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Existentially positive Mustafin theories of S-acts over a group

The paper is connected with the study of Jonsson spectrum notion of the fixed class of models of S-acts signature, assuming a group as a monoid of S-acts. The Jonsson spectrum notion is effective when describing theoretical-model properties of algebras classes whose theories admit joint embedding and amalgam properties. It is usually sufficient to consider universal-existential sentences true on models of this class. Up to the present paper, the Jonsson spectrum has tended to deal only with Jonsson theories. The authors of this study define the positive Jonsson spectrum notion, the elements of which can be, non-Jonsson theories. This happens because in the definition of the existentially positive Mustafin theories considered in a given paper involve not only isomorphic embeddings, but also immersions. In this connection, immersions are considered in the definition of amalgam and joint embedding properties. The resulting theories do not necessarily have to be Jonsson. We can observe that the above approach to the Jonsson spectrum study proves to be justified because even in the case of a non-Jonsson theory there exists regular method for finding such Jonsson theory that satisfies previously known notions and results, but that will also be directly related to the existentially positive Mustafin theory in question.

Keywords: Jonsson theory, perfect Jonsson theory, positive model theory, Jonsson spectrum, positive Jonsson theory, immersion, S-acts, Jonsson S-acts theory, $\exists PM$-theory, cosemanticity.

Introduction

This study is a continuation of previous works by the first two authors of the given paper, related to the study of the theoretical-model properties of positive Jonsson theories [1–5] and Jonsson spectrum of models classes of fixed signature [6–8]. Note that the Jonsson theories form a subclass of inductive theories and, by virtue of their definition, are not, complete. However, they distinguish a rather wide class of classical algebras, such as groups, abelian groups, fixed characteristic fields, Boolean algebras, S-acts, etc. More information about Jonsson theories can be found in [9–17]. The famous American mathematician J. Keisler in his article [18] has conventionally allocated two directions of Model Theory, «western» and «eastern», the names of which are connected with the geographical place of residence of two different directions founders of the model theory A. Robinson and A. Tarsky. It can be noted that the «Western» model theory predominantly studies complete theories and the «Eastern» Jonsson theories and each direction has its own special concepts and methods. In Jonsson writings [19, 20], classes of models of an arbitrary signature satisfying certain well-known theoretical-model and algebraic properties, in the study of which the notion of Jonsson theory has emerged originally, have been defined [18; 80]. It is clear that Jonsson theories define a class of incomplete theories and the interest in studying such theories is also fuelled by the difference between the definitions of the «Western» and «Eastern» model theories concerning the notions of model's universality and homogeneity. In consequence of this difference, which was first noticed by E.A. Palyutin [21], T.G. Mustafin has identified perfect Jonsson theories that eliminate this difference. Subsequently, T.G. Mustafin defined and studied the generalised Jonsson theories [22] and using the technique defined in this direction. In paper [22], he described generalised Jonsson theories of Boolean algebras. In a further study of Jonsson theories, several new classes of positive Jonsson theories were defined [23–25]. Interest in positivity theory arose after the appearance of the works [26–28]. In these works, it was shown that the whole classical first-order model

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theory is a special case of the positive model theory defined under these works. Subsequently, in this framework of positivity [28], there were identified positive Jonsson theories [3].

In the present paper, we do not go into positivity in the sense of work [3], we remain in the first-order model theory framework, but generalise the concepts of those classes of theories that have been considered in [22–25]. Let us focus on a brief description of some key concepts from the works [26, 27], which we need in this paper. Namely, the notions of minimal fragment and those morphisms that coincide with and are used in the positive Jonsson theories study in the works [23–25].

The main result of the paper is related to the study of properties of the positive Jonsson spectrum of a S-acts theory models' class over a group. Interest in the study of the theoretical-model aspects of S-acts theory has arisen relatively recently and is related to the works of W. Gould [29] and T.G. Mustafin [30, 31]. In work [31], T.G. Mustafin proved the fact that any complete theory is similar in some sense to some S-acts theory. Jonsson theories are also closely related to S-acts theory. Thus, in paper [32], there has been derived a connection between an existentially complete perfect Jonsson theory and some Jonsson S-acts theory. In paper [33], a description of Jonsson S-acts theories over a group was obtained. This paper obtains results generalizing the results from [33] as part of a positive spectrum study of $\exists PM$-theories of S-acts over a group.

1 Necessary concepts and results of positive model theory

Let us recall the basic definitions of the positive logic concepts and the results obtained in [26, 27].

A positive fragment (in L) is a subset $\Delta \subseteq L$ containing all atomic formulas and closed with respect to variable substitution, positive Boolean combinations and subformulas. For a given Δ the following sets of formulas are defined:

$$\Sigma = \Sigma(\Delta) = \{\exists y \varphi(x, y) : \varphi \in \Delta\},$$

$$\Pi = \Pi(\Delta) = \{\forall y \varphi(x, y) : \varphi \in \Delta\} = \{\neg\psi : \psi \in \Sigma(\Delta)\}.$$

Definition 1. ([27]) Let M and N be a structures of the language, $A \subseteq M$ and $f : A \to N$ be a map (that is, $f : M \to N$ is a partial map with $dom(f) = A$). Then f is a partial Δ-homomorphism if for every $a \in A$ and every formula $\varphi(x) \in \Delta$ from $M \models \varphi(a)$ follows that $N \models \varphi(f(a))$.

If $dom(f) = M$, then $f : M \to N$ is a Δ-homomorphism; if $M = N$, then f is a (partial) endomorphism.

Definition 2. ([26]) A Π-theory is a set of Π-sentences, closed with respect to deducibility.

Definition 3. ([27]) Let κ be a relatively large cardinal (at least $\kappa > |\Delta|$), and U the structure of the language. Then U is κ-universal domain if it satisfies the following properties:

1) κ-homogeneity: Let $f : U \to U$ be partial endomorphism U, and suppose that $|dom(f)| < \kappa$. Then f extends to automorphism U.

2) κ-compactness: Let $\Gamma \subset \Delta$ such that $|\Gamma| < \kappa$ and suppose that every finite subset of the set Γ is realizable in U. Then Γ is realizable in U.

Definition 4. ([26]) A model $M \models T$ is existentially closed if every Δ-homomorphism $f : M \to N$ such that $N \models T$, is a Σ-embedding.

Definition 5. ([27]) Let U be a universal domain and $T = Th_{\Pi}(U)$. Then we say that U is a universal domain for T.

Definition 6. ([27]) Π-theory T is complete if it is equal to $Th_{\Pi}(M)$ for some structure M of the language of the theory T.

If T is not complete, then the completion of the theory T is a minimal (with respect to inclusion) complete Π-theory containing T. In this case, a universal domain of the theory T is any universal domain of its extensions, i.e., a universal domain whose Π-theory is a complement of the theory T.

Lemma 1. ([27]) Let T be Π-theory. Then for every model $M \models T$ there exists an existentially closed model N and morphism $M \to N$.

Theorem 1. ([27])

1) The completion of the Π-theory are exactly the Π-theories of its various existentially closed models.

2) A Π-theory is positive Robinson if and only if all its completions are positive Robinson.

3) A complete Π-theory is positive Robinson if and only if it has a universal domain.

Theorem 2. ([27]) For Π-theory T the following conditions are equivalent:

1. T is positive Robinson theory.
2. The class of existentially closed models of theory T is axiomatic.

2 Existentially positive Mustafin theories and their properties

Let us define the notion of existentially positive Mustafin theory ($\exists PM$-theory). The main difference of this concept from the classical notion of the theory is that only positive sentences are involved in the axioms defining the theory. Thus, this class of theories is persistent with respect to homomorphisms. If at some fixed Δ, the considered $\exists PM$-theory is Jonsson in the classical sense, then we apply to it all notations and results known earlier, e.g., as in [9].

Let L be a first-order language, At be the set of atomic formulas of L, $B^+(At)$ be the closed set of relatively positive Boolean combinations (conjunctions and disjunctions) of all atomic formulas, their subformulas and substitution of variables. $Q(B^+(At))$ is the set of formulas in prenex normal form obtained by applying quantifiers ($\forall$ and $\exists$) to $B^+(At)$. We call a formula positive if it belongs to the set $Q(B^+(At)) = L^+$. A theory is called positively axiomatizable if its axioms are positive. $B(L^+)$ is an arbitrary Boolean combination of formulas from L^+. It is easy to see that $\Pi(\Delta) \subseteq B(L^+)$ when $\Delta = B^+(At)$, where $\Pi(\Delta)$ is such as described earlier.

Following [26, 27] define Δ-morphisms between structures.

Let M and N be structures of the language, $\Delta \subseteq B(L^+)$. A map $h : M \to N$ is called Δ-homomorphism (symbolically $h : M \to_\Delta N$) if for any $\varphi(\overline{x}) \in \Delta$, $\forall \overline{a} \in M$ from the fact that $M \models \varphi(\overline{a})$, it follows that $N \models \varphi(h(\overline{a}))$. The model M is called the beginning in N and we say that M continues in N, with $h(M)$ called the continuation of M. If the map h is injective, then we say that the map h immerses M into N (symbolically $h : M \leftrightarrow_\Delta N$).

Hereafter we will use the term Δ-extension and Δ-immersion. Within this definition (Δ-homomorphism), it is easy to see that isomorphic embedding and elementary embedding are Δ-imbeddings when $\Delta = B(At)$ and $\Delta = L$, correspondingly.

Definition 7. If C is a class of L-structures, then we note that an element M of C is Δ-positively existentially closed in C if every Δ-homomorphism from M to any element of C is Δ-immersion. We denote the class of all Δ-positively existentially closed models by $(E_C^\Delta)^+$; if $C = ModT$ for some theory T, then by E_T, $(E_T^\Delta)^+$ we mean respectively the class of existentially closed and Δ-positively existentially closed models of that theory. If $\Delta = L$ we obtain a class of positively existentially closed models of this theory and denote it by E_T^+.

Hereinafter throughout the paper $\Delta = B^+(At)$ and in the case where the considered theory is not Jonsson due to the considered positivity (since, n-immersion is not the same as n-embedding), we will use the universal domain from [26] instead of the semantic model considered theory. $\Delta = B^+(At)$, consistent with the above definitions, satisfies the minimal fragment from [26] and is consistent with the definition of $\exists PM$-theory.

Let $0 \leq n \leq \omega$. Π_n^+-formula be a formula of language L^+ whose prenex normal form has n variable quantifiers and begins with $\forall$-quantifier. Similarly, Σ_n^+-formula is a formula of L^+ whose prenex normal form has n variable quantifiers and begins with quantifier $\exists$.

Definition 8. Model A of theory T will be called positively existentially closed with respect to Σ_n-formulas if $\forall\varphi(x) \in \Sigma_n^+$, $\forall a \in A$, for any model $B \supset A$, from the fact that $B \models \varphi(a)$ follows that $A \models \varphi(a)$.

The set of all positive existentially closed with respect to Σ_n-formulas of models of the theory T we will denote as ${}_nE_T^+$.

Definition 9. We consider that theory T admits $\exists_n JEP$, if for any two $A, B \in {}_nE_T^+$ there exists $C \in {}_nE_T^+$ and Δ-homomorphisms $h_1 : A \to_\Delta C$, $h_2 : B \to_\Delta C$.

Definition 10. We say that theory T admits $\exists_n AP$, if for any $A, B, C \in {}_nE_T^+$ such that $h_1 : A \to_\Delta C$, $g_1 : A \to_\Delta B$, where h_1, g_1 are Δ-homomorphisms, there exists $D \in {}_nE_T^+$ and $h_2 : C \to_\Delta D$, $g_2 : B \to_\Delta D$, where h_2, g_2 are Δ-homomorphisms, such that $h_2 \circ h_1 = g_2 \circ g_1$.

If we consider only Δ-immersions as Δ-homomorphisms, then we get the definition of the so-called $\exists PM$-theory.

Definition 11. Let $0 \leq n \leq \omega$. The theory T is called an existentially positive Mustafin ($\exists PM$-theory) if

1) the theory T has infinite models,
2) theory T is Π_{n+2}^+-axiomatizable,
3) theory T admits $\exists_n JEP$,
4) theory T admits $\exists_n AP$.

Definition 12. The $\exists PM$-theory at $n = 0$ will be called the $\exists PJ$-theory.

Hereafter, all definitions of concepts relating to Jonsson theories (in the ordinary sense) are considered to be known and can be extracted, for example, from [9].

In the study of Jonsson theories the main tool of their investigation is the semantic method, which consists in the following: The elementary properties of the centre of Jonsson theory are «translated» onto the theory itself. In this case, the elementary theory of the semantic model of Jonsson theory is similar to the positive Robinson theory, and is invariant to this Jonsson theory because all semantic models of the same Jonsson theory are elementary equivalent to each other. In this connection, if $\exists PJ$-theory is not Jonsson in the classical sense, then by its semantic model we will mean any of its universal domain U (as in [26]) and by the centre T^* we will mean the following set of sentences $T^0 = Th_{\forall\exists}(U)$.

Note the following fact from the work [34].

Fact 1. ([34]) Inductive theory T is Jonsson if and only if there is a semantic model of theory T.

Definition 13. If $\exists PJ$-theory T is Jonsson, then its semantic model is T-$\exists PJ$-universal T-$\exists PJ$-homogeneous model of theory T of cardinality κ, where κ is a fixed unreachable cardinal.

Definition 14. $\exists PJ$-Jonsson theory T is called perfect if its semantic model C is a saturated model of the theory $Th(C)$.

Let us recall the following fact, which describes the perfect Jonsson theories:

Theorem 3. ([9]) Let T be a perfect Jonsson theory. Then the following conditions are equivalent:

1) T^* is model companion T;
2) $Mod(T^*) = E_T = E_{T^*}$;
3) $T^* = T^f = T^0$,

where $T^* = Th(C)$ is the center of theory T (C is semantic model of theory T), T^0 is Kaiser hull (maximal $\forall\exists$-theory mutually model-consistent with T), $T^f = Th(F_T)$, where F_T is class of generic models of the theory T (in terms of Robinson finite forcing).

The positive Robinson theory in the sense of [26, 27] is a generalization of the Kaiser hull concept T^0 for the Jonsson theory T. It follows from the Theorem 3 that when $\Delta = B(At)$ and $\exists PJ$-theory is perfect, the notion of semantic model and universal domain coincide.

Definition 15. Let A be some infinite model of signature σ. A is called $\exists PJ$-model if the set of sentences $Th_{\forall\exists^+}(A)$ is $\exists PJ$-theory.

In all the following, we will denote the $Th_{\forall\exists^+}(A)$ theory by $\forall\exists^+(A)$.

The following result generalizes Proposition 1 of [35].

Lemma 2. Let T be $\exists PJ$-theory complete for existential sentences. Then any infinite model of theory T is a $\exists PJ$-model.

Definition 16. Models A and B will be called $\exists PJ$-equivalent and denoted by $A \equiv_{\exists PJ} B$ if for any $\exists PJ$-theory T $A \models T \Leftrightarrow B \models T$.

The following result generalises Theorem 1 of [35].

Lemma 3. Let A and B be models of signature σ. Then the following conditions are equivalent:

1) $A \equiv_{\exists PJ} B$,

2) $\forall\exists^{+}(A) = \forall\exists^{+}(B)$.

Definition 17. Two $\exists PJ$-theories T_1 and T_2 are called $\exists PJ$-cosemantic ($T_1 \bowtie_{\exists PJ} T_2$) if they have the same semantic model, in case if T_1 and T_2 are Jonsson theories; and they have the same universal domain, in case they are not Jonsson.

Definition 18. ([9]) Models A and B of the signature σ are called $\exists PJ$-cosemantic ($A \bowtie_{\exists PJ} B$), if for any $\exists PJ$-theory T_1 such that $A \models T_1$, there is a $\exists PJ$-theory T_2, $\exists PJ$-cosemantic with T_1, such that $B \models T_2$. And vice versa.

Lemma 4. For any models A and B, the following implication is true:

$$A \equiv B \Rightarrow A \equiv_{\exists PJ} B \Rightarrow A \bowtie_{\exists PJ} B.$$

Similarly, the notion of $\exists PM$-cosemanticity between $\exists PM$-theories and respectively their models is defined.

The following convention is paramount. We will talk about the semantic aspect of $\exists PJ$-theory. If $\exists PJ$-theory T is Jonsson, then we work with E_T as a class of models of some Jonsson theory. If $\exists PJ$-theory T is not Jonsson, then we consider as E_T the class of its positively existentially closed models E_T^{+}. Such an approach for the class E_T, a class of existentially closed models of an arbitrary universal theory T, has been considered in [36].

Since two cases are possible with respect to Jonsson theories: perfect and imperfect, we will stick to the following. According to [9], if a Jonsson theory T is perfect, then the class of its existentially closed models E_T is elementary and coincides with E_{T^*}, where T^* is its center. If the theory T is imperfect, we do as in [36], i.e., instead of E_T work with the class E_T^{+}.

When an arbitrary $\exists PJ$-theory T is considered, the class E_T^{+} is considered an extension of E_T (both classes always exist), and depending on the perfection or imperfection of the theory T, the theoretical-model properties of the class E_T^{+} are of special interest.

For any theory T we will denote by $T_{\forall+}$ the theory which axioms are positive universal corollaries of the theory T.

Lemma 5. Let T_1 and T_2 be $\exists PJ$-theories, with C_1 being the semantic model of T_1 and C_2 the semantic model of T_2. If $(T_1)_{\forall+} = (T_2)_{\forall+}$, then $T_1 \bowtie_{\exists PJ} T_2$.

Theorem 4. Let T_1 and T_2 be $\exists PJ$-theories, with C_1 being the semantic model of T_1 and C_2 being the semantic model of T_2. Then the following conditions are equivalent:

1) $C_1 \bowtie_{\exists PJ} C_2$,

2) $C_1 \equiv_{\exists PJ} C_2$,

3) $C_1 = C_2$.

3 Positive Jonsson spectrum of $\exists PM$-theories of a fixed class of S-acts theory models over a group. Main results

The main result of the paper will be the characterization of Jonsson spectra $\exists PM$-theories of S-acts over a group with respect to cosemanticity by means of some invariants which have been defined in paper [33].

Let us give the basic definitions and statements from [33] necessary to formulate and prove the results of the paper.

Let us recall the definition of a S-act.

Definition 19. ([33]) Let A be non-empty set, $\langle S; \cdot, e\rangle$ is monoid. Algebraic system $\langle A; \langle f_\alpha : \alpha \in S\rangle\rangle$ with unary operations f_α, $\alpha \in S$, is called a S-act over S, if the following conditions hold:

$f_e(a) = a$ for all $a \in A$;

$f_{\alpha\beta}(a) = f_\alpha(f_\beta(a))$ for all $a \in A$ and all $\alpha, \beta \in S$.

Let $a \in A$, then $S_a = \{f_\alpha(a) : \alpha \in S\}$; if $\bar{a}$ is tuple of elements from A, then $S_{\bar{a}} = \bigcup\limits_{a_i \in \bar{a}} S_{a_i}$. The set $C_a = \{b \in A : b \in S_a \text{ or } a \in S_b\}$ is called a component.

Proposition 1. ([33]) If T is a S-act theory and for any $f : S_{\overline{a}} \backsimeq S_{\overline{b}}$ there exists a $g \supset f$ such that $g : C_{\overline{a}} \backsimeq C_{\overline{b}}$, then T admits the elimination of the quantifiers.

Hereafter, we consider S-acts over the group G and correspondingly the theory of S-acts over the group.

If A is a S-act over the group G, $a \in A$, then

$$id(a) = \{g \in G : f_g(a) = a\}; \quad \mathfrak{p}(G) = \{H : H \preceq G\}.$$

If $H \preceq G$, then $\mathfrak{F}(H) = |\{gH : g \in G, \{\varphi \in G : \varphi gH = gH\} = H\}|$.

Definition 20. ([33]) 1) If Γ is a family or type of sentence, then $T_\Gamma = \{\psi : \{\varphi \in \Gamma : T \vdash \varphi\} \vdash \psi\}$;

2) $\nabla = \Pi_1 \cup \Sigma_1$, i.e., ∇ is the family of all universal or existential formulas.

Definition 21. ([33]) If $T = T_\nabla$, then the theory T will be called a primitive.

Let us write a known fact about primitives.

Fact 2. ([33]) For a complete theory T the following conditions are equivalent:

1) T is a primitive;

2) if $\mathfrak{A}, \mathfrak{B} \models T$ and $\mathfrak{A} \subseteq \mathfrak{C} \subseteq \mathfrak{B}$, then $\mathfrak{C} \models T$

Definition 22. ([33]) An expression of the form $g \in X$ will be called an atomic figure, where $g \in G$, X is a fixed symbol. A figure is any formal Boolean combination of atomic figures. Denote by Φ the set of all figures. For each figure $\varphi(X)$ we define by induction $U(\varphi) \subseteq \mathfrak{p}(G)$ and formula $\theta(\varphi, a)$ S-act language for any element a of any S-act

1) if $\varphi(X) = g \in X$, then $U(\varphi) = \{H \preceq G : g \in H\}$, $\theta(\varphi, a) = (f_g(a) = a)$;

2) if $\varphi(X) = \neg\psi(X)$, then $U(\varphi) = \mathfrak{p}(G) - U(\psi)$, $\theta(\varphi, a) = \neg\theta(\psi, a)$;

3) if $\varphi(X) = \psi_1(X)\&\psi_2(X)$, then $U(\varphi) = U(\psi_1) \cap U(\psi_2)$, $\theta(\varphi, a) = \theta(\psi_1, a)\&\theta(\psi_2, a)$.

Let us use the following notations from [33].

Let [ ] be a closure operator induced by a topology over $\mathfrak{p}(G)$ which base of open neighborhood is $\{U(\varphi) : \varphi \in \Phi\}$. If $\mathfrak{h} \subseteq \mathfrak{p}(G)$, then

$$< \mathfrak{h} >= \{gHg^{-1} : g \in G, H \in \mathfrak{h}\}.$$

Let ( ) denote the Poizat operator, i.e., the smallest closure operator on $\mathfrak{p}(G)$ with property $(\mathfrak{h}) \supseteq [\mathfrak{h}] \cup < \mathfrak{h} >$.

$Q = \{H \preceq G : \exists\varphi \in \Phi(U(\varphi) = [H]) \text{ and } \mathfrak{F}(H) < \infty\}$.

Definition 23. ([33]) A pair $< \mathfrak{h}, \varepsilon >$ is called a characteristic if $\mathfrak{h} \subseteq \mathfrak{p}(G), \mathfrak{h} = (\mathfrak{h}), \varepsilon : Q \to [\infty] \cup \omega$ and $\varepsilon(H) = 0 \Leftrightarrow H \notin \mathfrak{h}$.

Definition 24. ([33]) If $n < \omega$, T is a S-act theory, then

$$T^{(n)}(G) = \{< H_1, ..., H_n >\in G^n : \exists\mathfrak{A} \models T, < a_1, ..., a_n >\in \mathfrak{A}^n(\&_{m=1}^n H_m = id(a_m))\}.$$

Definition 25. ([33]) If T is S-act theory, then $\varepsilon_T : Q \to [\infty] \cup \omega$ such that

$$\varepsilon_T(H) = \begin{cases} k, \text{ if } k = max\{|\{G_a : a \in \mathfrak{A}, id(a) = H\}| : \mathfrak{A} \models T\} < \omega; \\ \\ \infty, \text{ if no such maximum exists.} \end{cases}$$

Let $ch(T) =< T^1(G), \varepsilon_T >$.

Proposition 2. ([33]) $ch(T)$ is a characteristic.

Theorem 5. ([33]) Let S-acts theory T have an infinite model. Then

(1) T is inductive;

(2) if T has the property of joint embedding, then it also has the property of amalgamation;

(3) if T is complete, then it admits the elimination of quantifiers and is primitive.

Theorem 6. ([33]) 1) Every α-Jonsson theory of S-acts is perfect and is Jonsson, $0 \leq \alpha \leq \omega$.

2) The S-acts theory T is a Jonsson $\Leftrightarrow \forall 1 \leq n \leq \omega$ $(T^{(n)}(G) = (T^{(1)}(G))^n)$.

Similarly to Theorem 6, let us formulate and prove the following result.

Theorem 7. For every $\exists PM$-theory T of S-acts over a group two cases are possible:

1. a) T is a Jonsson theory, then T is perfect;

b) $\exists PJ$-theory T of S-acts is a Jonsson $\Leftrightarrow \forall 1 \leq n \leq \omega$ $(T^{(n)}(G) = (T^{(1)}(G))^n)$.

2. T is not a Jonsson theory. Then there exists some $\exists PM$-theory T' such that T' is a Jonsson theory and is a Kaiser hull for theory T.

Let us first prove the lemma.

Lemma 6. Let T be $\exists PM$-theory of S-acts over a group and all completions of T admit the elimination of quantifiers. Then

(1) T is perfect;

(2) T is $\exists PJ$-theory.

Proof. (1) Let C be the semantic model of theory T, $T^* = Th(C)$ and C^* is saturated model of theory T^*. $C^* \subseteq_{\Sigma_n^+} C$, $C^* \in E_T^+$ and $D(C^*) = D(C)$. From homogeneity and equality of diagrams follows that $C \cong C^*$, i.e., T is perfect.

(2) Let C be the semantic model for T (saturated for T^*). Obviously C is $\exists PJ$-universal, we have to show that C is $\exists PJ$-homogeneous. Let $A, B \in E_T^+$, with $A \cong B$ by f. Suppose the contrary, that is, the model C is not $\exists PJ$-homogeneous and there exist such existentially closed submodels A' and B' of the semantic model C such that $A \subseteq A'$ and $B \subseteq B'$. This means that there exists an existential formula $\varphi(x)$ such that $A' \models \varphi(x)$ but $B' \not\models \varphi(x)$. It follows that $A \models \varphi(x)$ and $B \not\models \varphi(x)$ due to existential closure of A and B, which contradicts isomorphism f. By virtue of the fact that T^* admits the quantifier elimination then $(C, a)_{a \in A} \equiv (C, f(a))_{a \in A}$, which means that f is an automorphism.

Proof of Theorem 7.

1. a) It follows from Lemma 6.

1. b) It is easy to show that from the condition $\forall n < \omega, T^{(n)}(G) = (T^{(1)}(G))^n$ follows the joint embedding property and vice versa.

2. Let T be $\exists PM$-theory not Jonsson, then since $\Delta = B^+(At)$, we can use the universal domain U for the minimal fragment $\Delta = B^+(At)$ from [26]. Consider all $\forall\exists$-sequences true in U, that is, consider the theory $Th_{\forall\exists}(U) = \Delta$. There are 2 possible cases: $U \in E_\Delta^+$ and $U \notin E_\Delta^+$.

If $U \in E_\Delta^+$, let us consider the theory $Th_{\forall\exists}(U) = \Delta$. Let us show that this theory is Jonsson. To do this, we will use Fact 1. The semantic model of Δ will be the family of maximal components of the theory of all S-acts over the group. It is easy to see that by virtue of Theorem 6, this model is saturated in its cardinality, hence Δ is a perfect Jonsson $\exists PM$-theory and is a Kaiser hull for theory T.

If $U \notin E_\Delta^+$, then, since Δ is an inductive theory, there exists a model $D \in E_\Delta^+$ such that U is isomorphically embedded in D. Consider the theory $\Delta' = Th_{\forall\exists}(D)$. Similarly, it is easy to prove that Δ' is a perfect Jonsson $\exists PM$-theory and that Δ' is a Kaiser hull for theory T.

We will need the following definition and theorem from paper [33].

Definition 26. ([33]) If $< \mathfrak{h}, \varepsilon >$ is a characteristic, then

$T_1(\mathfrak{h},\varepsilon) = \{\forall y \neg\theta(\varphi, y) : (\varphi \in \Phi, U(\varphi) \cap \mathfrak{h} = \varnothing\} \cup \{\forall y_1, ..., y_{\varepsilon(H)\mathfrak{F}(H)+i}(\&_i\theta(\varphi, y_i) \to \bigvee_{i\neq j}(y_i = y_j)) : H \in Q \cap \mathfrak{h},\ \varphi \in \Phi,\ \varepsilon(H) < \infty,\ U(\varphi) = [H]\}$,

$T_2(\mathfrak{h},\varepsilon) = T_1(\mathfrak{h},\varepsilon) \cup \{\exists y_1, ..., y_{\varepsilon(H)\mathfrak{F}(H)}(\&_i\theta(\varphi, y_i)\&\&_{i\neq j}(y_i \neq y_j)) : H \in Q \cap \mathfrak{h},\ \varepsilon(H) < \infty,\ U(\varphi) = [H]\} \cup \{\exists y_1, ..., y_n(\&_i\theta(\varphi, y_i)) : U(\varphi) \cap (\mathfrak{h} - Q) \neq \varnothing \vee \exists H \in U(\varphi) \cap Q(\varepsilon(H) = \infty), n < \omega\}$.

Theorem 8. ([33]) 1) $ch(T_1(\mathfrak{h},\varepsilon)) = ch(T_2(\mathfrak{h},\varepsilon)) =< \mathfrak{h}, \varepsilon >$ for any characteristic $< \mathfrak{h}, \varepsilon >$;

2) Jonsson S-acts theories T_1 and T_2 are cosemantic $\Leftrightarrow ch(T_1) = ch(T_2)$;

3) T is Jonsson S-acts theory and $ch(T) =< \mathfrak{h}, \varepsilon >$ if and only if $T_1(\mathfrak{h},\varepsilon) \subseteq T \subseteq T_2(\mathfrak{h},\varepsilon)$.

Similar to Theorem 8, we have a result for the case of $\exists PM$-theory.

Theorem 9. Let T_1 and T_2 be $\exists PM$-theory of S-acts over group for fixed $0 \leq n \leq \omega$. Then:

(1) $\mathrm{ch}(T_1(\mathfrak{h},\varepsilon)) = ch(T_2(\mathfrak{h},\varepsilon)) =< \mathfrak{h}, \varepsilon >$ for any characteristic $< \mathfrak{h}, \omega >$;

(2) $T_1 \bowtie_{\exists PM} T_2 \Leftrightarrow ch(T_1) = ch(T_2)$;

(3) There is $\exists PM$-theory T of S-acts over group such that $ch(T_1) =< \mathfrak{h}, \varepsilon >$ iff $T_1(\mathfrak{h},\varepsilon) \subseteq T \subseteq T_2(\mathfrak{h},\varepsilon)$

The proof is the same as for Theorem 8.

The result of Theorem 9 has a natural continuation in the context of the theoretical-model properties study of the positive spectrum of a fixed class of S-acts over the group.

Let K be a class of structures of fixed signature σ. Consider positive spectrum of $\exists PM$-theories of class K:

$$PSp(K) = \{T \mid T is\ \exists PM\text{-theory in language } K \subseteq \mathrm{Mod}(T) \text{ for a fixed } 0 \leq n \leq \omega\}.$$

Note that the cosemanticity relation on a set of theories is an equivalence relation. Therefore, we can consider the factor set $PSp(K)/_{\bowtie_{\exists PM}}$ of the positive spectrum class K with respect to the relation $\bowtie_{\exists PM}$.

The result is as follows:

Theorem 10. Let K_Π be a class of all S-acts over group, $[T_1], [T_2] \in PSp(K_\Pi)/_{\bowtie_{\exists PM}}$. Then

1) if $[T_1]$ and $[T_2]$ are classes of Jonsson $\exists PM$-theories then $C_{[T_1]} \bowtie_{\exists PM} C_{[T_2]} \Leftrightarrow ch([T_1]^*) = ch([T_2]^*)$;

2) if $[T_1]$ and $[T_2]$ are classes of not Jonsson $\exists PM$-theories, then there are such classes of Jonsson $\exists PM$-theories $[\Delta_1], [\Delta_2] \in PSp(K_\Pi)/_{\bowtie_{\exists PM}}$, that Δ_i is the Kaiser hull for T_i, where $i = 1, 2$ $C_{[\Delta_1]} \bowtie_{\exists PM} C_{[\Delta_2]} \Leftrightarrow ch([\Delta_1]^*) = ch([\Delta_2]^*)$;

3) if $[T_1]$ is a class of Jonsson $\exists PM$-theories, and $[T_2]$ is a class of not Jonsson $\exists PM$-theories, then there is such Jonsson $\exists PM$-theory Δ, that $C_{[T_1]} \bowtie_{\exists PM} C_{[\Delta]} \Leftrightarrow ch([T_1]^*) = ch([\Delta]^*)$.

Proof.

1) $\Rightarrow$: Let $[T_1], [T_2] \in PSp(K_\Pi)/_{\bowtie_{\exists PM}}$ be classes of Jonsson $\exists PM$-theories and $C_{[T_1]} \bowtie_{\exists PM} C_{[T_2]}$. Since $[T_1]$ and $[T_2]$ are classes of Jonsson S-acts theories over a group, then $[T_1]$ and $[T_2]$ are classes of perfect Jonsson theories, hence, by Theorem 2.12 from [9], $[T_1]^*$ and $[T_2]^*$ are Jonsson S-acts theories over a group. Then according to 2) of Theorem 8 $ch([T_1]^*) = ch([T_2]^*)$ since $[T_1]^*$ and $[T_2]^*$ are complete theories.

$\Leftarrow$: Let $[T_1]$ and $[T_2]$ be classes of Jonsson $\exists PM$-theories of S-acts over a group and $ch([T_1]^*) = ch([T_2]^*)$. Then $[T_1]$ and $[T_2]$ are classes of perfect Jonsson theories, then $[T_1]^*$ and $[T_2]^*$ are complete Jonsson $\exists PM$-theories of S-acts over a group. Since $ch([T_1]^*) = ch([T_2]^*)$, it follows from 2) of Theorem 9 that $[T_1]^* \bowtie_{\exists PM} [T_2]^*$. From the definition of cosemanticity, it follows that $C_{[T_1]^*} = C_{[T_2]^*}$. However, since $[T_1]^*$ and $[T_2]^*$ are complete Jonsson $\exists PM$-theories, then $[T_1]^* \in [T_1]$ and $[T_2]^* \in [T_2]$, i.e., $C_{[T_1]} = C_{[T_2]}$, from which it follows that $C_{[T_1]} \bowtie_{\exists PM} C_{[T_2]}$.

2) Let $[T_1], [T_2] \in PSp(K_\Pi)/_{\bowtie_{\exists PM}}$ be classes of not Jonsson $\exists PM$-theories, $C_{[T_1]} = U_1, C_{[T_2]} = U_2$ and $[T_1]^* = Th_{\forall\exists}(U_1), [T_2]^* = Th_{\forall\exists}(U_2)$. Since $[T_1]^*$ and $[T_2]^*$ are inductive theories, there are positive

existentially closed models D_1 and D_2 of these theories such that U_1 is isomorphically embedded in D_1 and U_2 is isomorphically embedded in D_2. Consider the theories $\Delta_1 = Th_{\forall\exists}(D_1)$ and $\Delta_2 = Th_{\forall\exists}(D_2)$. They are Jonsson perfect $\exists PM$-theories. The existence of theories Δ_1 and Δ_2 follows from Theorem 7 and they are Kaiser hulls for T_1 and T_2 respectively. Then it follows from 1) of this theorem that $C_{[\Delta_1]} \bowtie_{\exists PM} C_{[\Delta_2]} \Leftrightarrow ch([\Delta_1]^*) = ch([\Delta_2]^*)$.

3) Let $[T_1]$ be the class of Jonsson $\exists PM$-theories and $[T_2]$ be the class of not Jonsson $\exists PM$-theories. Then, similarly to 2), using Theorem 7, we can find such a Jonsson $\exists PM$-theory Δ, which is a Kaiser hull for theory T_2 and according to 1) hold $C_{[T_1]} \bowtie_{\exists PM} C_{[\Delta]} \Leftrightarrow ch([T_1]^*) = ch([\Delta]^*)$.

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Группалар маңайындағы полигондардың экцистенциалды позитивті мұстафиндік теориясы

Мақала полигондар сигнатурасының бекітілген модельдер класының йонсондық спектрінің ұғымын зерттеумен байланысты. Сонымен бірге полигонның моноиды ретінде группа қарастырылған. Йонсондық спектр ұғымы алгебралар класын модельді-теоретикалық қасиеттерін сипаттау үшін эффективті болып табылады. Теориялар үйлесімді енгізілуге және амальгама қасиетіне ие. Бұл жағдайда, әдетте, осы модельдер класы бойынша ақиқат болатын әмбебап-экзистенциалды ұсыныстарды қарастыру жеткілікті. Осы уақытқа дейін йонсондық спектр, әдетте, тек йонсондық теорияларымен жұмыс істеді. Авторлар мақалада позитивті йонсондық спектрі түсінігін анықтайды, оның элементтері, жалпы алғанда, йонсондық емес теориялар болуы мүмкін. Бұл мақалада қарастырылатын теорияларды анықтауда изоморфтық енгізулер ғана емес, сонымен қатар батулар (яғни, экзистенциалды позитивті мұстафиндік теория) қатыстылығымен түсіндіріледі. Осыған байланысты амальгама қасиеттерін және бірлескен үйлесімді қасиеттерін анықтауда батулар қарастырылады. Нәтижесінде, теорияның осындай өзгерістеріне байланысты алынған теориялар йонсондық болуы міндетті емес. Осы мақаланың негізгі нәтижелерін талдай отырып, йонсондық емес спектрді зерттеудің жоғарыда аталған тәсілі, ең болмағанда, йонсондық емес теория жағдайында да, бұрын белгілі ұғымдар мен нәтижелерді қанағаттандыратын, бірақ сонымен бірге қарастырылатын экзистенциалды позитивті мұстафиндік теориясымен тікелей байланысты болатын йонсондық теорияны табудың тұрақты әдісі бар екені байқалады.

Кілт сөздер: йонсондық теория, кемел йонсондық теория, позитивті модельдер теориясы, йонсондық спектр, позитивті йонсондық теория, бату, полигон, полигондардың йонсондық теориясы, $\exists PM$-теория, косеманттылық.

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Экзистенциально позитивные мустафинские теории полигонов над группой

Статья связана с изучением понятия йонсоновского спектра фиксированного класса моделей сигнатуры полигонов, причём в качестве моноида полигона рассматривается группа. Понятие йонсоновского спектра является эффективно работающим при описании теоретико-модельных свойств классов алгебр, теории которых допускают свойства совместного вложения и амальгамы. При этом, как правило, достаточно рассматривать универсально-экзистенциальные предложения, истинные на моделях этого класса. До настоящей работы йонсоновский спектр, как правило, оперировал только йонсоновскими теориями. Авторами статьи определено понятие позитивного йонсоновского спектра, элементами которого могут быть, вообще говоря, не йонсоновские теории. Это происходит из-за того, что в определении рассматриваемых теорий в данной статье (а именно, экзистенциально позитивных мустафинских теорий) участвуют не только изоморфные вложения, но и погружения. В связи с этим в определении свойства амальгамы и свойства совместного вложения рассмотрены погружения. Как следствие, полученные в силу таких изменений теории не обязательно должны быть йонсоновскими. Анализируя основные полученные результаты данной статьи, мы можем заметить, что указанный выше подход к изучению йонсоновского спектра оказывается оправданным, хотя бы в силу того, что даже в случае не йонсоновской теории существует регулярный метод нахождения такой йонсоновской теории, которая удовлетворяет ранее известным понятиям и результатам, но которая также будет непосредственно связана с рассматриваемой экзистенциально позитивной мустафинской теорией.

Ключевые слова: йонсоновская теория, совершенная йонсоновская терия, позитивная теория моделей, йонсоновский спектр, позитивная йонсоновская теория, погружение, полигон, йонсоновская теория полигонов, $\exists PM$-теория, косемантичность.

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