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Boundary value problem for a system of partial differential equations with the Dzhrbashyan–Nersesyan fractional differentiation operators

A boundary value problem in a rectangular domain for a system of partial differential equations with the Dzhrbashyan–Nersesyan fractional differentiation operators with constant coefficients is studied in the case when the matrix coefficients of the system have complex eigenvalues. Existence and uniqueness theorems for the solution to the boundary value problem under study are proved. The solution is constructed explicitly in terms of the Wright function of the matrix argument.

Keywords: system of partial diffrential equations, fractional derivatives, Dzhrbashyan–Nersesyan operator, boundary value problem, fundamental solution, Wright matrix function.

Introduction

Consider the system of differential equations

$$Lu(x,y) \equiv D_{0x}^{\{\alpha_0,\alpha_1,...,\alpha_k\}}u(x,y) + AD_{0y}^{\{\beta_0,\beta_1,...,\beta_m\}}u(x,y) = Bu(x,y) + f(x,y), \tag{1}$$

in the domain $\Omega = \{(x,y):\ 0<x<a,\ 0<y<b\}$, $a,b<\infty$, where $D_{0x}^{\{\alpha_0,\alpha_1,...,\alpha_k\}}$ and $D_{0y}^{\{\beta_0,\beta_1,...,\beta_m\}}$ are the Dzhrbashyan–Nersesyan fractional differentiation operators [1] of orders $\alpha = \sum\limits_{i=0}^{k}\alpha_i - 1 > 0$ and $\beta = \sum\limits_{i=0}^{m}\beta_i - 1 > 0$, respectivly, $\alpha_i, \beta_j \in (0,1]$, $(i=\overline{0,k}, j=\overline{0,m})$; $f(x,y) = ||f_1(x,y),...,f_n(x,y)||$ and $u(x,y) = ||u_1(x,y),...,u_n(x,y)||$ are given and desired n-dimensional vectors, respectively, A and B are given constant real square matrices of order n.

The Dzhrbashyan–Nersesyan fractional differentiation operator $D_{0t}^{\{\gamma_0,\gamma_1,...,\gamma_k\}}$ of the order $\gamma = \sum\limits_{i=0}^{k}\gamma_i - 1 > 0$, $\gamma_i \in (0,1]$, $(i=\overline{0,k})$, associated with the sequence $\{\gamma_0,\gamma_1,...,\gamma_k\}$, is determined by the relation [1]

$$D_{0t}^{\{\gamma_0,\gamma_1,...,\gamma_k\}}v(t) = D_{0t}^{\gamma_k-1}D_{0t}^{\gamma_{k-1}}...D_{0t}^{\gamma_1}D_{0t}^{\gamma_0}v(t),$$

where D_{0t}^{γ} is the Riemann–Liouville fractional integro-differentiation operator [2; 9].

The operator $D_{0t}^{\{\gamma_0,\gamma_1,...,\gamma_k\}}$ was introduced in [1], where the form of the initial conditions for the ordinary differential equations with such an operator, and the Cauchy problem was studied. The Dzhrbashyan-Nersesyan operator generalizes a number of definitions of fractional derivatives, including the Riemann–Liouville and Gerasimov–Caputo derivatives.

A review of works related to the study of the equation (1) with Riemann–Liouville and Gerasimov–Caputo derivatives, including in the scalar case $n=1$, can be found in [3] and [4].

Equations of the order not higher than one containing operators of the form $D_{0t}^{\{\gamma_0,\gamma_1,...,\gamma_k\}}$ are studied in [5–10]. In [5], for a linear partial differential equation of fractional order with many independent variables a fundamental solution is constructed and a boundary problem is solved.

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In papers [6], [7], boundary value problems in a rectangular domain for first-order partial differential equations with variable coefficients are studied. In papers [8] and [9], a boundary value problem with an integral condition and a boundary value problem in a rectangle, respectively, are studied for equations with constant coefficients. The study [10] considers an equation containing the Dzhrbashyan–Nersesyan operators in two independent variables, and in one of the variables the equation includes a linear combination of two Dzhrbashyan–Nersesyan operators, the orders of which are associated with the sequences $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$. The question of the influence of the distribution of the values of these parameters on the setting of the initial conditions is studied.

We also note the papers [11, 12] where the unique solvability of initial problems for some classes of linear equations with operator coefficients in the Banach spaces are studied.

In papers [3], [4], [13–15], boundary value problems in rectangular domains and the Cauchy problem for systems with sign-definite eigenvalues of matrix coefficients in the main part, with Riemann–Liouville partial derivatives whose the order does not exceed one, are considered. For these systems, the situation with the formulation of boundary value problems is similar to the case of a single equation. In this paper, we extend the class of such systems to include systems with eigenvalues of the coefficients of the main part lying in some corner of the complex plane, and with more general Dzhrbashyan–Nersesyan operators of fractional differentiation.

1 Auxiliary assertions

The Riemann-Liouville fractional integro-differentiation operator D^{ν}_{ay} of order ν is defined as follows [2; 9]:

$$D^{\nu}_{ay} g(y) = \frac{\operatorname{sign}(y-a)}{\Gamma(-\nu)} \int\limits_{a}^{y} \frac{g(s)ds}{|y-s|^{\nu+1}}, \quad \nu < 0,$$

for $\nu \geq 0$ the operator D^{ν}_{ay} can be determined by recursive relation

$$D^{\nu}_{ay} g(y) = \operatorname{sign}(y-a) \frac{d}{dy} D^{\nu-1}_{ay} g(y), \quad \nu \geq 0.$$

The following series

$$\phi(\rho, \mu; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\rho k + \mu)}, \quad \rho > -1, \quad \mu \in \mathbb{C}$$

defines the Wright function [16], [17] which depends on two parameters ρ and μ.

The following relation holds [4]

$$\phi(\rho, \mu; z)\big|_{z=0} = \frac{1}{\Gamma(\mu)}. \tag{2}$$

If $\beta \in (0, 1)$, $\mu \in \mathbb{R}$, then the following estimate is valid [18]

$$|\phi(-\beta, \mu; -z)| \leq C \exp\left(-\sigma |z|^{\frac{1}{1-\beta}}\right), \tag{3}$$

where $C = C(\beta, \mu, \sigma)$ and

$$\sigma < (1-\beta)\beta^{\frac{\beta}{1-\beta}} \cos \frac{\arg z}{1-\beta}, \quad 0 \leq |\arg z| < \frac{1-\beta}{2}\pi.$$

Under the condition

$$\beta \in (0, 1), \quad 0 \leq |\arg \lambda| < \frac{1-\beta}{2}\pi, \tag{4}$$

the inequality

$$|y^{\mu-1}\phi(-\beta,\mu;-\lambda xy^{-\beta})| \le Cx^{-\theta}y^{\mu+\beta\theta-1}, \quad x>0, \quad y>0, \tag{5}$$

holds [18], where $C = C(\mu,\beta,\theta,\lambda)$, $\theta \ge 0$ for $\mu \ne 0,-1,-2,...$, and $\theta \ge -1$ for $\mu = 0,-1,-2,....$

The following differentiation formula is valid [17]:

$$\frac{d}{dz}\phi(\rho,\mu;z) = \phi(\rho,\mu+\rho;z), \quad \rho > -1. \tag{6}$$

Let $\beta \in (0,1)$, $\mu,\nu \in \mathbb{R}$, and the inequality (4) holds, then the formula [18]

$$D^{\nu}_{0y}y^{\mu-1}\phi(-\beta,\mu;-\lambda y^{-\beta}) = y^{\mu-\nu-1}\phi(-\beta,\mu-\nu;-\lambda y^{-\beta}) \tag{7}$$

is true. By (7), we have

$$D^{\{\gamma_0,...,\gamma_j\}}_{0y}y^{\mu-1}\phi(-\beta,\mu;-\lambda y^{-\beta}) = y^{\mu-\mu_j}\phi(-\beta,\mu-\mu_j+1;-\lambda y^{-\beta}), \tag{8}$$

where $\mu_j = \sum\limits_{i=0}^{j}\gamma_i$.

Formulas (6)–(8) give the equality

$$\left(\frac{\partial}{\partial x} + \lambda D^{\{\gamma_0,...,\gamma_j\}}_{0y}\right) y^{\mu-1}\phi(-\beta,\mu;-\lambda xy^{-\beta}) = 0, \quad \beta = \sum_{i=0}^{j}\gamma_i - 1 < 1. \tag{9}$$

Using the integration by parts formula and the relations (2) and (3), one can show that the equality

$$\int\limits_0^{\infty} t^n\phi(-\beta,\mu;-\lambda t)dt = \frac{n!}{\lambda^{n+1}\Gamma(\mu+(n+1)\beta)}, \quad n = 0,1,.... \tag{10}$$

holds under the condition (4).

For $\lambda = 1$, the equality (10) was obtained in [19].

2 Special solutions

2.1 Wright matrix function

In papers [3], [4] the Wright matrix function was defined

$$\phi(\rho,\mu;A) = \sum_{k=0}^{\infty}\frac{A^k}{k!\Gamma(\rho k+\mu)}, \quad \rho > -1, \quad \mu \in \mathbb{C}$$

and its following properties were established.

1. Let the matrix A be reduced with the help of the matrix H to the Jordan normal form $J(\lambda)$, i.e.

$$A = HJ(\lambda)H^{-1},$$

where $J(\lambda) = \mathrm{diag}[J_1(\lambda_1),...,J_p(\lambda_p)]$ is the quasidiagonal matrix with cells of the form

$$J_k \equiv J_k(\lambda_k) = \left\| \begin{matrix} \lambda_k & 1 & \dots & 0 \\ & \lambda_k & \dots & 0 \\ & 0 & \ddots & \vdots \\ & & & \lambda_k \end{matrix} \right\|, \quad k = 1,...,p,$$

$\lambda_1, ..., \lambda_p$ are the eigenvalues of the matrix A, $J_k(\lambda_k)$ are the square matrices of order r_k+1, $\sum\limits_{k=1}^{p} r_k + p = n$. Then the function $\phi(\rho,\mu;Az)$ can be represented as

$$\phi(\rho,\mu;Az) = H\phi(\rho,\mu;J(\lambda)z)H^{-1}, \tag{11}$$

where

$$\phi(\rho,\mu;J(\lambda)z) = \mathrm{diag}[\phi(\rho,\mu;J_1(\lambda_1)z), ..., \phi(\rho,\mu;J_p(\lambda_p)z)],$$

$$\phi(\rho,\mu;J_k(\lambda_k)z) = \begin{Vmatrix} \phi^0_{\rho,\mu}(\lambda_k z) & \phi^1_{\rho,\mu}(\lambda_k z) & \dots & \phi^{r_k}_{\rho,\mu}(\lambda_k z) \\ & \phi^0_{\rho,\mu}(\lambda_k z) & \dots & \phi^{r_k-1}_{\rho,\mu}(\lambda_i z) \\ & 0 & \ddots & \vdots \\ & & & \phi^0_{\rho,\mu}(\lambda_k z) \end{Vmatrix},$$

$$\phi^m_{\rho,\mu}(\lambda z) = \frac{1}{m!}\frac{\partial^m}{\partial\lambda^m}\phi(\rho,\mu;\lambda z) = \frac{z^m}{m!}\phi(\rho,\mu+\rho m;\lambda z).$$

2. Using the representation (11) and equality (2), we obtain

$$\phi(\rho,\mu;Az)\big|_{z=0} = \frac{1}{\Gamma(\mu)}I, \tag{12}$$

where I is the identity matrix of order n.

3. The following differentiation formula is valid

$$\frac{d}{dz}\phi(\rho,\mu;Az) = A\phi(\rho,\rho+\mu;Az). \tag{13}$$

Further, we assume that all eigenvalues $\lambda_1, ..., \lambda_p$ of the matrix A satisfy the condition

$$0 \le |\arg\lambda_i| < \frac{1-\beta}{2}\pi, \quad i = \overline{1,p}. \tag{14}$$

Due to the relations (2), (3), (5), (6), (7), (10), the following properties proved in [3], [4] remain valid under the condition (14).

4. Due to (7) and (11), for $\beta \in (0,1)$, $\mu,\nu \in \mathbb{R}$, we have

$$D^\nu_{0y} y^{\mu-1}\phi(-\beta,\mu;-A\tau y^{-\beta}) = y^{\mu-\nu-1}\phi(-\beta,\mu-\nu;-A\tau y^{-\beta}). \tag{15}$$

From (15) it follows

$$D^{\{\gamma_0,...,\gamma_j\}}_{0y} y^{\mu-1}\phi(-\beta,\mu;-A\tau y^{-\beta}) = y^{\mu-\mu_j}\phi(-\beta,\mu-\mu_j+1;-A\tau y^{-\beta}), \quad \mu_j = \sum_{i=0}^{j}\gamma_i. \tag{16}$$

5. The equalities (13), (15) and (16) imply the equality

$$\left(\frac{\partial}{\partial\tau} + AD^{\{\gamma_0,...,\gamma_j\}}_{0y}\right) y^{\mu-1}\phi(-\beta,\mu;-A\tau y^{-\beta}) = 0, \quad \beta = \sum_{i=0}^{j}\gamma_i - 1 < 1. \tag{17}$$

6. By virtue of (10) and (11) it follows

$$\int\limits_0^\infty \phi(-\beta,\mu;-Az)dz = \frac{1}{\Gamma(\mu+\beta)}A^{-1}. \tag{18}$$

7. Let $A(x,y)$ be the matrix with entries $a_{ij}(x,y)$. By $|A(x,y)|_*$ we denote a scalar function taking the maximum absolute value of entries of the matrix $A(x,y)$ for each (x,y), i.e., $|A(x,y)|_* = \max\limits_{i,j} |a_{ij}(x,y)|$. Likewise, for a vector $b(x,y)$ with components $b_i(x,y)$, we set $|b(x,y)|_* = \max\limits_i |b_i(x,y)|$.

From the estimate (5) it follows that

$$|y^{\nu-1}\phi(-\beta,\nu;-A\tau y^{-\beta})|_* \leq C\tau^{-\theta}y^{\nu+\beta\theta-1}, \quad \tau>0, \quad y>0, \tag{19}$$

where $\beta\in(0,1)$ and $\theta\geq 0$ for $\nu\neq 0,-1,-2,...$; and $\theta\geq -1$ for $\nu=0,-1,-2,\ldots$.

8. Formulas (3) and (11) yields the estimate

$$|\phi(-\delta,\varepsilon;-Az)|_* \leq C\exp\left(-\sigma|z|^{\frac{1}{1-\delta}}\right), \quad z\geq 0, \tag{20}$$

where $\delta\in(0,1)$, $\varepsilon\in\mathbb{R}$, $\sigma<(1-\delta)\delta^{\frac{1}{1-\delta}}\lambda_0^{\frac{\delta}{1-\delta}}$, $\lambda_0=\min\limits_{1\leq i\leq p}\{|\lambda_i|\}$, $\lambda_1,...,\lambda_p$ are the eigenvalues of the matrix A.

2.2 Properties of the function $\Phi^{\mu,\nu}_{\alpha,\beta}(x,y)$

In [3], the following function is defined

$$\Phi^{\mu,\nu}_{\alpha,\beta}(x,y) \equiv \int\limits_0^\infty e^{B\tau}x^{\mu-1}\phi(-\alpha,\mu;-\tau x^{-\alpha})y^{\nu-1}\phi(-\beta,\nu;-A\tau y^{-\beta})d\tau. \tag{21}$$

The estimates (3) and (5) imply the convergence of the integral (21) for any $\mu,\nu\in\mathbb{R}$, and $x^2+y^2\neq 0$.

The following assertions are true.

Lemma 2.1. For all $\mu,\nu\in\mathbb{R}$ the following equalities hold:

$$D^{\varepsilon}_{0x}\Phi^{\mu,\nu}_{\alpha,\beta}(x,y)=\Phi^{\mu-\varepsilon,\nu}_{\alpha,\beta}(x,y), \quad \alpha+\mu>0, \tag{22}$$

$$D^{\delta}_{0y}\Phi^{\mu,\nu}_{\alpha,\beta}(x,y)=\Phi^{\mu,\nu-\delta}_{\alpha,\beta}(x,y), \quad \beta+\nu>0. \tag{23}$$

Lemma 2.1 follows from the formulas (7), (15), (21).

Lemma 2.1 implies the equalities

$$D^{\{\varepsilon_0,\ldots,\varepsilon_j\}}_{0x}\Phi^{\mu,\nu}_{\alpha,\beta}(x,y)=\Phi^{\mu-\mu_j+1,\nu}_{\alpha,\beta}(x,y), \quad \alpha+\mu_i>0, \quad \mu_j=\sum_{i=0}^{j}\varepsilon_i, \tag{24}$$

$$D^{\{\delta_0,\ldots,\delta_j\}}_{0y}\Phi^{\mu,\nu}_{\alpha,\beta}(x,y)=\Phi^{\mu,\nu-\nu_j+1}_{\alpha,\beta}(x,y), \quad \beta+\nu_j>0, \quad \nu_j=\sum_{i=0}^{j}\delta_i. \tag{25}$$

Lemma 2.2. The estimate

$$\left|\Phi^{\mu,\nu}_{\alpha,\beta}(x,y)\right|_* \leq Cx^{\alpha+\mu-\alpha\theta-1}y^{\nu+\beta\theta-1}, \quad \theta\in[\theta_1,\theta_2) \tag{26}$$

holds for all $x\in[0;x_0]$, where $\theta_1=\begin{cases} 0, & -\nu\notin\mathbb{N}_0, \\ -1, & -\nu\in\mathbb{N}_0, \end{cases}$ $\theta_2=\begin{cases} 1, & \mu\neq 0, \\ 2, & \mu=0, \end{cases}$ $\mathbb{N}_0=\{0,1,2,...\}$, and the constant C depends on x_0.

The validity of Lemma 2.2, under the condition (14), follows from the formulas (3) and (19), similarly to the case when all eigenvalues of the matrix A are positive [3].

Lemma 2.3. Let $AB = BA$, $\sum\limits_{i=0}^{k_1} \varepsilon_i = \alpha + 1$, $\sum\limits_{i=0}^{m_1} \delta_i = \beta + 1$, then the following equality holds:

$$\left(D_{0x}^{\{\varepsilon_0,\ldots,\varepsilon_{k_1}\}} + A D_{0y}^{\{\delta_0,\ldots,\delta_{m_1}\}} - B \right) \Phi_{\alpha,\beta}^{\mu,\nu}(x,y) = \frac{x^{\mu-1} y^{\nu-1}}{\Gamma(\mu)\Gamma(\nu)} I. \tag{27}$$

Proof. Let us denote

$$h_\alpha^\mu(x,\tau) = x^{\mu-1}\phi(-\alpha,\mu;-\tau x^{-\alpha}), \quad h_\beta^\nu(y,\tau) = y^{\nu-1}\phi(-\beta,\nu;-A\tau y^{-\beta}).$$

Using the fact that due to (9)

$$\left(D_{0x}^{\{\varepsilon_0,\ldots,\varepsilon_{k_1}\}} + \frac{\partial}{\partial\tau} - B \right) e^{B\tau} h_\alpha^\mu(x,\tau) = 0,$$

the integration by parts formula and relations (2), (3), (12) and (20), we obtain

$$D_{0x}^{\{\varepsilon_0,\ldots,\varepsilon_{k_1}\}} \Phi_{\alpha,\beta}^{\mu,\nu}(x,y) = \int\limits_0^\infty e^{B\tau} D_{0x}^{\{\varepsilon_0,\ldots,\varepsilon_{k_1}\}} h_\alpha^\mu(x,\tau) h_\beta^\nu(y,\tau) d\tau =$$

$$= \frac{x^{\mu-1} y^{\nu-1}}{\Gamma(\mu)\Gamma(\nu)} I + B\Phi_{\alpha,\beta}^{\mu,\nu}(x,y) + \int\limits_0^\infty e^{B\tau} h_\alpha^\mu(x,\tau) \frac{\partial}{\partial\tau} h_\beta^\nu(y,\tau) d\tau. \tag{28}$$

By virtue of (25), we get

$$D_{0y}^{\{\delta_0,\ldots,\delta_{m_1}\}} \Phi_{\alpha,\beta}^{\mu,\nu}(x,y) = \int\limits_0^\infty e^{B\tau} h_\alpha^\mu(x,\tau) D_{0y}^{\{\delta_0,\ldots,\delta_{m_1}\}} h_\beta^\nu(y,\tau) d\tau. \tag{29}$$

By (28) and (29), taking into account the equality

$$\left(A D_{0y}^{\{\delta_0,\ldots,\delta_{m_1}\}} + \frac{\partial}{\partial\tau} - B \right) e^{B\tau} h_\beta^\nu(y,\tau) = 0,$$

which follows from (17), we get (27). Lemma 2.3 is proven.

3 Problem statement and main theorem

Let all eigenvalues $\lambda_1, ..., \lambda_p$ of the matrix A satisfy the condition (14). We formulate a boundary value problem for the system (1).

Problem 3.1. Find a solution $u(x,y)$ of system (1) with the boundary conditions

$$\lim_{x\to 0} D_{0x}^{\{\alpha_0,\alpha_2,\ldots,\alpha_i\}} u = \varphi_i(y), \quad 0 \le i \le k-1, \quad 0 < y < b, \tag{30}$$

$$\lim_{y\to 0} D_{0y}^{\{\beta_0,\beta_1,\ldots,\beta_j\}} u = \psi_j(x), \quad 0 \le j \le m-1, \quad 0 < x < a, \tag{31}$$

where $\varphi_i(y)$ and $\psi_j(x)$ are given n-vectors functions.

A regular solution of system (1) in the domain Ω is defined as a vector function $u(x, y)$ satisfying at all points $(x, y) \in \Omega$ the system (1) and the inclusions

$$D_{0x}^{\{\alpha_0,\dots,\alpha_k\}}u, D_{0y}^{\{\beta_0,\dots,\beta_m\}}u \in C(\Omega); \tag{32}$$

$$\begin{gathered} D_{0x}^{\{\alpha_0,\dots,\alpha_i\}}u \in C(\Omega \cup \{x=0\}), \quad D_{0y}^{\{\beta_0,\dots,\beta_j\}}u \in C(\Omega \cup \{y=0\}), \\ \frac{\partial}{\partial x}D_{0x}^{\{\alpha_0,\dots,\alpha_i\}}u, \frac{\partial}{\partial y}D_{0y}^{\{\beta_0,\dots,\beta_j\}}u \in C(\Omega) \cap L(\Omega), \quad (i=\overline{0,k-1}, j=\overline{0,m-1}); \end{gathered} \tag{33}$$

$x^{1-\varepsilon}y^{1-\delta}u(x, y) \in C(\overline{\Omega})$, for some $\varepsilon > 0$ and $\delta > 0$.

We accept the following notation:

$$\mu_j = \sum_{p=0}^{j}\alpha_p, \quad \bar{\mu}_j = \sum_{p=j}^{k}\alpha_p, \quad \mu_j^i = \sum_{p=j}^{i}\alpha_p; \quad \nu_i = \sum_{p=0}^{i}\beta_p, \quad \bar{\nu}_i = \sum_{p=i}^{m}\beta_p, \quad \nu_i^j = \sum_{p=i}^{j}\beta_p.$$

Theorem 3.1. Let $AB = BA$, all eigenvalues $\lambda_1, ..., \lambda_p$ of the matrix A satisfy the condition (14), $\alpha_0 + \alpha_k > 1$, $\beta_0 + \beta_m > 1$,

$$\begin{gathered} \varphi_j(y) = D_{0y}^{-\rho_j}\varphi_j^*(y), \quad y^{1-\eta_j}\varphi_j^*(y), \, y^{1-\nu}\varphi_{k-1}(y) \in C[0, b], \\ \eta_j > 0, \quad \rho_j > \max\left\{\mu_{j+1}^{k-1}\frac{\beta}{\alpha}, 1-\beta_m\right\}, \quad j = \overline{0, k-2}; \end{gathered} \tag{34}$$

$$\begin{gathered} \psi_i(y) = D_{0x}^{-\sigma_i}\psi_i^*(x), \quad x^{1-\xi_i}\psi_i^*(x), \, x^{1-\mu}\psi_{m-1}(x) \in C[0, a], \\ \xi_i > 0, \quad \sigma_i > \max\left\{\nu_{i+1}^{m-1}\frac{\alpha}{\beta}, 1-\alpha_k\right\}, \quad i = \overline{0, m-2}; \end{gathered} \tag{35}$$

$$\begin{gathered} f(x, y) = D_{0x}^{-\sigma}D_{0y}^{-\rho}f^*(x, y), \quad x^{1-\xi}y^{1-\eta}f^*(x, y) \in C(\overline{\Omega}), \\ \sigma > 1-\alpha_k, \quad \rho > 1-\beta_m, \quad \xi > 0, \quad \eta > 0; \end{gathered} \tag{36}$$

whereinto $\varepsilon < \min\{\alpha_0, \sigma_i + \xi_i, \sigma + \xi, \mu\}$, $\delta < \min\{\beta_0, \rho_i + \eta_i, \rho + \eta, \nu\}$. Then there exists a unique regular solution to problem (1), (30), (31) in Ω. The solution has the form

$$u(x, y) = \sum_{j=0}^{k-1}\int_0^y D_{0x}^{\{\alpha_k,\alpha_{k-1},\dots,\alpha_{j+1}\}}G(x, y-s)\varphi_j(s)ds+$$

$$+\sum_{i=0}^{m-1}\int_0^x D_{0y}^{\{\beta_m,\beta_{m-1},\dots,\beta_{i+1}\}}G(x-t, y)A\psi_i(t)dt + \int_0^y\int_0^x G(x-t, y-s)f(t, s)dtds, \tag{37}$$

where

$$G(x, y) = \Phi_{\alpha,\beta}^{0,0}(x, y).$$

Remark 3.1. If we put

$$\sigma_i = \sigma = \mu = \alpha_0, \quad \rho_j = \rho = \nu = \beta_0, \quad \xi_i = \eta_j = \xi = \eta = 1,$$

then $\varepsilon < \alpha_0$, $\delta < \beta_0$, and the conditions (34)–(36) will take the form

$$\begin{gathered} \varphi_j(y) = D_{0y}^{-\beta_0}\varphi_j^*(y), \quad \varphi_j^*(y), \, y^{1-\beta_0}\varphi_{k-1}(y) \in C[0, b], \\ \beta_0 > \max\left\{\beta - \frac{\alpha_0+\alpha_k-1}{\alpha}\beta, 1-\beta_m\right\}, \quad j = \overline{0, k-2}; \end{gathered} \tag{38}$$

$$\begin{gathered} \psi_i(y) = D_{0x}^{-\alpha_0}\psi_i^*(x), \quad \psi_i^*(x), \, x^{1-\alpha_0}\psi_{m-1}(x) \in C[0, a], \\ \alpha_0 > \max\left\{\alpha - \frac{\beta_0+\beta_m-1}{\beta}\alpha, 1-\alpha_k\right\}, \quad i = \overline{0, m-2}; \end{gathered} \tag{39}$$

$$f(x,y) = D_{0x}^{-\alpha_0} D_{0y}^{-\beta_0} f^*(x,y), \quad f^*(x,y) \in C(\overline{\Omega}). \tag{40}$$

Remark 3.2. In the case of a system with Riemann–Liouville derivatives, i.e., when $k = m = 1$, $D_{0x}^{\{\alpha_0,1\}} = D_{0x}^{\alpha_0}$, $D_{0y}^{\{\beta_0,1\}} = D_{0y}^{\beta_0}$, the conditions (38)–(40) will take the form

$$y^{1-\beta_0}\varphi_0(y) \in C[0,b], \quad x^{1-\alpha_0}\psi_0(x) \in C[0,a], \quad f(x,y) = D_{0x}^{-\alpha_0} D_{0y}^{-\beta_0} f^*(x,y), \quad f^*(x,y) \in C(\overline{\Omega}).$$

The solution has the form

$$u(x,y) = \int\limits_0^y G(x,y-s)\varphi_0(s)ds + \int\limits_0^x G(x-t,y)A\psi_0(t)dt +$$

$$+ \int\limits_0^y \int\limits_0^x G(x-t,y-s)f(t,s)dtds.$$

Remark 3.3. In the case of a system with Gerasimov–Caputo derivatives, i.e., when $k = m = 1$, $D_{0x}^{\{1,\alpha_1\}} = \partial_{0x}^{\alpha_1}$, $D_{0y}^{\{1,\beta_1\}} = \partial_{0y}^{\beta_1}$, the conditions (38)–(40) will take the form

$$\varphi_0(y) \in C[0,b], \quad \psi_0(x) \in C[0,a], \quad f(x,y) = D_{0x}^{-1} D_{0y}^{-1} f^*(x,y), \quad f^*(x,y) \in C(\overline{\Omega}).$$

The solution has the form

$$u(x,y) = \int\limits_0^y D_{0x}^{\alpha_1-1}G(x,y-s)\varphi_0(s)ds + \int\limits_0^x D_{0y}^{\beta_1-1}G(x-t,y)A\psi_0(t)dt +$$

$$+ \int\limits_0^y \int\limits_0^x G(x-t,y-s)f(t,s)dtds.$$

In what follows, for brevity, we will denote

$$u_{\varphi_j}(x,y) = \int\limits_0^y D_{0x}^{\{\alpha_k,\ldots,\alpha_{j+1}\}}G(x,y-s)\varphi_j(s)ds,$$

$$u_{\psi_i}(x,y) = \int\limits_0^x D_{0y}^{\{\beta_m,\ldots,\beta_{i+1}\}}G(x-t,y)A\psi_i(t)dt,$$

$$u_f(x,y) = \int\limits_0^x \int\limits_0^y G(x-t,y-s)f(t,s)dsdt.$$

3.1 Representation of solutions

Lemma 3.1. Every regular solution $u(x,y)$ to problem (1), (30), (31) in Ω can be represented in the form (37).

Proof. Let $u(x,y)$ be a solution to problem (1), (30), (31), and matrix $V \equiv V(x-t,y-s)$ be a solution to the equation

$$L^*V \equiv D_{xt}^{\{\alpha_k,\alpha_{k-1},\ldots,\alpha_0\}}V + D_{ys}^{\{\beta_m,\beta_{m-1},\ldots,\beta_0\}}VA = VB + I, \tag{41}$$

satisfying the conditions

$$\lim_{t\to x} D_{xt}^{\{\alpha_k,\alpha_{k-1},\ldots,\alpha_i\}} V = 0, \quad 1 \le i \le k, \tag{42}$$

$$\lim_{s\to y} D_{ys}^{\{\beta_m,\beta_{m-1},\ldots,\beta_j\}} V = 0, \quad 1 \le j \le m, \tag{43}$$

where I is the identity matrix.

Lemmas 2.2 and 2.3 show that $V(x-t, y-s) = \Phi_{\alpha,\beta}^{1,1}(x-t, y-s)$ is the solution to the problem (41)–(43). From (22) and (23) we see that

$$V_{xy}(x, y) = G(x, y). \tag{44}$$

We have the following formula [20]

$$\int_0^x \left[h(x,t) D_{0t}^{\{\alpha_0,\alpha_1,\ldots,\alpha_m\}} g(t) - D_{xt}^{\{\alpha_m,\alpha_{m-1},\ldots,\alpha_0\}} h(x,t) \cdot g(t) \right] dt =$$

$$= \sum_{i=1}^{m} D_{xt}^{\{\alpha_m,\alpha_{m-1},\ldots,\alpha_{m+1-i}\}} h(x,t) \cdot D_{0t}^{\{\alpha_0,\alpha_1,\ldots,\alpha_{m-i}\}} g(t) \Big|_{t=0}^{t=x}. \tag{45}$$

By (1) and (41) we get

$$V(x-t, y-s) Lu(t,s) - L^* V(x-t, y-s) \cdot u(t,s) = V(x-t, y-s) f(t,s) - u(t,s),$$

or

$$\left(V D_{0t}^{\{\alpha_0,\ldots,\alpha_k\}} u - D_{xt}^{\{\alpha_k,\ldots,\alpha_0\}} V \cdot u \right) + \left(V A D_{0s}^{\{\beta_0,\ldots,\beta_m\}} u - D_{ys}^{\{\beta_m,\ldots,\beta_0\}} V A \cdot u \right) = Vf - u.$$

Integrating the last equality, taking into account the formula (45), we obtain

$$\int_0^x \int_0^y u(t,s) ds dt = \int_0^x \int_0^y V(x-t, y-s) u(t,s) ds dt -$$

$$- \sum_{i=1}^{k} \int_0^y D_{xt}^{\{\alpha_k,\alpha_{k-1},\ldots,\alpha_{k+1-i}\}} V(x, y-s) D_{0t}^{\{\alpha_0,\alpha_1,\ldots,\alpha_{k-i}\}} u(t,s) \Big|_{t=0}^{t=x} ds -$$

$$- \sum_{j=1}^{m} \int_0^x D_{ys}^{\{\beta_m,\beta_{m-1},\ldots,\beta_{m+1-j}\}} V(x-t, y) A D_{0s}^{\{\beta_0,\beta_1,\ldots,\beta_{m-j}\}} u(t,s) \Big|_{s=0}^{s=y} dt. \tag{46}$$

Therefore, differentiating (46) with respect to x and with respect to y, taking into account (30), (31), (42), (43) and (44), and then changing the order of summation, we get (37). Lemma 3.1 is proved.

3.2 Properties of the fundamental solution

Lemma 3.2. [3] Let $AB = BA$, then the equality

$$\left(D_{0x}^{\alpha} + A D_{0y}^{\beta} - B \right) G(x, y) = 0 \tag{47}$$

holds.

Lemma 3.3. Let the vectors $\psi_i(x)$ $(i = \overline{0, k-1})$, and $\varphi_j(y)$ $(j = \overline{0, m-1})$, satisfy the conditions of Theorem 3.1, then the relations

$$\lim_{x\to 0} D_{0x}^{\{\alpha_0,\ldots,\alpha_i\}} u_{\varphi_j}(x,y) = \begin{cases} 0, & i \neq j, \\ \varphi_j(y), & i = j, \end{cases} \quad , \quad y > \varepsilon > 0, \tag{48}$$

$$\lim_{y\to 0} D_{0y}^{\{\beta_0,\ldots,\beta_j\}} u_{\psi_j}(x,y) = \begin{cases} 0, & i \neq j, \\ \psi_i(y), & i = j, \end{cases} \quad , \quad x > \varepsilon > 0, \tag{49}$$

$$\lim_{y\to 0} D_{0y}^{\{\beta_0,\ldots,\beta_j\}} u_{\varphi_j}(x,y) = 0, \quad x > \varepsilon > 0, \tag{50}$$

$$\lim_{x\to 0} D_{0x}^{\{\alpha_0,\ldots,\alpha_i\}} u_{\psi_j}(x,y) = 0, \quad y > \varepsilon > 0, \tag{51}$$

hold, where the limits (48) and (51) are uniform on any closed subset $(0; b)$, and the limits (49) and (50) on any closed subset $(0; a)$.

Proof. Using (22), we write

$$u_{\varphi_j}(x,y) = \int_0^y \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1},0}(x, y-s)\varphi_j(s)ds.$$

By virtue of the formula (24) and the estimate (26), we obtain

$$D_{0x}^{\{\alpha_0,\ldots,\alpha_i\}} \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1},0}(x,y) = \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1}+1-\mu_i,0}(x,y),$$

$$\left|\Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1}+1-\mu_i,0}(x,y)\right|_* \leq C x^{1+\alpha-\bar{\mu}_{j+1}-\mu_i-\alpha\theta} y^{\beta\theta-1}, \quad \theta \in [-1, 1].$$

Let $i < j$, then, taking into account the fact that $\sum\limits_{s=i+1}^{j} \alpha_s < \alpha$ for $\alpha_0 + \alpha_k > 1$, we get that there exists $\theta \in (0, 1)$, such that

$$1 + \alpha - \bar{\mu}_{j+1} - \mu_i - \alpha\theta = \sum_{s=0}^{k} \alpha_s - \sum_{s=0}^{i} \alpha_s - \sum_{s=j+1}^{k} \alpha_s - \alpha\theta = \sum_{s=i+1}^{j} \alpha_s - \alpha\theta > 0.$$

By virtue of the last relations, for $i < j$, we obtain

$$D_{0x}^{\{\alpha_0,\ldots,\alpha_i\}} u_{\varphi_j}(x,y) = \int_0^y \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1}+1-\mu_i,0}(x, y-s)\varphi_j(s)ds \in C(\Omega \cup \{x = 0\}), \tag{52}$$

$$\lim_{x\to 0} D_{0x}^{\{\alpha_0,\ldots,\alpha_i\}} u_{\varphi_j}(x,y) = 0. \tag{53}$$

Consider now the case $i = j$. Taking into account that $1 - \bar{\mu}_{j+1} + 1 - \mu_j = 1 - \alpha$ we obtain

$$D_{0x}^{\{\alpha_0,\ldots,\alpha_j\}} u_{\varphi_j}(x,y) = \int_0^y \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1}+1-\mu_j,0}(x, y-s)\varphi_j(s)ds = \int_0^y \Phi_{\alpha,\beta}^{1-\alpha,0}(x, y-s)\varphi_j(s)ds.$$

Hence, in view of the equality [3]

$$\lim_{x\to 0} \int_0^y \Phi_{\alpha,\beta}^{1-\alpha,0}(x, y-s)q(s)ds = q(y),$$

which under condition (14) is proved using equality (18), we get

$$\lim_{x\to 0} D_{0x}^{\{\alpha_0,\dots,\alpha_j\}} u_{\varphi_j}(x,y) = \varphi_j(y). \tag{54}$$

For $i > j$, i.e. $1 \le i \le k-1$, and $0 \le j \le k-2$, due to (24) and equality (47) we obtain

$$D_{0x}^{\{\alpha_0,\dots,\alpha_i\}} u_{\varphi_j}(x,y) = D_{0x}^{\{\alpha_{j+1},\dots,\alpha_i\}} \int\limits_0^y D_{0x}^{\alpha} G(x,y-s)\varphi_j(s)ds =$$

$$= D_{0x}^{\{\alpha_{j+1},\dots,\alpha_i\}} \int\limits_0^y \left(B - AD_{ys}^{\beta}\right) G(x,y-s)\varphi_j(s)ds = I_1(x,y) - I_2(x,y). \tag{55}$$

It follows from the relation $\alpha > \alpha_j + 1 + ... + \alpha_i$ that there exists $\theta \in (0,1)$, such that $\alpha(1-\theta) > \alpha_j + 1 + ... + \alpha_i$. Therefore, by the estimate

$$\left|\Phi_{\alpha,\beta}^{1-\mu_{j+1}^i,0}(x,y-s)\right|_* \le Cx^{\alpha-\alpha\theta-\mu_{j+1}^i} y^{\beta\theta-1}, \quad \theta\in[-1,1),$$

we get the relations

$$I_1(x,y) = B\int\limits_0^y \Phi_{\alpha,\beta}^{1-\mu_{j+1}^i,0}(x,y-s)\varphi_j(s)ds \in C(\Omega\cup\{x=0\}), \tag{56}$$

$$\lim_{x\to 0} I_1(x,y) = 0. \tag{57}$$

Let us consider the second term

$$I_2(x,y) = A\int\limits_0^y D_{0x}^{\mu_{j+1}^i-1} D_{ys}^{\beta} G(x,y-s) D_{0s}^{-\rho_j}\varphi_j(s)ds = A\int\limits_0^y \Phi_{\alpha,\beta}^{1-\mu_{j+1}^i,\rho_j-\beta}(x,y-s)\varphi_j^*(s)ds.$$

In view of the estimate

$$\left|\Phi_{\alpha,\beta}^{1-\mu_{j+1}^i,\rho_j-\beta}(x,y)\right|_* \le Cx^{\alpha-\alpha\theta-\mu_{j+1}^i} y^{\rho_j-\beta+\beta\theta-1}, \quad \theta\in[0,1),$$

we get that the integral $I_2(x,y)$ converges under the condition $\begin{cases} \alpha-\alpha\theta-\mu_{j+1}^i > 0, \\ \rho_j-\beta+\beta\theta > 0, \end{cases}$ i.e., when θ satisfies the condition $\frac{\mu_{j+1}^i}{\alpha} < 1-\theta < \frac{\rho_j}{\beta}$, at that

$$\lim_{x\to 0} I_2(x,y) = 0. \tag{58}$$

By (53), (54), (55), (57) and (58) we get (48).

The relation (49) is proved similarly.

Let us prove the relation (50). Formulas (25) and (26) give the equalities

$$\frac{\partial^s}{\partial y^s} D_{0y}^{\{\beta_0,\dots,\beta_i\}} \Phi_{\alpha,\beta}^{1-\bar\mu_{j+1},\rho_j}(x,y) = \Phi_{\alpha,\beta}^{1-\bar\mu_{j+1},\rho_j-\nu_i+1-s}(x,y), \quad s=0,1,$$

and

$$\left|\Phi_{\alpha,\beta}^{1-\bar\mu_{j+1},\rho_j-\nu_i+1-s}(x,y)\right|_* \le Cx^{\alpha-\alpha\theta-\bar\mu_{j+1}} y^{\rho_j-\nu_i+\beta\theta-s}, \quad s=0,1, \quad \theta\in[0,1).$$

By the last estimate, taking into account that due to $\rho_j > 1 - \beta_m > 1 - \bar{\nu}_{i+1}$ one can choose θ sufficiently close to 1, so that $\rho_j - \nu_i + \beta\theta = \rho_j + \bar{\nu}_{i+1} - 1 + \beta(\theta - 1) > 0$, we get

$$D_{0y}^{\{\beta_0,\dots,\beta_i\}} u_{\varphi_j}(x,y) = \int\limits_0^y \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1},\rho_j+1-\nu_i}(x,y-s)\varphi_j^*(s)ds \in C(\Omega \cup \{y=0\}), \tag{59}$$

$$\frac{\partial}{\partial y} D_{0y}^{\{\beta_0,\dots,\beta_i\}} u_{\varphi_j}(x,y) = \int\limits_0^y \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1},\rho_j-\nu_i}(x,y-s)\varphi_j^*(s)ds \in C(\Omega) \cup L(\Omega), \tag{60}$$

and

$$\lim_{y\to 0} D_{0y}^{\{\beta_0,\dots,\beta_i\}} u_{\varphi_j}(x,y) = 0, \quad 0 \le i \le m-1.$$

The relation (51) is proved similarly. Lemma 3.3 is proved.

Lemma 3.4. The function (37) is a solution (1) satisfying the inclusions (32) and (33).

Proof. Using estimates

$$\left|\Phi_{\alpha,\beta}^{-\mu_{j+1}^{k-1},0}(x,y)\right|_* \le C x^{\alpha-\alpha\theta-\mu_{j+1}^{k-1}-1} y^{\beta\theta-1}, \quad \theta \in [0,1),$$

$$\left|\Phi_{\alpha,\beta}^{-\mu_{j+1}^{k-1},\rho_j-\beta}(x,y)\right|_* \le C x^{\alpha-\alpha\theta-\mu_{j+1}^{k-1}-1} y^{\rho_j-\beta+\beta\theta-1}, \quad \theta \in [0,1),$$

and inequalities $\alpha - \alpha\theta - \mu_{j+1}^{k-1} > 0$, $\rho_j - \beta + \beta\theta > 0$, by (55) with $i = k-1$ we get

$$\frac{\partial}{\partial x} D_{0x}^{\{\alpha_0,\dots,\alpha_{k-1}\}} u_{\varphi_j}(x,y) = B\int\limits_0^y \Phi_{\alpha,\beta}^{-\mu_{j+1}^{k-1},0}(x,y-s)\varphi_j(s)ds -$$

$$-A\int\limits_0^y \Phi_{\alpha,\beta}^{-\mu_{j+1}^{k-1},\rho_j-\beta}(x,y-s)\varphi_j^*(s)ds \in C(\Omega) \cap L(\Omega). \tag{61}$$

By (61) it follows that

$$D_{0x}^{\{\alpha_0,\dots,\alpha_k\}} u_{\varphi_j}(x,y) = B\int\limits_0^y \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1},0}(x,y-s)\varphi_j(s)ds -$$

$$-A\int\limits_0^y \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1},\rho_j-\beta}(x,y-s)\varphi_j^*(s)ds \in C(\Omega \cup \{x=0\}). \tag{62}$$

Due to the estimate

$$\left|\Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1},\rho_j-\nu_i}(x,y)\right|_* \le C x^{\alpha-\alpha\theta-\bar{\mu}_{j+1}} y^{\rho_j-\nu_i+\beta\theta-1}, \quad \theta \in [0,1),$$

and the inequality $\rho_j + \beta_m > 1$, one can always choose θ sufficiently close to 1, so that

$$\rho_j - \nu_i + \beta\theta > \beta_{i+1} + \dots + \beta_{m-1} + (\theta-1)\beta > 0,$$

so

$$\frac{\partial}{\partial y} D_{0y}^{\{\beta_0,\dots,\beta_i\}} u_{\varphi_j}(x,y) = \int\limits_0^y \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1},\rho_j-\nu_i}(x,y-s)\varphi_j^*(s)ds \in C(\Omega) \cap L(\Omega). \tag{63}$$

From (63) for $i = m-1$ it follows that

$$D_{0y}^{\{\beta_0,\ldots,\beta_m\}} u_{\varphi_j}(x,y) = \int\limits_0^y \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1},\rho_j-\beta}(x,y-s)\varphi_j^*(s)ds \in C(\Omega). \tag{64}$$

It can be seen from (62) and (64) that $u_{\varphi_j}(x,y)$ are solutions of the homogeneous system

$$D_{0x}^{\{\alpha_0,\ldots,\alpha_k\}} u_{\varphi_j}(x,y) + AD_{0y}^{\{\beta_0,\ldots,\beta_m\}} u_{\varphi_j}(x,y) = Bu_{\varphi_j}(x,y).$$

The proof for $u_{\psi_i}(x,y)$ is similar.

Let us show that

$$u_f(x,y) = \int\limits_0^x \int\limits_0^y \Phi_{\alpha,\beta}^{\sigma,\rho}(x-t,y-s)f^*(t,s)dsdt$$

system solution (1). In view of

$$\left|\Phi_{\alpha,\beta}^{\sigma-\mu_j+1,\rho}(x,y)\right|_* \leq Cx^{\bar{\mu}_{j+1}-\alpha\theta+\sigma-1}y^{\rho+\beta\theta-1},$$

we get

$$\left|D_{0x}^{\{\alpha_0,\ldots,\alpha_j\}} u_f(x,y)\right|_* \leq Cx^{\sigma+\xi-(1-\bar{\mu}_{j+1})-\alpha\theta}y^{\rho+\eta+\beta\theta-1}.$$

The inequality $\sigma+\xi > 1-\alpha_k$, implies $\sigma+\xi-(1-\bar{\mu}_{j+1})-\alpha\theta > 0$ and the relations

$$D_{0x}^{\{\alpha_0,\ldots,\alpha_j\}} u_f(x,y) = \int\limits_0^x \int\limits_0^y \Phi_{\alpha,\beta}^{\sigma-\mu_j+1,\rho}(x-t,y-s)f^*(t,s)dsdt \in C(\Omega\cup\{x=0\}), \tag{65}$$

$$\lim_{x\to 0} D_{0x}^{\{\alpha_0,\ldots,\alpha_j\}} u_f(x,y) = 0, \quad 0 \leq j \leq k-1. \tag{66}$$

It follows from (48)–(51) and (66) that the boundary conditions (30) and (31) hold.

By the estimate

$$\left|\Phi_{\alpha,\beta}^{\sigma-\mu_j,\rho}(x,y)\right|_* \leq Cx^{\alpha-\mu_j-\alpha\theta+\sigma-1}y^{\rho+\beta\theta-1},$$

and inequalities $\alpha-\mu_j-\alpha\theta+\sigma = \bar{\mu}_{j+1}+\sigma-1-\alpha\theta > 0$, which follow from the inequality $\sigma > 1-\alpha_k$, we get

$$\frac{\partial}{\partial x}D_{0x}^{\{\alpha_0,\ldots,\alpha_j\}} u_f(x,y) = \int\limits_0^x \int\limits_0^y \Phi_{\alpha,\beta}^{\sigma-\mu_j,\rho}(x-t,y-s)f^*(t,s)dsdt. \tag{67}$$

By (67), estimate

$$\left|\frac{\partial}{\partial x}D_{0x}^{\{\alpha_0,\ldots,\alpha_j\}} u_f(x,y)\right|_* \leq Cx^{\sigma+\xi+\bar{\mu}_{j+1}-1-\alpha\theta}y^{\rho+\eta+\beta\theta-1},$$

due to the inequality $\sigma > 1-\alpha_k$, we obtain

$$\frac{\partial}{\partial x}D_{0x}^{\{\alpha_0,\ldots,\alpha_j\}} u_f(x,y) \in C(\Omega)\cap L(\Omega), \quad 0 \leq j \leq k-1. \tag{68}$$

Therefore

$$D_{0x}^{\{\alpha_0,\ldots,\alpha_k\}} u_f(x,y) = \int\limits_0^x \int\limits_0^y \Phi_{\alpha,\beta}^{\sigma-\alpha,\rho}(x-t,y-s)f^*(t,s)dsdt \in C(\Omega). \tag{69}$$

Similarly, we get

$$D_{0x}^{\{\beta_0,\ldots,\beta_m\}} u_f(x,y) = \int_0^x \int_0^y \Phi_{\alpha,\beta}^{\sigma,\rho-\beta}(x-t,y-s) f^*(t,s)dsdt \in C(\Omega). \tag{70}$$

By (69) and (70), taking into account Lemma 2.3, we obtain

$$\left(D_{0x}^{\{\alpha_0,\ldots,\alpha_k\}} + AD_{0x}^{\{\beta_0,\ldots,\beta_m\}} - B\right) u_f(x,y) =$$

$$= \int_0^x \int_0^y \left(D_{0x}^{\{\alpha_0,\ldots,\alpha_k\}} + AD_{0x}^{\{\beta_0,\ldots,\beta_m\}} - B\right) \Phi_{\alpha,\beta}^{\sigma,\rho}(x-t,y-s) f^*(t,s)dsdt =$$

$$= \int_0^x \int_0^y \frac{(x-t)^{\sigma-1}(y-s)^{\rho-1}}{\Gamma(\sigma)\Gamma(\rho)} f^*(t,s)dsdt = D_{0x}^{-\sigma} D_{0y}^{-\rho} f^*(t,s) = f(t,s).$$

From (52), (56), (59)–(65), (68)–(70), it follows that (37) satisfies the inclusions (32), (33). Lemma 3.4 is proved.

3.3 Proof of the main theorem

Using the estimate (26) and the conditions of Theorem 3.1 on the functions $\psi_i(x)$, $\varphi_j(y)$ and $f(x,y)$, we obtain the estimates

$$x^{1-\varepsilon} y^{1-\delta} |u_{\varphi_j}(x,y)|_* \le C x^{\alpha-\alpha\theta-\bar{\mu}_{j+1}+1-\varepsilon} y^{\rho_j+\eta_j+\beta\theta-\delta}, \quad \theta \in (0,1), \tag{71}$$

$$x^{1-\varepsilon} y^{1-\delta} |u_{\psi_i}(x,y)|_* \le C x^{\alpha-\alpha\theta+\sigma_i+\xi_i-\varepsilon} y^{\beta\theta-\bar{\nu}_{i+1}+1-\delta}, \quad \theta \in (0,1), \tag{72}$$

$$x^{1-\varepsilon} y^{1-\delta} |u_f(x,y)|_* \le C x^{\alpha-\alpha\theta+\sigma+\xi-\varepsilon} y^{\beta\theta+\rho+\eta-\delta}, \quad \theta \in (0;1). \tag{73}$$

Considering that $\alpha + 1 - \bar{\mu}_{j+1} = \mu_j > \varepsilon$, due to $\alpha_0 > \varepsilon$, and the fact that $\rho_j + \eta_j > \varepsilon$, by (71) we get $x^{1-\varepsilon} y^{1-\delta} u_{\varphi_j} \in C(\overline{\Omega})$. Taking into account the inequalities $\sigma_i + \xi_i > \varepsilon$, and the fact that $\beta\theta - \bar{\nu}_{i+1} + 1 = \beta(\theta - 1) + \nu_i > \delta$, due to $\beta_0 > \delta$, by (72) we get $x^{1-\varepsilon} y^{1-\delta} u_{\psi_i} \in C(\overline{\Omega})$. It follows from (73) and the inequalities $\sigma + \xi > \varepsilon$ and $\rho + \eta > \delta$ that the inclusion $x^{1-\varepsilon} y^{1-\delta} u_f \in C(\overline{\Omega})$.

The above together with Lemmas 3.2, 3.3, and 3.4 proves the existence of a regular solution to the problem (1), (30), (31). The uniqueness of the solution to the problem follows by Lemma 3.1. Theorem 3.1 is proved.

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Джрбашян–Нерсесян бөлшек дифференциалдау операторы бар дербес туындылы теңдеулер жүйесіне арналған шеттік есеп

Тікбұрышты облыста жүйенің матрицалық коэффициенттері күрделі меншікті мәндерге ие болған жағдайда тұрақты коэффициенттері бар Джрбашян-Нерсесян бөлшек дифференциалдау операторы бар дербес туындылы теңдеулер жүйесіне арналған шеттік есеп зерттелді. Зерттелетін шеттік есептердің шешімінің бар болуы және жалғыздық теоремалары дәлелденді. Шешім матрицалық аргументтің Райт функциясы тұрғысынан анық түрде құрастырылған.

Кілт сөздер: дербес туындылы теңдеулер жүйесі, бөлшек ретті туынды, Джрбашян-Нерсесян операторы, шеттік есеп, іргелі шешім, матрицалық аргументтің Райт фукнциясы.

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Краевая задача для системы уравнений в частных производных с операторами дробного дифференцирования Джрбашяна–Нерсесяна

Исследована краевая задача в прямоугольной области для линейной системы уравнений с частными операторами дробного дифференцирования Джрбашяна–Нерсесяна с постоянными коэффициентами в случае, когда матричные коэффициенты системы имеют комплексные собственные значения. Доказаны теоремы существования и единственности решения исследуемой краевой задачи. Решение построено в явном виде в терминах функции Райта матричного аргумента.

Ключевые слова: система уравнений с частными производными, производные дробного порядка, оператор Джрбашяна–Нерсесяна, краевая задача, фундаментальное решение, функция Райта матричного аргумента.

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