. 2. — P. 349–361.

30 Malik S.A. An inverse source problem for a two parameter anomalous diffusion equation with nonlocal boundary conditions / S.A. Malik, S. Aziz // Computers and Mathematics with Applications. — 2017. — 73. — No. 12. — P. 2548–2560.

31 Serikbaev D. A source inverse problem for the pseudo-parabolic equation with the fractional Sturm–Liouville operator / D. Serikbaev, N. Tokmagambetov // Bulletin of the Karaganda university. Mathematics series. — 2020. — No. 4 (99). — P. 143–151.

32 Sadarangani K.B. A non-local problem with discontinuous matching condition for loaded mixed type equation involving the Caputo fractional derivative / K.B. Sadarangani, O.Kh. Abdullaev // Advances in Difference Equations. — 2016. — No. 241. — P. 1–10.

33 Sadarangani K.B. About a Problem for Loaded Parabolic-Hyperbolic Type Equation with Fractional Derivatives / K.B. Sadarangani, O.Kh. Abdullaev // International Journal of Differerential Equations. — 2016. — ID 9815796. — P. 1–6.

34 Yuldashev T.K. Inverse boundary value problem for a fractional differential equations of mixed type with integral redefinition conditions / T.K. Yuldashev, B.J. Kadirkulov // Lobachevskii Journal of Mathematics. — 2021. — 42. — No. 3. — P. 649–662.

35 Kim M.Ha. Operational method for solving multi-term fractional differential equations with the generalized fractional derivatives / M.Ha. Kim, Ri.G. Chol, O.H. Chol // Fractional Calculus and Applied Analyses. — 2014. — 17. — No. 1. — P. 79–95.

36 Джрбашян М.М. Интегральные преобразования и представления функций в комплексной области. — М.: Наука, 1966. — 671 с.

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Жоғары ретті Гильфер типті дифференциалдық теңдеу үшін аралас есеп туралы

Үшөлшемді облыста аралас туындыларда бастапқы шарттары және шағын оң параметрлері бар жұп ретті Гильфер типті дербес туындылы теңдеу үшін аралас есептің шешілу мүмкіндігі қарастырылған. Бұл жоғары ретті бөлшек дифференциалдық теңдеудің шешімі тұрақты функциялар класында зерттелген. Бөлшек операторының реті $1 < \alpha < 2$ тең болатын жағдай зерттелді. Фурье қатарларының әдісі қолданылып, қарапайым дифференциалдық теңдеулердің есептелетін жүйесі алынды. Бастапқы есеп қарапайым дифференциалдық теңдеу ретінде интегралданады, ал интегралдық тұрақтылар берілген бастапқы шарттарды пайдалана отырып табылады. Коши-Шварц теңсіздігі мен Бессель теңсіздігін пайдаланып, алынған Фурье қатарының абсолютті және біркелкі жинақтылығы дәлелденді. Берілген функцияларға қатысты есептің шешімінің тұрақтылығы да зерттелді.

Кілт сөздер: бөлшек реті, Гильфер операторы, аралас есеп, Фурье қатары, бастапқы шарттар, бірегей шешім.

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О смешанной задаче для дифференциального уравнения типа Хильфера высшего порядка

В трехмерной области рассмотрена разрешимость смешанной задачи для дифференциального уравнения в частных производных типа Хильфера четного порядка с начальными условиями и малыми положительными параметрами в смешанных производных. Решение этого дробного дифференциального уравнения высшего порядка изучено в классе регулярных функций. Исследован случай, когда порядок дробного оператора равен $1 < \alpha < 2$. Применен метод рядов Фурье, и получена счетная система обыкновенных дифференциальных уравнений. Начальная задача интегрируется как обыкновенное дифференциальное уравнение, и интегральные константы находятся с помощью заданных начальных условий. С помощью неравенства Коши–Шварца и неравенства Бесселя доказана абсолютная и равномерная сходимость полученного ряда Фурье. Изучена также устойчивость решения задачи по заданным функциям.

Ключевые слова: дробный порядок, оператор Хильфера, смешанная задача, ряды Фурье, начальные условия, однозначная разрешимость.

References

1 Assanova, A.T. (2021). Ob odnoznachnoi razreshimosti nachalno-kraevoi zadachi dlia sistemy differentsialnykh uravnenii v chastnykh proizvodnykh tretego poriadka [Unique solvability of an initial-boundary value problem for a system of third-order partial differential equations]. *Differentsialnye uravneniia – Differential Equations, 57*, 1, 125–129 [in Russian].

2 Assanova, A.T. (2019). O reshenii nachalno-kraevoi zadachi dlia sistemy differentsialnykh uravnenii v chastnykh proizvodnykh tretego poriadka [Solution to initial-boundary value problem for a system of partial differential equations of the third order]. *Izvestiia vuzov. Matematika – Russian Mathematics (Izv. VUZ)*, 4, 15–26 [in Russian].

3 Assanova, A.T., & Abildayeva, A.D. (2020). Sabalakhova A.P. An initial-boundary value problem for a higher-order partial differential equation. *News of the National Academy of Sciences of the Republic of Kazakhstan. Series Physico-Mathematical, 302*, 2, 133–141.

4 Assanova, A.T., Vasilina, G.K., & Imanchiev, A.E. (2021). Initial-boundary-value problem for an integro-differential equation of the third order. *Journal of Mathematical Sciences, 253*, 2, 181–203.

5 Assanova, A.T., Imanchiyev, A.E., & Kadirbayeva, Zh.M. (2020). A nonlocal problem for loaded partial differential equations of fourth order. *Bulletin of the Karaganda University. Mathematics series, 97*, 1, 6–16.

6 Baranovskaya, S.N., & Yurchuk, N.I. (2019). Zadacha Koshi i vtoraia smeshannaia zadacha dlia parabolicheskikh uravnenii s potentsialom Diraka, sosredotochennym v konechnom chisle zadannykh tochek [Cauchy problem and the second mixed problem for parabolic equations with Dirac potential concentrated in a finite number of given points]. *Differentsialnye uravneniia – Differential Equations, 55*, 3, 356–360 [in Russian].

7 Il'in, V.A. (2008). Edinstvennost obobshchennykh reshenii smeshannykh zadach dlia volnovogo uravneniia s nelokalnymi granichnymi usloviiami [Uniqueness of generalized solutions to mixed problems for the wave equation with nonlocal boundary conditions]. *Differentsialnye uravneniia – Differential Equations, 44*, 5, 672–680 [in Russian].

8 Ilin, V.A., & Kuleshov, A.A. (2014). Ob ekvivalentnosti dvukh opredelenii obobshchennogo iz klassa L_p resheniia smeshannoi zadachi dlia volnovogo uravneniia [Equivalence of two definitions of a generalized L_p solution to the initial–boundary value problem for the wave equation]. *Trudy MIAN – Proceedings of Steklov Mathematical Institute, 284,* 163–168.

9 Ilin, V.A., & Luferenko, P.V. (2009). Smeshannye zadachi, opisyvaiushchie pridolnye kolebaniia sterzhnia, sostoiashchego iz dvukh uchastkov, imeiushchikh raznye plotnosti i raznye uprugosti, no odinakovye impedansy [Mixed problems describing longitudinal oscillations of a rod consisting of two segments with different densities and different elasticities but equal impedances]. *Doklady Rossiiskoi akademii nauk – Reports of Academy of Sciences, 428*, 1, 12–15 [in Russian].

10 Jenaliev, M.T. (1994). K obobshchennoi razreshimosti nagruzhennogo volnovogo uravneniia [To the generalized solvability of a loaded wave equation]. *Differentsialnye uravneniia – Differential Equations, 30*, 4, 723–724.

11 Yuldashev, T.K. (2016). Nelokalnaia smeshannaia zadacha dlia integro-differentsialnogo uravneniia tipa Bussineska s vyrozhdennym yadrom [Nonlocal mixed-value problem for a Boussinesq-type integro-differential equation with degenerate kernel]. *Ukrainskii matematicheskii zhurnal — Ukrainian Mathematical Journal, 68*, 8, 1115–1131 [in Russian].

12 Yuldashev, T.K. (2017). Smeshannaia zadacha dlia psevdoparabolicheskogo integro-differentsialnogo uravneniia s vyrozhdennym yadrom [Mixed problem for pseudoparabolic integro-differential equation with degenerate kernel]. *Differentsialnye uravneniia – Differential Equations, 53*, 1, 101–110 [in Russian].

13 Mainardi, F. (1997). Fractional calculus: some basic problems in continuum and statistical mechanics. In: Carpinteri, A., Mainardi, F. (Eds.). Fractals and Fractional Calculus in Continuum Mechanics. Springer, Wien, 1997.

14 Area, I., Batarfi, H., Losada, J., Nieto, J.J., Shammakh, W., & Torres, A. (2015). On a fractional order Ebola epidemic model. *Advances in Difference Equations, 2015*, 1, ID 278, 1–12.

15 Hussain, A., Baleanu, D., & Adeel, M. (2020). Existence of solution and stability for the fractional order novel coronavirus (nCoV-2019) model. *Advances in Difference Equations, 2020*, 384.

16 Ullah, S., Khan, M.A., Farooq, M., Hammouch, Z., & Baleanu, D. (2020). A fractional model for the dynamics of tuberculosis infection using Caputo–Fabrizio derivative. *Discrete and Continuous Dynamical Systems, Ser. S, 13*, 3, 975–993.

17 Tenreiro Machado, J.A. (Ed.). (2019). Handbook of Fractional Calculus with Applications. Vols. 1-8. Berlin, Boston: Walter de Gruyter GmbH, 2019.

18 Kumar, D., & Baleanu, D. (2019). Editorial: fractional calculus and its applications in physics. *Frontiers Physics, 7*, 6.

19 Sun, H., Chang, A., Zhang, Y., & Chen, W. (2019). A review on variable-order fractional differential equations: mathematical foundations, physical models, numerical methods and applications. *Fractional Calculus and Applied Analysis, 22*, 1, 27–59.

20 Patnaik, S., Hollkamp, J.P., & Semperlotti, F. (2020). Applications of variable-order fractional operators: a review. *Proceedings of the Royal Society, A., 476*, 2234, 1–32.

21 Yuldashev, T.K., & Kadirkulov, B.J. (2020). On solvability of an initial value problem for Hilfer type fractional differential equation with nonlinear Maxima. *Daghestan Electronic Mathematical Reports, 14,* 48–65.

22 Yuldashev, T.K., & Kadirkulov, B.J. (2020). Boundary value problem for weak nonlinear partial differential equations of mixed type with fractional Hilfer operator. *Axioms, 9*, 2, ID 68, 1–19.

23 Yuldashev, T.K., & Kadirkulov, B.J. (2020). Nonlocal problem for a mixed type fourth-order differential equation with Hilfer fractional operator. *Ural Mathematical Journal, 6*, 1, 153–167.

24 Yuldashev, T.K., & Karimov, E.T. (2020). Inverse problem for a mixed type integro-differential equation with fractional order Caputo operators and spectral parameters. *Axioms, 9*, 4, ID 121, 1–24.

25 Abdullaev, O.Kh., & Sadarangani, K.B. (2016). Nonlocal problems with integral gluing condition for loaded mixed type equations involving the Caputo fractional derivative. *Electronic Journal of Differential Equations, 2016*, 164, 1–10.

26 Berdyshev, A.S., & Kadirkulov, B.J. (2016). Ob odnoi nelokalnoi zadache dlia parabolicheskogo uravneniia chetvertogo poriadka s drobnym operatorom Dzhrbashiana–Nersesiana [On a nonlocal problem for a fourth-order parabolic equation with the fractional Dzhrbashyan–Nersesyan operator]. *Differentsialnye uravneniia – Differential Equations, 52*, 1, 123–128 [in Russian].

27 Islomov, B.I., & Ubaydullayev, U.Sh. (2020). On a boundary value problem for a parabolic-hyperbolic equation with fractional order Caputo operator in rectangular domain. *Lobachevskii Journal of Mathematics, 41*, 9, 1801–1810.

28 Karimov, E.T. (2020). Frankl-type problem for a mixed type equation with the Caputo fractional derivative. *Lobachevskii Journal of Mathematics, 41*, 9, 1829–1836.

29 Karimov, E., Mamchuev, M., & Ruzhansky, M. (2020). Nonlocal initial problem for second order time-fractional and space-singular equation. *Hokkaido Mathematical Journal, 49*, 2, 349–361.

30 Malik, S.A., & Aziz, S. (2017). An inverse source problem for a two parameter anomalous diffusion equation with nonlocal boundary conditions. *Computers and Mathematics with Applications, 73*, 12, 2548–2560.

31 Serikbaev, D., & Tokmagambetov, N. (2020). A source inverse problem for the pseudo-parabolic equation with the fractional Sturm-Liouville operator. *Bulletin of the Karaganda University. Mathematics series, 4*(99), 143–151.

32 Sadarangani, K.B., & Abdullaev, O.Kh. (2016). A nonlocal problem with discontinuous matching condition for loaded mixed type equation involving the Caputo fractional derivative. *Advances in Difference Equations, 2016*, 241, 1–12.

33 Sadarangani, K.B., & Abdullaev, O.Kh. (2016). About a Problem for Loaded Parabolic-Hyperbolic Type Equation with Fractional Derivatives. *International Journal of Differential Equations, 2016*, ID 9815796, 1–6.

34 Yuldashev, T.K., & Kadirkulov, B.J. (2021). Inverse boundary value problem for a fractional differential equations of mixed type with integral redefinition conditions. *Lobachevskii Journal of Mathematics, 42*, 3, 649–662.

35 Kim, M.Ha., Chol, Ri.G., & Chol, O.H. (2014). Operational method for solving multi-term fractional differential equations with the generalized fractional derivatives. *Fractional Calculus and Applied Analyses, 17*, 1, 79–95.

36 Dzhrbashian, M.M. (1966). Integralnye preobrazovaniia i predstavleniia funktsii v kompleksnoi oblasti [Integral transformations and representations of functions in complex domain]. Moscow: Nauka [in Russian].