DOI 10.31489/2022M/86-97
UDC 517.9

A. S. Kassym, L. K. Kussainova*

L.N.Gumilyov Eurasian National University, Astana, Kazakhstan
(E-mail: kassym.aizhan@gmail.com, kussainova.leili@gmail.com)

On the approximation of solutions of one singular differential equation on the axis

In this paper we study the problem of the best approximation by linear methods of solutions to one Triebel-type equation. This problem was solved by using estimates of the linear widths of the unit ball in corresponding spaces of differentiable functions. According to the definition, linear widths give the best estimates for the approximation of compact sets in a given normed space by linear methods which are implemented through finite-dimensional operators. The problem includes answers to the questions about the solvability of the studied equation, the construction of the corresponding weighted space of differentiable functions, the development of a method for estimating linear widths of compact sets in weighted polynomial Sobolev space. In this work, conditions are obtained under which the considered operator has a bounded inverse. The weighted Sobolev space corresponding to the posed problem is determined. Upper estimates are obtained for the counting function for a sequence of linear widths, which correspond to the posed problem. One example is constructed in which two-sided estimates of linear widths are given. The method for solving this problem can be applied to the numerical solution of non-standard ordinary differential equations on an infinite axis.

Keywords: differential equations, Triebel equations, approximation of sets by linear methods, widths of sets, weighted Sobolev spaces.

1 Introduction and Main results

In this paper, we consider the problem of the best linear approximations of solutions to the equation

$$Ty \equiv -\rho_0^{\mu}(x)y'' + q_1(x)y' + (q_0(x) + \rho_0^{\nu}(x))y_0 = f \tag{1}$$

with the right side in the Hilbert space $L_2(I)$, T is an operator satisfying conditions from the Triebel class $U^1_{\mu,\nu}(I,\rho_0)$ $(\nu > \mu + 2,\ \mu > 0)$, where $I = [0,\infty)$, on $+\infty$, i.e. [1]: $\rho_0 \geq 1$ and q_i $(i = 0, 1)$ are functions infinitely differentiable in I such that

i) $\lim\limits_{x\to\infty} \rho_0(x) = \infty$,

ii) $|\rho_0^{(k)}(x)| \leq O(\rho_0^{1+k}(x)),\ k = 0, 1, ...,$

iii) $q_0^{(k)}(x) = o(\rho_0^{\nu+k}(x)),\ q_1^{(k)}(x) = o(\rho_0^{(\nu+\mu)/2+k}(x))$ for $x \to \infty$ $(k = 0, 1, 2, ...)$.

To solve the problem, we used a modified method of localization of estimates for the widths of compact sets in weighted spaces of differentiable functions [1; 104], [2–7], as well as coercive estimates for differential operators [8, 9]. The method of local estimates developed in this paper on intervals of adjustable variable length can be used in the theory of numerical solutions of a certain class of singular differential equations on an infinite axis. All results presented in this paper are new.

We denote $V^2_{p,(\mu,\nu)}(I)$ the completion of the class $C_0^{\infty}(I)$ of functions infinitely differentiable and finite in I with respect to the norm

$$\left\| y; V^2_{p,(\mu,\nu)}(I) \right\| = \left[\sum_{k=0}^{2} \int_0^{\infty} \left| \rho_0^{l_k} y^{(k)} \right|^p dx \right]^{1/p},\ 1 \leq p < \infty,$$

*Corresponding author.
E-mail: kussainova.leili@gmail.com

where $l_k = 2^{-1}((2-k)\nu + k\mu)$ for $k = 0, 1, 2$. Let $V = V^2_{2,(\mu,\nu)}(I)$.

If

$$\sup_{x\geq 0} \left| q_0(x)\rho^{-\nu}(x) \right| = \beta_0 < 1, \tag{2}$$

$$\sup_{x\geq 0} |q_1(x)|\rho_0^{(\nu+\mu)/2}(x) = \beta < \infty,$$

then for the minimal operator

$$T_0 y = Ty,\ y \in C_0^\infty(I),$$

the following inequality holds

$$\|T_0 y; L_2(I)\| \leq (2+\beta)\, \|y\|_V\, .$$

Therefore, the operator T_0 has a closed extension

$$\widetilde{T}y = Ty,\ y \in D(\widetilde{T}) = V^2_{2,(\mu,\nu)} \subset L_2(I).$$

We defined the operator $T \overset{det}{=} \widetilde{T}$ in (1), where the norm $\|T\| \leq 2+\beta$.

Let y_i be a sequence of functions $y_i \in C_0^\infty(I)$ fundamental in the norm $\|\cdot\|_V$. Then each of the sequences $\{a_r y_j^{(r)}\}$, $\{y_j^{(r)}\}$ ($a_r = \rho_0^{lr},\ r = 0, 1, 2$) are fundamental in $L_2(I)$. Therefore, $y = \lim y_j$ in $L_2(I)$ has finite a.e. in I derivatives $y'(x)$, $y''(x)$, and $\|y\|_V < \infty$.

Theorem 1. Let condition (2) be satisfied and

$$\frac{1}{2(1-\beta_0)} \sup_{x\geq 0} \left[\left((\rho_0^\mu)'(x) \right)^2 + (q_1(x))^2 \right] \rho_0^{-(\nu+\mu)}(x) = \beta_1 < 1, \tag{3}$$

$$C_{\mu,\nu} = \frac{1}{(1-\beta_0)(1-\beta_1)}.$$

Then the operator T in (1) has an inverse T^{-1}. Therefore

$$\left\| T^{-1} \right\| \leq C_{\mu,\nu}. \tag{4}$$

Let $F = \{y \in V : \|Ty; L_2(I)\| \leq 1\}$. From (4) it follows that

$$F \subset \{y \in V : \|y\|_V \leq c\},\ c = c_{\mu,\nu}.$$

Let $\mathbb{C}$ be a bounded set in a Banach space X, containing 0, $\mathfrak{U}_k(\mathbb{C}, X)$ be the class of all continuous linear operators $U : X \to X$ of dimension $\leq k$ and such that $\mathbb{C} \subset D(U)$. The value

$$\lambda_k(\mathbb{C}, X) = \inf_{U\in\mathfrak{U}_k(X)} \sup_{x\in\mathbb{C}} \|x - \Lambda x\|_X$$

is called the linear k-width of $\mathbb{C}$ in X [10; 16].

The widths $\lambda_k(\mathbb{C}, X)$ are related to the problem of the best (linear) method for approximating the set $\mathbb{C}$ in X.

Let $\mathcal{N}(\lambda; \mathbb{C}, X) = \sum\limits_{\lambda_k(\mathbb{C},X)>\lambda} 1$ (number of widths $\lambda_k(\mathbb{C}, X) > \lambda$). In this paper, we obtain an estimate (from above) for the counting function $\mathcal{N}(\lambda; F, L_2(I))$.

Let Ω be a (Lebesgue) measurable set in $\mathbb{R}$. Here and below BX is the unit ball of space X, $L_p(\Omega)$ is the space of functions f in Ω with the seminorm $\|f; L_p(\Omega)\| = \left(\int_\Omega |f(x)|^p\, dx\right)^{1/p} < \infty$, $L_{p,loc}(I)$ is the space of all functions f in I such that $f \in L_p(G)$ for any compact $G \subset I$, $L^+_{loc}(I) = L^+_{1,loc}(I)$ is the class of non-negative locally summable (weight) functions f in I for which Lebesgue measure

$|\{x \geq t : f(x) > 0\}| > 0$ for all $t > 0$. Let $\rho, v \in L^+_{loc}(I)$, $\rho > 0$, $\rho^{-1} = 1/\rho \in L_{loc}(I)$, $\delta \in (0,1)$. Assume that

$$S_{(\delta)}(x, h; v) = \inf_{\{e\}_\delta} \int\limits_{\Delta \setminus e} v dt, \ \Delta = [x, x+h],$$

where infimum is taken over the set $\{e\}_\delta$ of all closed $e \subset \Delta$ with measure $|e| \leq \delta \, |\Delta| = \delta h$,

$$\mathcal{M}_{(\delta)}(x, h; \rho, v) = h \left(\int\limits_\Delta \rho^{-1} \right)^{1/2} \left(S_{(\delta)}(x, h; v) \right)^{1/2}.$$

Let $h(\cdot)$ be a finite positive function in I. The function $h(\cdot)$ is called the length function in I (with respect to the pair (ρ, v)), if

$$\mathcal{M}_{(\delta)}(x, h(x); \rho, v) \geq 1, \ (x \in I). \tag{5}$$

We set

$$h_{(\delta)}(x; \rho, v) = \sup \left\{ h > 0 : \ \mathcal{M}_{(\delta)}(x, h; \rho, v) \leq 1 \right\}. \tag{6}$$

Proposition 1. a) If

$$0 < h_{(\delta)}(x; \rho, v) < \infty, \tag{7}$$

then

$$M_{(\delta)}(x, h_{(\delta)}(x; \rho, v); \rho, v) = 1. \tag{8}$$

b) If $\rho = \rho_0^{2\mu}$, $v = (q_0 + \rho_0^\nu)^2$, then

$$h_{(\delta)}(x) = h_{(\delta)}(x; \rho, v) < \infty \ (x \geq 0).$$

Remark 1. The equality (8) implies the realization of condition (5). Then every finite positive function $h_{(\delta)}(x; \rho, v)$ is the length function with respect to the pair (ρ, v).

Remark 2. If $h(\cdot)$ is a length function with respect to the pair (ρ, v), then $0 < h_{(\delta)}(x; \rho, v) \leq h(x)$, $(x \geq 0)$.

We introduce a maximal operator with respect to the interval basis associated with the length function $h(\cdot)$ in I. Let

$$\mathcal{B} = \bigcup_{y \geq 0} \{\Delta = [\alpha, \beta) : y \leq \alpha < \beta \leq y + h(y)\}, \ \mathcal{B}_x = \{\Delta = \mathcal{B} : x \in \Delta\}.$$

We define a maximal operator with respect to basis $\mathcal{B}$ [11; 43].

$$M^* f(x, h(\cdot)) = \sup_{\Delta \in \mathcal{B}_x} \frac{1}{|\Delta|} \int\limits_\Delta |f(t)| dt, \ f \in L_{loc}(I).$$

Let

$$M^*_{(\delta)} f(x) = M^* f(x, h_{(\delta)}(\cdot)),$$

$$\mathcal{K}_{(\delta)}(x) = (h_{(\delta)}(x))^{3/2} \left[\int\limits_\Delta \rho_0^{-2\mu}(t) dt \right]^{1/2}, \ \Delta = [x, x + h_{(\delta)}(x)].$$

Theorem 2. Let
$$\lim_{x\to\infty} K_{(\delta)}(x) = 0 \ (0 < \delta < 1/2).$$
There is $c(\delta) > 1$, such that
$$\mathcal{N}(\lambda; F, L_2(I)) \le (c^{-1}\lambda)^{-1/2} \int\limits_{G(c^{-1}\lambda)} \left(M^*_{(\delta)}\rho_0^{-2\mu}\right)^{1/4} dx,$$
where
$$G(\lambda) = \{x > 0 : h_{(\delta)}(x) \left(M^*_{(\delta)}\rho_0^{-2\mu}\right)^{1/4} > \lambda^{1/2}\},$$
$c = 4c(\delta)c(\mu,\nu)$.

Example 1. Consider the equation
$$Ty \equiv -(3+x)^{\mu}y'' + q_1(x)y' + (q_0(x) + (3+x)^{2\nu})y = f \tag{9}$$
under the following conditions: $q_i \in C^{\infty}(I)$ $(i = 0, 1)$ satisfy conditions iii) with respect to $\rho_0(x) = 3+x$, and also
$$\sup_{x\ge 0} |q_0(x)|(3+x)^{-\nu} = \beta_0 \le 3/4,$$
$$\sup_{x\ge 0} |q_1(x)|(3+x)^{(\nu+\mu)/2} = \beta < 1/2,$$
$$\frac{1}{2} < \mu < 2 < \nu - \mu.$$
Then $\beta_1 < \frac{3}{5}$ and, by virtue of Theorem 1, for the operator T in (9) there exists T^{-1} with the norm
$$\left\|T^{-1}\right\| \le ((1-\beta_0)(1-\beta_1))^{-1} < 4 \cdot \frac{5}{2} = 10.$$
Therefore, the solution set of the equation (9) with the right side $f \in L_2(I)$ is contained in the ball $10BV$.

Let $c_{\mu} = (3^{\mu}4)^{1/2}$, $u = (4c_{\mu}^2(10\lambda^{-1}))^{1/2}$. By virtue of Theorem 2 we have
$$\mathcal{N}(c\lambda; 10BV, L_2(I)) = \mathcal{N}(10^{-1}c\lambda; BV; L_2(I)) \le$$
$$\le c_{\mu}(10^{-1}\lambda)^{-1/2} \int\limits_{G(10^{-1}\lambda)} (\beta + x)^{-\mu/2} dx \le c_{\mu}(10^{-1}\lambda)^{-1/2} \int\limits_0^u x^{-\mu/2} dx = \tag{10}$$
$$= \frac{1}{2-\mu}\left(3^{\mu} + 6(10^{-1}\lambda)^{-1}\right)^{(\nu-\mu+2)/2\nu}.$$
Let $c\lambda = \lambda_n(10BV, L_2(I))$. By (10) for any solution y of the equation (9) with $f \in BL_2(I)$ we have
$$\inf_{U\in\mathfrak{U}_n(L_2(I))} \|y - Uy; L_2(I)\| \le \lambda_n\left(10BV; L_2(I)\right) \le \kappa n^{-2(1-\frac{2-\mu}{2-\mu+\nu})},$$
where $\kappa = 160c3^{\mu}(\frac{1}{2-\mu})^{27/(242-\mu)}$.

2 Proof of main results

Proof of Theorem 1. Let $a_0(x) = q_0(x) + \rho_0^{\nu}(x)$, $a_1(x) = q_1(x)$, $a_2(x) = \rho_0^{\mu}(x)$.

1. Let $y \in C_0^{\infty}(I)$. In this case

$$(Ty, y) = \int_0^{\infty} \left[\left(\sqrt{a_2(x)} y' \right)^2 + \left(a_2'(x) + a_1(x) \right) y' y + a_0(x) y^2 \right] dx =$$

$$= \int_0^{\infty} \left[\sqrt{a_2(x)} y' + \frac{a_2'(x) + a_1(x)}{2\sqrt{a_2(x)}} y \right]^2 dx + \int_0^{\infty} w(x) y^2 dx \geq \int_0^{\infty} w(x) y^2 dx,$$

where $w(x) = a_0(x) - (a_2'(x) + a_1(x))^2 / 4a_2(x)$. By conditions (2) and (3) we get

$$a_0(x) = \rho^{\nu}(x)(1 + q_0(x)\rho^{-\nu}(x)) \geq (1 - \beta_0)\rho_0^{\nu},$$

$$\frac{(a_2'(x) + a_1(x))^2}{4a_0(x)a_2(x)} \leq \frac{1}{2(1-\beta_0)} \left[\left((\rho_0^{\mu})'(x) \right)^2 + (q_1(x))^2 \right] \rho^{-(\nu+\mu)} \leq \beta_1,$$

where

$$\inf_{x>0} w(x) \geq 1 - \beta_0,$$

$$\| Ty; L_2(I) \| \geq (1 - \beta_1) \| y; L_2(I) \| . \tag{11}$$

It follows from (11) that the operator

$$T_0 y = Ty, \ y \in C_0^{\infty}(I),$$

has a bounded inverse operator T_0^{-1} c $D(T_0^{-1}) \subset L_2(I)$. Wherein

$$\left\| T_0^{-1} \right\| \leq C_{\mu,\nu}. \tag{12}$$

The estimate (4) follows from (12).

2. Let $y \in V^2_{2,(\mu,\nu)}$, $\{y_j\}$ be a sequence from $C_0^{\infty}(I)$ converging to y in $V^2_{2,(\mu,\nu)}$. Since

$$\| Ty - Ty_j; L_2(I) \| \leq 2(2 + \beta) \| y - y_j \|_V \ \ (j \geq 1).$$

By (11) we get

$$\| Ty; L_2(I) \| = \lim_{j \to \infty} \| Ty_j; L_2(I) \| \geq C_{\mu,\nu}^{-1} \| y; L_2(I) \| .$$

Therefore, there exists an inverse operator T^{-1} and the estimate (4) takes place.

Proof of Proposition 1. Let the function $h_{(\delta)}(x; \rho, v)$ in (6) satisfies the condition (7).

a) There is a sequence $h_j > 0$ $(j \in \mathbb{N})$ converging to $h = h_{\delta}(x)$, such that $\mathcal{M}_{(\delta)}(x, h_j; \rho, v) \leq 1$. Let $\Delta_j = [x, x + h_j]$. Passing to the limit (for $j \to \infty$) in the estimate

$$h_j^2 \int_{\Delta_j} \rho^{-1} d\xi S_{(\delta)}(x, h; \rho, v) \leq \mathcal{M}^2_{(\delta)}(x, h; \rho, v) + h_j^2 \int_{\Delta_j} \rho^{-1} d\xi \int_{x+h_j}^{x+h} v d\xi,$$

leads to inequality

$$M_{(\delta)}(x, h_{(\delta)}(x; \rho, v); \rho, v) \leq 1.$$

On the other hand, there is a sequence $h'_j > h$, $M_{(\delta)}(x, h''_j; \rho, v) > 1$. Let $\Delta'_j = [x, x+h]$ with measure $|e| \leq \delta h$

$$(h'_j)^2 \int\limits_{\Delta'_j} \rho^{-1} d\xi \int\limits_{\Delta'_j \backslash e} v d\xi > 1. \tag{13}$$

Passing to the limit in (13) (for $j \to \infty$) leads to the estimates

$$h^2 \int\limits_{(\Delta)} \rho^{-1} d\xi \int\limits_{\Delta \backslash e} v d\xi \geq 1,$$

$$M_{(\delta)}(x, h_{(\delta)}(x; \rho, v)) \geq 1.$$

b) The assertion follows from the estimates

$$(1-\delta)h < S_{(\delta)}(x, h; \rho_0^{2\nu}) \leq \int\limits_x^{x+h} \rho_0^{2\nu} d\xi.$$

Let

$$K_{(\delta)}(x, h; \rho) = h^{3/2} \left(\int\limits_x^{x+h} \rho^{-1} d\xi \right)^{1/2}.$$

Lemma 1. Let $M_{(\delta)}(x, h; \rho, v) \geq 1$. There is $c(\delta) > 1$, such that for all $y \in C_0^\infty(I)$

$$\left(\int\limits_x^{x+h} |y|^2 d\xi \right)^{1/2} \leq c(\delta) K_{(\delta)}(x, h; \rho, v) \left[\int\limits_x^{x+h} (\rho(\xi) \left| y'' \right|^2 + v(\xi) \left| y \right|^2) d\xi \right]^{1/2}. \tag{14}$$

The proof of Lemma 1 is essentially a repetition of the main lemma in [2].

Let on intervals $\Delta \subset I$ $(1 \leq \rho < \infty)$ the following equality holds

$$\|y; W(\Delta)\| = \left\| \rho^\mu y''; L_2(\Delta) \right\| + \|\rho^\nu y; L_2(\Delta)\|.$$

Let $W(\Delta)$ be the space $C^\infty(\Delta)$ with the norm $W(\Delta)$. $W = W^2_{2,(\mu,\nu)}$ denote the completion of the class $C_0^\infty(I)$ in the norm $\|y; W(I)\|$. It is easy to see that

$$K_{(\delta)}(x) = K_{(\delta)}(x, h_{(\delta)}(x); \rho^{2\mu}) \leq (1-\delta)^{-1/2}. \tag{15}$$

Indeed, taking $h = h_{(\delta)}(x)$, $\Delta = [x, x+h]$, from (8) we derive

$$h \left(\int\limits_\Delta \rho^{2\mu} d\xi \right)^{1/2} = \left(\inf_{\{e\}_\delta} \int\limits_{\Delta \backslash e} \rho^{2\nu} \right)^{-1/2} \leq [(1-\delta)h]^{-1/2},$$

which inplies (15).

Lemma 2. The following estimate is true

$$\|y; L_2(I)\| \leq c(\delta) \left\| y; W^2_{2,(\mu,\nu)}(I) \right\|, \ y \in C_0^\infty(I). \tag{16}$$

Proof. Let supp$y \subset [\xi_0, \xi_1]$, $0 \le \xi_0 < \xi_1 < \infty$. Let us show that

$$\inf_{\xi_0 \le x \le \xi_1} h_\delta(x) \ge \gamma > 0. \tag{17}$$

If $h_\delta(x) < 1$ $(x \in [\xi_0, \xi_1])$, then by (8)

$$h_\delta(x) \ge \left[\int\limits_{\xi_0}^{\xi_1+1} \rho^{2\nu} d\xi \right]^{-1/2} = b > 0.$$

We take $\gamma = \min\{1, b\}$. From (17) it follows that $[\xi_0, \xi_1] \subset \bigcup_{j=1}^{\mathbb{N}} \Delta(x_j)$ $(N < \infty)$, where $x_1 = \xi_0$, $x_{j+1} = x_j + h_{(\delta)}(x_j)$, $\Delta(x_j) = [x_j, x_{j+1})$. Using the estimates (14) and (15), we derive the inequality (16) namely:

$$\|y; L_2(I)\| \le \left(\sum_{j=1}^{N} \int\limits_{\Delta(x_j)} |y|^2 \, d\xi \right)^{1/2} \le c(\delta) \left\| y; W^2_{2,(\mu,\nu)}(I) \right\|.$$

Lemma 3. The following statements are true: a)

$$0 < \widetilde{h}(x, \lambda) = \sup\{h > 0; K_{(\delta)}(x, h; \rho_0^{2\mu}) \le \lambda\} < \infty \ (x > 0),$$

$$\widetilde{h}(x, \lambda) < h_{(\delta)}(x), \text{ if } K_{(\delta)}(x) > \lambda, \tag{18}$$

$$K_{(\delta)}(x, \widetilde{h}(x, \lambda); \rho^{2\mu}) = \lambda, \tag{19}$$

b) on each $\widetilde{\Delta} = \widetilde{\Delta}(x; \lambda) = [x, x + \widetilde{h}(x, \lambda)]$ the counting function

$$\mathcal{N}(\lambda; BW(\widetilde{\Delta}), L_2(\widetilde{\Delta})) \le 1. \tag{20}$$

Proof. a) The estimates $0 < \widetilde{h}(x, \lambda) < \infty$ follow from the limit equalities

$$\lim_{h \to 0+} K_{(\delta)}(x, h; \rho_0) = 0, \ \lim_{h \to \infty} K_{(\delta)}(x, h; \rho_0) = \infty.$$

The statement (18) is trivial. Equality (19) is proved as equality (8).

b) Let $U_x y(t) = y(x) + y'(x)(t - x)$, $y \in W(\widetilde{\Delta})$. The operator $U_x \in \mathfrak{U}_2(BW(\widetilde{\Delta}), L_2(\widetilde{\Delta}))$, because $\dim U_x \le 2$ and the norm

$$\left\| U_x y; L_2(\widetilde{\Delta}) \right\| \le (|\widetilde{\Delta}|^{1/2} + |\widetilde{\Delta}|^{3/2})(|y(x)| + |y'(x)|) = b_x < \infty.$$

Use the Taylor formula with integral remainder

$$\left\| y - U_x y; L_2(\widetilde{\Delta}) \right\| = \left[\int\limits_{\widetilde{\Delta}} \left| \int\limits_x^t (t - \xi) y''(\xi) d\xi \right|^2 dt \right]^{1/2} \le \widetilde{h}(x, \lambda) \left[\int\limits_{\widetilde{\Delta}} \rho_0^{-2\mu} d\xi \right]^{1/2} \left[\int\limits_{\widetilde{\Delta}} \left| \rho_0^{\mu} y'' \right|^2 d\xi \right]^{1/2} \le \lambda.$$

Therefore, estimate (20) holds.

Lemma 4. Let $K_{(\delta)}(x) \to 0$ for $x \to \infty$. Let

$$\Delta_j = \begin{cases} [x_j, x_j + h_{(\delta)}(x_j)), & \text{if } K_{(\delta)}(x_j) \le \lambda, \\ [x_j, x_j + \widetilde{h}(x_j, \lambda)), & \text{if } K_{(\delta)}(x_j) > \lambda, \end{cases} \ (j = 1, 2, ...), \ x_1 = 0.$$

There are estimates

$$\mathcal{N}(2c\lambda; BW(I)) \le \sum_{j:K_{(\delta)}(x_j)>\lambda} \mathcal{N}(\lambda; BW(\Delta_j), L_2(\Delta_j)), \tag{21}$$

where $c = c(\delta)$ is the constant from Lemma 1.

Proof. Let $K_{(\delta)}(x_j) \le \lambda$. By (14)

$$n_j = \mathcal{N}(c\lambda; BW(\Delta_j), L_2(\Delta_j)) = 0.$$

Let $\Lambda = \{j \in \mathbb{N} : K_{(\delta)}(x_j) > \lambda\}$. Since $K_{(\delta)}(x) \to 0$ for $x \to \infty$, then $\Lambda \subset \{1, 2, ..., m\}$, where $m \in \mathbb{N}$ suffices big. If $n_j > 0$ $(j \in \Lambda)$, then for all $n \ge n_j$

$$\inf_{U \in \mathfrak{U}_{n_j}(BW(\Delta_j), L_2(\Delta_j))} \sup_{y \in BW(\Delta_j)} \|y - Uy; L_2(\Delta_j)\| \le c\lambda.$$

Therefore, for an arbitrarily small $\eta > 0$ there is an operator $U_j \in \mathfrak{U}_{n_j}(BW(\Delta_j), L_2(\Delta_j))$, for which

$$\sup_{y \in BW(\Delta_j)} \|y - U_j y; L_2(\Delta_j)\| \le (1 + \eta)c\lambda. \tag{22}$$

Let χ_j be the characteristic function of the interval $[x_j, x_j + h(x_j, \lambda))$, $\Lambda_+ = \{j \in \Lambda : n_j > 0\}$. Operator

$$Uy = \sum_{j \in \Lambda_+} \chi_j U_j(\chi, y), \ y \in L_2(I_\epsilon),$$

has finite dimension

$$\dim U \le \sum_{j \in \Lambda_+} n_j.$$

Moreover, for any $y \in BW$ it follows from Lemma 1 and (22) that

$$\int_0^\infty |y - Uy|^2 \, dx = \sum_{j \in \Lambda_+} \int_{\Delta_j} |\chi_j y - U_j(\chi, y)|^2 \, dx + \sum_{j \notin \Lambda_+} \int_{\Delta_j} |y|^2 dx \le$$

$$\le \sum_{j \in \Lambda_+} ((1+\eta)c\lambda)^2 \|y; W(\Delta_j)\|^2 + \sum_{j \notin \Lambda_+} (c\lambda)^2 \|y; W(\Delta_j)\|^2 \le ((2+\eta)c\lambda)^2 \|y\|_W^2 \le ((2+\eta)\lambda)^2. \tag{23}$$

The passage to the limit in (23) leads to the following estimates:

$$\|y - Uy; L_2(I)\| \le 2c\lambda, \ y \in BW,$$

$$\lambda_n(BW, L_2(I)) \le 2c\lambda, \text{ if } n \ge \sum_{j \in \Lambda_+} n_j,$$

$$\mathcal{N}(2c\lambda; BW, L_2(I)) \le \sum_{j \in \Lambda_+} n_j = \sum_{j:K_0(x_j)>\lambda} \mathcal{N}(c\lambda; BW(\Delta_j), L_2(\Delta)).$$

Proof of Theorem 2. It follows from (19), (21)

$$N(2c\lambda; BW, L_2(I)) \leq \lambda^{-1/2} \sum_{j\in\Lambda} \Big[K(x_j, h_j; \rho_0^{2\mu})\Big]^{1/2}, \tag{24}$$

where $\Lambda = \{j \in \mathbb{N} : K_{(\delta/2)}(x_j) > \lambda\}$, $\widetilde{h}_j = \widetilde{h}(x_j, \lambda)$ and $\Delta_j = [x_j; x_j + h_j)$ $(j \in \Lambda)$ do not intersect. Since $\Delta'_j = [x_j; x_j + h_j/2) \in B_{t,(\delta/2)}$ for all $t \in \Delta'_j$, then

$$h_j^{-1} \int\limits_{\Delta_j} \rho_0^{-2\mu} d\xi \leq M^*_{(\delta/2)} \rho_0^{-2\mu}(t),\ t \in \Delta'_j. \tag{25}$$

Therefore

$$(K(x_j, h_j))^{1/2} = h_j \left(\frac{1}{|\Delta_j|} \int\limits_{\Delta_j} \rho_0^{-2\mu} d\xi\right)^{1/4} \leq 2 \int\limits_{\Delta'_j} \left(M^*_{(\delta/2)} \rho_0^{-2\mu}\right)^{1/4} d\xi,$$

and by (24) and embedding $BV \subset BW$ we have

$$\mathcal{N}(2c\lambda; BV, L_2(I)) \leq 2\lambda^{-1/2} \sum_{j\in\Lambda} \int\limits_{\Delta'_j} \left(M^*_{(\delta/2)} \rho_0^{-2\mu}\right)^{1/4} d\xi. \tag{26}$$

Let $\Delta = [x, x+h)$ $(h > 0)$, $E_\delta(\Delta) = \{e : e = \overline{e} \subset [x, x+h] \text{ and } |e| \leq \delta h\}$. With $0 < \delta < 1/2$

$$E_\delta(\Delta) \subset E_{2\delta}(\Delta), \tag{27}$$

and for $t \in [x, x+h/2]$, $\Delta_t = [t, t+h/2)$

$$\{e_t = e \cap [t, t+h/2],\ e \in E_\delta(\Delta)\} \subset E_{2\delta}(\Delta_t). \tag{28}$$

(27) and (28) allow us to show (using simple reasoning) that

$$h_{(\delta)}(x) \geq h_{(\delta/2)}(x), \tag{29}$$

$$M^*_{(\delta)} f(x) \geq M^*_{(\delta/2)} f(x), \tag{30}$$

$$h_{(\delta)}(t) \geq h_{(\delta/2)}(x)/2,\ \text{if}\ t \in [x, x+h/2]. \tag{31}$$

Now from (25), (29)–(31) we deduce that

$$\Delta'_j \subset G(\lambda/2), \tag{32}$$

namely that for all $t \in \Delta'_j$ we have

$$(h_\delta(t))^2 \left(M^*_{(\delta)} \rho_0^{-2\mu}(t)\right)^{1/2} \geq \frac{1}{4} h_j^2 \left(h_j^{-1} \int\limits_{\Delta_j} \rho_0^{-2\mu} d\xi\right)^{1/2} = \frac{1}{4}\lambda.$$

Since $F \subset aBV \subset aBW$, $a = c_{\mu,\nu}$, then by (26) and (32)

$$\mathcal{N}(\lambda; F, L_2(I)) \leq 2(\frac{ac}{\lambda})^{1/2} \int\limits_{G(\lambda/ac)} \left(M^*_{(\delta)} \rho_0^{-2\mu}\right)^{1/4} d\xi.$$

Acknowledgments

This study was funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP08856104).

References

1 Мынбаев К.Т. Весовые функциональные пространства и спектр дифференциальных операторов / К.Т. Мынбаев, М.О. Отелбаев. — М.: Наука, 1988. — 288 с.

2 Айтенова М.С. Об асимптотике распределения аппроксимативных чисел вложения весовых классов / М.С. Айтенова, Л.К. Кусаинова // Мат. журн. — 2002. — 2. — № 1. — С. 3–9.

3 Айтенова М.С. Об асимптотике распределения аппроксимативных чисел вложения весовых классов / М.С. Айтенова, Л.К. Кусаинова // Мат. журн. — 2002. — 2. — № 2. — С. 7–14.

4 Отелбаев М.О. Оценки спектра одного класса дифференциальных операторов / М.О. Отелбаев, Л.К. Кусаинова, А. Булабаев // Збірник працъiнституту математики НАН Украіни. — 2009. — 6. — № 1. — С. 165–190.

5 Муратбеков М.Б. Существование решений и оценки поперечников множеств, связанных с решениями нелинейных уравнений неклассического типа / М.Б. Муратбеков, А.Б. Шыракбаев, Б.А. Баулыбаева // Вестн. Караганд. ун-та. Сер. Математика. — 2012. — 66. — № 2.— С. 81–86.

6 Кусаинова Л.К. Аппроксимативные оценки для одного дифференциального оператора в весовом гильбертовом пространстве / Л.К. Кусаинова, Я.Т. Султанаев, Г.К. Мурат // Дифференциальные уравнения. — 2019. — 55. — № 12. — С. 1644–1651.

7 Кусаинова Л.К. Об одном обратимом расширении несамосопряженного сингулярного дифференциального оператора на полуоси / Л.К. Кусаинова, Я.Т. Султанаев, А.С. Касым // Дифференциальные уравнения. — 2021. — 57. — № 10. — С. 1431–1435.

8 Гаибов Д.С. Коэрцитивные оценки и разделимость дифференциальных операторов класса Трибеля: автореф. — Душанбе, 2008.

9 Муратбеков М.Б. Разделимость и оценки поперечников множеств, связанных с областью определения нелинейного оператора типа Шрёдингера / М.Б. Муратбеков // Дифференциальные уравнения. — 1991. — 27. — № 6. — С. 1034–1042.

10 Тихомиров В.М. Некоторые вопросы теории приближений / В.М. Тихомиров. — М.: Изд-во МГУ, 1976. — 305 с.

11 Гусман М.—Дифференцирование интегралов в $\mathbb{R}_n$ / М. Гусман. М.: Мир, 1978. — 200 с.

А. С. Қасым, Л. К. Кусаинова

Л.Н. Гумилев атындағы Еуразия ұлттық университеті, Астана, Қазақстан

Бір сингулярлы дифференциалдық теңдеудің шешімдерінің аппроксимациясы туралы

Мақалада Трибел типті дифференциалдық теңдеудің шешімдерін сызықтық әдістермен ең жақсы жуықтау мәселесі зерттелді. Бұл есептер дифференциалданатын функциялардың сәйкес кеңістіктеріндегі бірлік шарының көлденең сызығын бағалау арқылы шешілді. Анықтамаға сәйкес, көлденең сызық берілген нормаланған кеңістіктегі компакт жиындарды сызықтық ақырлыөлшемді операторлар арқылы жүзеге асырылатын сызықтық әдістермен жуықтаудың ең жақсы бағалауын береді. Тапсырмада зерттелетін теңдеудің шешілетіндігі туралы, сәйкес дифференциалданатын функциялардың салмақты кеңістігін құру, Соболев салмақты полиномиалды кеңістігіндегі компакт жиындардың көлденең сызығын бағалау үшін әдістемесін құру туралы мәселелер қамтылды. Бұл жұмыста қарастырылған оператордың шектелген кері операторы болудың шарттары алынды. Қойылған мәселеге сәйкес Соболев салмақты кеңістігі анықталды, көлденең сызық тізбегі үшін санау функциясының жоғарғы бағалаулары алынды. Сонымен қатар, көлденең сызықтың екі жақты бағалаулары берілген мысал құрастырылды. Бұл есептің шешу әдісін шексіз осьте стандартты емес кәдімгі дифференциалдық теңдеулерді сандық түрде шешу үшін қолдануға болады.

Кілт сөздер: дифференциалдық теңдеулер, Трибел теңдеулері, жиындарды сызықтық әдістермен жуықтау, көлденең жиындар, Соболевтің салмақты кеңістіктері.

А.С. Касым, Л.К. Кусаинова

Евразийский национальный университет имени Л.Н.Гумилева, Астана, Казахстан

Об аппроксимации решений одного сингулярного дифференциального уравнения на оси

В статье исследована задача о наилучшем приближении линейными методами решений одного уравнения типа Трибеля. Эта задача решалась с помощью оценок линейных поперечников единичного шара в соответствующих пространствах дифференцируемых функций. Согласно определению, линейные поперечники дают наилучшие оценки аппроксимации компактов в заданном нормированном пространстве линейными методами, реализуемыми через конечномерные операторы. Задача включает ответы на вопросы о разрешимости изучаемого уравнения, построение соответствующего весового пространства дифференцируемых функций, разработку метода для оценки линейных поперечников компактов в весовом полиномиальном пространстве Соболева. В работе получены условия, при которых рассматриваемый оператор становится ограниченно обратным. Определено весовое пространство Соболева, соответствующее поставленной задаче. Получены верхние оценки считающей функции для последовательности линейных поперечников, соответствующих поставленной проблеме. Построен один пример, в котором даны двусторонние оценки линейных поперечников. Метод решения этой задачи может быть применен к численному решению нестандартных обыкновенных дифференциальных уравнений на бесконечной оси.

Ключевые слова: дифференциальные уравнения, уравнения Трибеля, аппроксимация множеств линейными методами, поперечники множеств, весовые пространства Соболева.

References

1 Mynbaev, K.T., & Otelbaev, M.O. (1988). *Vesovye funktsionalnye prostranstva i spektr differentsialnykh operatorov [Weighted function spaces and the spectrum of differential operators]*. Moscow: Nauka [in Russian].

2 Aitenova, M.S., & Kusainova, L.K. (2002). Ob asimptotike raspredeleniia approksimativnykh chisel vlozheniia vesovykh klassov [On the asymptotics of the distribution of approximate embedding numbers of weight classes]. *Matematicheskii zhurnal — Mathematical Journal, 2*(1), 3–9 [in Russian].

3 Aitenova, M.S., & Kusainova, L.K. (2002). Ob asimptotike raspredeleniia approksimativnykh chisel vlozheniia vesovykh klassov [On the asymptotics of the distribution of approximate embedding numbers of weight classes]. *Matematicheskii zhurnal — Mathematical Journal, 2*(2), 7–14 [in Russian].

4 Otelbaev, M.O., Kusainova, L.K., & Bulabaev, A. (2009). Otsenki spektra odnogo klassa differentsialnykh operatorov [Estimates of the spectrum of a class of differential operators]. *Zbirnik pratsieshnstitutu matematiki NAN Ukraini — Collection of Prats shnstitutu Mathematics NAS of Ukraine, 6*(1), 165–190 [in Russian].

5 Muratbekov, M.B., Shyrakbaev, A.B., & Baulybaeva, B.A. (2012). Sushchestvovanie reshenii i otsenki poperechnikov mnozhestv sviazannykh s resheniiami nelineinykh uravnenii neklassicheskogo tipa [The existence of solutions and evaluate widths set of solutions of nonlinear equations of non-convent]. *Vestnik Karagandinskogo universiteta. Seriia Matematika. — Bulletin of the Karaganda university. Mathematics Series, 2(66)*, 81–86 [in Russian]..

6 Kusainova, L.K., Sultanaev, Ya.T., & Murat, G.K. (2019). Approksimativnye otsenki dlia odnogo differentsialnogo operatora v vesovom gilbertovom prostranstve [Approximate estimates for one differential operator in a weighted Hilbert space] *Differentsialnye uravneniia — Differential Equations, 55*(12), 1644–1651 [in Russian].

7 Kusainova, L.K., Sultanaev, Ya.T., & Kassym, A.S. (2021). Ob odnom obratimom rasshirenii nesamosopriazhennogo singuliarnogo differentsialnogo operatora na poluosi [On a reversible extension of a non-self-adjoint singular differential operator on the semiaxis] *Differentsialnye uravneniia — Differential Equations, 57*(10), 1431–1435 [in Russian].

8 Gaibov, D.S.(2008.) *Koertsitivnye otsenki i razdelimost differentsialnykh operatorov klassa Tribelia [Coercive Estimates and Separability of Triebel Class Differential Operators]*. Dushanbe [in Russian].

9 Muratbekov, M.B. (1991). Razdelimost i otsenki poperechnikov mnozhestv, sviazannykh s oblastiu opredeleniia nelineinogo operatora tipa Shredingera [Separability and estimates of the widths of sets related to the domain of a nonlinear operator of the Schrödinger type] *Differentsialnye uravneniia — Differential Equations, 27*(6), 1034–1042 [in Russian].

10 Tikhomirov, V.M. (1976). *Nekotorye voprosy teorii priblizhenii [Some questions in the theory of approximations]*. Moscow: Izdatelstvo Moskovskogo gosudarstvennogo universiteta [in Russian].

11 Gusman, M. (1978). *Differentsirovanie integralov v* $\mathbb{R}_n$ *[Differentiation of integrals in* $\mathbb{R}_n$*]*. Moscow: Mir [in Russian].