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Existence and smoothness of solutions of a singular differential equation of hyperbolic type

This paper investigates the question of the existence of solutions to the semiperiodic Dirichlet problem for a class of singular differential equations of hyperbolic type. The problem of smoothness of solutions is also considered, i.e., maximum regularity of solutions. Such a problem will be interesting when the coefficients are strongly growing functions at infinity. For the first time, a weighted coercive estimate was obtained for solutions to a differential equation of hyperbolic type with strongly growing coefficients.

Keywords: resolvent, hyperbolic type equation, maximal regularity, unbounded domain.

1 Introduction. Formulation of results

Considered on the strip

$$\overline{\Omega} = \{(x, y) : -\pi < x < \pi, -\infty < y < \infty\}$$

next problem:

$$(L + \lambda I)u = u_{xx} - u_{yy} + a(y)u_x + c(y)u + \lambda u = f(x, y) \in L_2(\Omega), \tag{1}$$

$$u(-\pi, y) = u(\pi, y), u_x(-\pi, y) = u_x(\pi, y), -\infty < y < \infty. \tag{2}$$

Further, we assume that the coefficients $a(y)$, $c(y)$ satisfy the conditions:

i) $|a(y)| \geq \delta_0 > 0$, $c(y) \geq \delta > 0$ are continuous functions in $R(-\infty, \infty)$;

ii) $\mu_0 = \sup\limits_{|y-t|\leq 1} \frac{a(y)}{a(t)} < \infty$, $\mu = \sup\limits_{|y-t|\leq 1} \frac{c(y)}{c(t)} < \infty$;

iii) $c(y) \leq c_0 \cdot c^2(y)$ $for\ all\ y \in R, c_0 > 0$ is a constant number.

Here it has to be noted that $a(y)$ and $c(y)$ can be unlimited functions at infinity.

The existence and uniqueness, as well as the qualitative behavior, of solutions for differential equations of hyperbolic type, were studied in [1–14]. In these works, Darboux and Goursat problems and the Cauchy problem, periodic and some boundary value problems for differential equations of hyperbolic type with constant or variable bounded coefficients were examined.

In this paper, in the space $L_2(\Omega)$, we study questions about the existence, uniqueness of solutions, and also the smoothness of solutions to a periodic problem without initial conditions [13] for a differential equation of hyperbolic type with strongly increasing coefficients at infinity.

In our previous paper [14], we studied a differential operator of hyperbolic type in the space $L_2(R^2)$.

In contrast to [14], in this paper, on a strip, we consider the so-called periodic problem without initial conditions. Here we note that in the future, this work will allow us to study questions about the compactness of the resolvent, about estimates for the singular (s-numbers) and eigenvalues of a differential operator of hyperbolic type corresponding to problem (1)–(2).

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Definition 1. We say that function $u(x,y) \in L_2(\Omega)$ is a strong solution to problem (1)–(2), if there is a sequence $\{u_n\} \subset C^{\infty}_{0,\pi}(\Omega)$ such that

$$\|u_n - u\|_{L_2(\Omega)} \to 0, \ \ \|(L+\lambda I)u_n - f\|_{L_2(\Omega)} \to 0 \ \ for \ \ n \to \infty,$$

where $C^{\infty}_{0,\pi}(\Omega)$ is the set consisting of infinitely differentiable finite functions with respect to variable y and satisfying conditions (2) with respect to variable x.

Theorem 1. Let the condition $i)$ be fulfilled. Then for $\lambda \geq 0$ for any $f(x,y) \in L_2(\Omega)$ there is a unique strong solution to the problem (1)–(2), and the equality is true

$$u(x,y) = (L+\lambda I)^{-1} f = \sum_{n=-\infty}^{\infty} (l_n + \lambda I)^{-1} f_n(y) \cdot e^{inx},$$

where $f(x,y) \in L_2(\Omega)$, $f(x,y) = \sum\limits_{n=-\infty}^{\infty} f_n(y) \cdot e^{inx}$, $f_n(y) =< f(x,y), e^{inx} >$ are fourier coefficients, $i^2 = -1$, $< \cdot, \cdot >$ is the scalar product in $L_2(\Omega)$,

$$(l_n + \lambda I)u = -u^{''}(y) + (-n^2 + ina(y) + c(y) + \lambda)u(y), u \in D(l_n).$$

Theorem 2. Let the condition $i)$ be fulfilled. Then for $\lambda \geq 0$ for any $f(x,y) \in L_2(\Omega)$ there is a unique strong solution to the problem (1)–(2), and the equality is true

$$\|u_{xx} - u_{yy}\|_2 + \|u_y\|_2 + \|a(y)u_x\|_2 + \|c(y)u\|_2 \leq c \cdot \|f\|_2,$$

where $c > 0$ is constant number.

2 Proof of theorems 1-2

Using the Fourier method, we reduce problem (1)–(2) to the study of the following differential operator with negative discrete parameter n $(n = 0, \pm 1, \pm 2, ...)$:

$$(l_n + \lambda I)u = -u^{''}(y) + (-n^2 + ina(y) + c(y) + \lambda)u(y), u \in D(l_n),$$

where $D(l_n)$ is the domain of the operator l_n.

Consider two cases:

I. Let be $(n = 0)$. In this case, the operator l_0 is the Sturm–Liouville operator.

This operator has been studied thoroughly in [15–21].

II. Let be $n \neq 0$. In this case, it is easy to see that the first term in the coefficient $(-n^2 + ina(y) + c(y) + \lambda)$ tends to $-\infty$, i.e. $-n^2 \to -\infty$.

In this case, the l_n operator is not a semi-bounded operator. Consequently, the methods that have been worked out for the Sturm–Liouville operator $L = -\frac{d^2}{dx^2} + q(x)u$ turn out to be poorly adapted to the study of the Sturm–Liouville operator with a negative parameter.

Let us take $\{\varphi_j\}$ the set of non-negative functions from $C^{\infty}_0(R)$ such that

$$\sum_j \varphi_j^2 \equiv 1, \sup \ p \ \varphi_j \subset \triangle_j, \ \bigcup_j \triangle_j = R,$$

where $\triangle_j = (j-1, j+1)$, $j = 0, \pm 1, \pm 2, ...$ the multiplicity of the intersection of which is not higher than three. The existence of such a covering follows from the results of [22].

Continue $a(y)$, $c(y)$ from $\triangle_j$ for all R . The resulting functions will be denoted by $a_j(y)$ and $c_j(y)$.

These functions are bounded and periodic functions. Denote by $(l_{n,j,\alpha}+\lambda I)$ the closure of operator

$$(l_{n,j,\alpha}+\lambda I)u=-u^{''}(y)+(-n^2+in(a_j(y)+\alpha)+c_j(y)+\lambda)u$$

defined on $C_0^\infty(R)$. We introduced the real number α to evaluate the norm of the operator $D_y(l_{n,j,\alpha}+\lambda I)^{-1}$ in the space $L_2(\Omega)$, where $D_y=\frac{\partial}{\partial y}$. The sign of the number α is chosen as follows: $\alpha\cdot b(y)>0$ at $y\in R$.

In the course of the proof using Lemma 3, we will get rid of this number.

Lemma 1. Suppose that the coefficients of the operator $l_{n,j,\alpha}+\lambda I$ satisfy condition i). Then for $\lambda\geq 0$:

1) for the differential operator $l_{n,j,\alpha}+\lambda I$ at $\lambda\geq 0$, there is a bounded inverse operator $(l_{n,j,\alpha}+\lambda I)^{-1}$ defined at all $L_2(R)$.

2) the resolvent of the operator $l_{n,j,\alpha}+\lambda I$ satisfies the following estimates:

a) $\|(l_{n,j,\alpha}+\lambda I)^{-1}\|_{2\to 2}\leq\frac{c}{(\delta+\lambda)^{\frac{1}{2}}}$, $c>0$ — constant number independent of n,j,α;

b) $\|\frac{d}{dy}(l_{n,j,\alpha}+\lambda I)^{-1}\|_{2\to 2}\leq\frac{c}{(\delta+\lambda)^{\frac{1}{4}}}$, $c>0$ — constant number independent of n,j,α;

c) $\|(l_{n,j,\alpha}+\lambda I)^{-1}\|_{2\to 2}\leq\frac{1}{|n|\cdot|a(\tilde{y}_j)|}$, $n\neq 0$, $c>0$ — constant number independent of n,j,α;

d) $\|(l_{n,j,\alpha}+\lambda I)^{-1}\|_{2\to 2}\leq\frac{2\cdot c}{c(\overline{y}_j)+\lambda}$, $c>0$ — constant number independent of n,j,α;

where $\|\cdot\|_{2\to 2}$ — is the norm of the operator $(l_{n,j,\alpha}+\lambda I)^{-1}$ in space $L_2(R)$, $|a(\tilde{y}_j)|=\min\limits_{y\in\overline{\triangle}_j}|a(y)|$, $|c(\overline{y}_j)|=\min\limits_{y\in\overline{\triangle}_j}|c(y)|$.

Lemma 1 is proved using functionals $<(l_{n,j,\alpha}+\lambda I)u,u>$, $<(l_{n,j,\alpha}+\lambda I)u,-inu>$ $(n=0,\pm1,\pm2,...)$ and repeating the calculations and arguments for these functionals, which were used in the proof of Lemma 2.1 [22] and Lemmas 4–6 [23].

Now, consider the differential operator

$$(l_{n,\alpha}+\lambda I)u=-u^{''}+(-n^2+in(a(y)+\alpha)+c(y)+\lambda)\cdot u,$$

which is a closure in $L_2(R)$ of the following operator originally defined on $C_0^\infty(R)$:

$$(l_{n,\alpha}+\lambda I)u=-u^{''}+(-n^2+in(a(y)+\alpha)+c(y)+\lambda)\cdot u.$$

We introduce the operator

$$K_{\lambda,\alpha}f=\sum_{\{j\}}\varphi_j(l_{n,j,\alpha}+\lambda I)^{-1}\varphi_jf, f\in L_2(R).$$

The following lemma is proved with the help of calculations and arguments that were used in the proof of Theorem 1.1–1.3 in [22] and Theorem 1 in [23].

Lemma 2. Suppose that the coefficients of the operator $l_{n,j,\alpha}+\lambda I$ satisfy condition i). Then there is a number $\lambda_0>0$ such that for the operator $l_{n,j,\alpha}+\lambda I$ for $\lambda\geq\lambda_0$ there is a resolvent and the equality

$$(l_{n,j,\alpha}+\lambda I)^{-1}f=K_{\lambda,\alpha}(I-M_{\lambda,\alpha})^{-1}f, f\in L_2(R)$$

holds, where $M_{\lambda,\alpha}f=\sum\limits_{\{j\}}\varphi_j^{''}(l_{n,j,\alpha}+\lambda I)^{-1}\varphi_jf+2\sum\limits_{\{j\}}\varphi_j^{'}\frac{d}{dy}(l_{n,j,\alpha}+\lambda I)^{-1}\varphi_jf,\ f\in L_2(R)$.

Lemma 3. Suppose that the coefficients of the operator $l_{n,j,\alpha}+\lambda I$ satisfy condition i). Then there is a number $\lambda_0>0$ such that for the operator $l_n+\lambda I$ $(n=0,\pm1,\pm2,...)$ for $\lambda\geq\lambda_0$ there is a resolvent and the equality

$$(l_n+\lambda I)^{-1}f=(l_{n,\alpha}+\lambda I)^{-1}(I-M_{\lambda,\alpha})^{-1}f, f\in L_2(R)$$

holds, where $M_{\lambda,\alpha} = in\alpha(l_{n,\alpha}+\lambda I)^{-1}$ $(n = 0, \pm1, \pm2, ...)$ and the operator's norm $M_{\lambda,\alpha}$: $\|M_{\lambda,\alpha}\|_{2\to 2} < 1$.

Using the method used in the proof of Lemma 9 in [23], we obtain the proof of Lemma 3.

Proof of Theorem 1. Using the scalar product $< (L + \lambda I)u, u >$ for all $u \in D(L)$ and taking into account the condition i), we obtain that

$$\|(L + \lambda I)u\|_2 \geq c \cdot \|u\|_2,$$

where $c(\delta) > 0$ is a constant number. Further, repeating the calculations and arguments used in the proof of Theorem 1 in [23], we obtain the proof of Theorem 1.

Proof of Theorem 2. Taking into account conditions ii)–iii), and also using the method used in the proof of Theorems in [24–26], we obtain the proof of Theorem 2.

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References

1 Адамар Ж. Задача Коши для линейных уравнений с частными производными гиперболического типа / Ж. Адамар. — М.: Наука, 1978. — 352 с.

2 Friedrichs K. Symmetric hyperbolic linear differential equations / K. Friedrichs // Comm. Pure and Appl. Math. — 1954. — 7. — P. 345–392.

3 Соболев С.Л. Избранные труды / С.Л. Соболев. — Новосибирск: Изд-во Института математики, 2006. — Т. 2.

4 Garding L. Linear hyperbolic partial differential equations with constant coefficients /L. Garding // Acta Math. — 1951. — 85. — P. 1–62.

5 Ладыженская О.А. О разрешимости основных краевых задач для уравнений гиперболического и параболического типов / О.А. Ладыженская // ДАН СССР. — 1954. — 97. — № 3. — С. 395–398.

6 Бицадзе А.В. Некоторые классы уравнений в частных производных / А.В. Бицадзе. — М.: Наука, 1981.

7 Лере Ж. Гиперболические дифференциальные уравнения / Ж. Лере. — М.: Наука, 1984.

8 Нахушев А.М. Задачи со смещением для уравнений в частных производных / А.М. Нахушев. — М.: Наука, 2006.

9 Кальменов Т.Ш. К теории начально-краевых задач для дифференциальных уравнений / Т.Ш. Кальменов. — Алматы, 2013.

10 Кигурадзе Т.И. О периодических краевых задачах для линейных гиперболических уравнений / Т.И. Кигурадзе // Дифф. уравн. — 1993. — 29. – № 2. — С. 281–297.

11 Асанова А.Т. О корректной разрешимости нелокальной краевой задачи для систем гиперболических уравнений / А.Т. Асанова, Д.С. Джумабаев // Докл. РАН. — 2003. — 391. — № 3. — С. 295–297.

12 Filinovskii A.V. Hyperbolic equations with growing coefficients in unbounded domains / A.V. Filinovskii // J. Math. Sci. (N. Y.). — 2014. — 197. — № 3. — P. 435–446.

13 Tikhonov A.N. Equations of mathematical physics / A.N. Tikhonov, A.A. Samarskii. — New York: Dover Publications, Inc., 1990.

14 Muratbekov M.B. Existence and maximal regularity of solutions in $L_2(R^2)$ for a hyperbolic type differential equation with quickly growing coefficients / M.B. Muratbekov, Ye.N. Bayandiyev // Eurasian Mathematical Journal. — 2020. — 11. — № 1. — P. 95–100.

15 Като Т. Теория возмущений линейных операторов / Т. Като. — М.: Мир, 1972. — 740 с.

16 Рид М. Методы современной математической физики. Т. 2. Гармонический анализ. Самосопряженность / М. Рид, Б. Саймон. — М.: Мир, 1978. — 430 с.

17 Титчмарш Э.Ч. Разложения по собственным функциям, связанные с дифференциальными уравнениями второго порядка — Т. 1,2. / Э.Ч. Титчмарш. — М.: ИЛ, 1961. — 278 с.

18 Мазья В.Г. О (p, l)-емкости, теоремах вложения и спектре самосопряженного эллиптического оператора / В.Г. Мазья // Изв. АН СССР. Сер мат. — 1973. — Т. 37. — С. 356–385.

19 Отелбаев М. Теоремы вложения пространств с весом и их применения к изучению спектра оператора Шредингера / М. Отелбаев // Тр. Мат. ин-та им. В.А. Стеклова АН СССР. — 1979. — 150. — С. 265–305.

20 Lorenzi L. On Schrodinger type operators with unbounded coefficients: Generation and heat kernel estimates / L. Lorenzi, A. Rhandi // J. Evol. Equ. — 2015. — 15. — № 1. — P. 53–88.

21 Metafune G. Uniqueness for elliptic operators on with unbounded coefficients / G. Metafune, D. Pallara, P.J. Rabier, R. Schnaubelt // Forum Math. — 2010. — 22. — P. 583–601.

22 Muratbekov M. On the existence of a resolvent and separability for a class of singular hyperbolic type differential operators on an unbounded domain / M. Muratbekov, M. Otelbaev // Eurasian Mathematical Journal. — 2016. — 7. — No. 1. — P. 50–67.

23 Muratbekov M.B. On the compactness of the resolvent of a Schrodinger type singular operator with a negative parameter / M.B. Muratbekov, M.M. Muratbekov // Chaos, Solitons and Fractals. — 2021. — 151. — P. 111–248.

24 Muratbekov M.B. Estimates of Eigenvalues of a Semiperiodic Dirichlet Problem for a Class of Degenerate Elliptic Equations / M.B. Muratbekov, S. Igissinov // Symmetry. — 2022. — 14. — P. 692.

25 Muratbekov M.B. On the bounded invertibility of a Schrodinger operator with a negative parameter in the space $L_2(R^n)$ / M.B. Muratbekov, M.M. Muratbekov // Bulletin of the Karaganda University-Mathematics. — 2019. — 93. — № 1. — P. 36–47.

26 Muratbekov M.B. On the existence of the resolvent and separability of a class of the Korteweg-de Vriese type linear singular operators / M.B. Muratbekov, A.O. Suleimbekova // Bulletin of the Karaganda University-Mathematics. — 2021. — 101. — № 1. — P. 87–97.

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Гиперболалық типтегі сингулярлық дифференциалдық теңдеудің шешімдерінің бар болуы және тегістігі

Мақалада гиперболалық типті сингулярлық дифференциалдық теңдеулер класы үшін жартылай периодтық Дирихле есебінің шешімдерінің бар екендігі туралы мәселе зерттелген. Сонымен қатар шешімдердің тегістігі туралы мәселе, яғни шешімдердің максималды регулярлығы қарастырылған. Коэффициенттері шексіздікте жылдам өсетін функциялар болғанда мұндай есеп қызықты болады. Осы жұмыста бірінші рет коэффициенттері жылдам өсетін гиперболалық типті дифференциалдық теңдеудің шешімдері үшін салмақты коэрцитивті бағалаулар алынған.

Кілт сөздер: резольвента, гиперболалық типтес теңдеу, максималды регулярлық, шексіз облыс.

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Существование и гладкость решений сингулярного дифференциального уравнения гиперболического типа

В статье изучен вопрос о существовании решений полупериодической задачи Дирихле для одного класса сингулярных дифференциальных уравнений гиперболического типа. Также рассмотрена задача о гладкости решений, т.е. максимальная регулярность решений. Данная задача будет интересной, когда коэффициенты являются сильно растущими функциями на бесконечности. По-видимому, в настоящей работе впервые получена весовая коэрцитивная оценка решений дифференциального уравнения гиперболического типа с сильно растущими коэффициентами.

Ключевые слова: резольвента, уравнение гиперболического типа, максимальная регулярность, бесконечная область.

References

1 Adamar, Zh. (1978). *Zadacha Koshi dlia lineinykh uravnenii s chastnymi proizvodnymi giperbolicheskogo tipa [The Cauchy problem for hyperbolic linear partial differential equations]*. Moscow: Nauka [in Russian].

2 Friedrichs, K.O. (1954). Symmetric hyperbolic linear differential equations. *Communications on Pure and Applied Mathematics, 7,* 345–392.

3 Sobolev, S.L. (2006). *Izbrannye trudy [Selected works]*. Novosibirsk: Izdatelstvo Instituta matematiki [in Russian].

4 Gårding, L. (1951). Linear hyperbolic partial differential equations with constant coefficients. *Acta Mathematica, 85*, 1–62.

5 Ladyzhenskaia, O.A. (1951). O razreshimosti osnovnykh kraevykh zadach dlia uravnenii giperbolicheskogo i parabolicheskogo tipov [On the solvability of the basic boundary value problems for the equation of hyperbolic and parabolic types]. *Doklady Akademii Nauk SSSR — Reports of the Academy of Sciences of the USSR, 97*(3), 395–398 [in Russian].

6 Bitsadze, A.V. (1981). *Nekotorye klassy uravnenii v chastnykh proizvodnykh [Some classes of partial differential equations]*. Moscow: Nauka [in Russian].

7 Leray, J. (1984). *Giperbolicheskie differentsialnye uravneniia [Hyperbolic differential equations]*. Moscow: Nauka [in Russian].

8 Nakhushev, A.M. (2006). *Zadachi so smeshcheniem dlia uravnenii v chastnykh proizvodnykh [Offset problems for partial differential equations]*. Moscow: Nauka [in Russian].

9 Kalmenov, T.Sh. (2013). *K teorii nachalno-kraevykh zadach dlia differentsialnykh uravnenii [On the theory of initial boundary value problems for differential equations]*. Almaty [in Russian].

10 Kiguradze, T.I. (1993). O periodicheskikh kraevykh zadachakh dlia lineinykh giperbolicheskikh uravnenii [On periodic boundary value problems for linear hyperbolic equations]. *Differentsialnye uravneniia — Differential equations, 29*(2), 281–297 [in Russian].

11 Assanova, A.T., & Dzhumabaev, D.S. (2003). O korrektnoi razreshimosti nelokalnoi kraevoi zadachi dlia sistem giperbolicheskikh uravnenii [On the correct solvability of a nonlocal boundary value problem for systems of hyperbolic equations]. *Doklady RAN — RAS reports, 391*(3), 295–297 [in Russian].

12 Filinovskii, A.V. (2014). Hyperbolic Equations with Growing Coefficients in Unbounded Domains. *J. Math. Sci., 197*, 435–446.

13 Tikhonov, A.N., & Samarskii, A.A. (1990). *Equations of mathematical physics.* New York: Dover Publications, Inc.

14 Muratbekov, M.B., & Bayandiyev, Ye.N. (2020). Existence and maximal regularity of solutions in $L_2(R^2)$ for a hyperbolic type differential equation with quickly growing coefficients. *Eurasian Mathematical Journal, 11*(1), 95–100.

15 Kato, T. (1972). *Teoriia vozmushchenii lineinykh operatorov [Perturbation theory for linear operators].* Moscow: Mir [in Russian].

16 Reed, M., & Simon, B. (1978). *Metody sovremennoi matematicheskoi fiziki. Tom 2. Garmonicheskii analiz. Samosopriazhennost. [Methods of Modern Mathematical Physics II: Fourier Analysis. Self-Adjointness].* Moscow: Mir [in Russian].

17 Titchmarsh, E.Ch. (1961). *Razlozheniia po sobstvennym funktsiiam, sviazannye s differentsialnymi uravneniiami vtorogo poriadka [Eigenfunction expansions associated with second-order differential equations].* Moscow: Inostrannaia literatura [in Russian].

18 Mazia, V.G. (1973). O (p,l)-emkosti, teoremakh vlozheniia i spektre samosopriazhennogo ellipticheskogo operatora [On $O(p,l)$-capacity, inbedding theorems, and the spectrum of a selfadjoint elliptic operator]. *Izvestiia AN SSSR. Seriia matematika — News of Academy of Sciences of the USSR. Mathematics series, 37*, 356–385 [in Russian].

19 Otelbaev, M. (1979). Teoremy vlozheniia prostranstv s vesom i ikh primeneniia k izucheniiu spektra operatora Shredingera [Imdedding theorems for spaces with a weight and their application to the study of the spectrum of the Schrodinger operator]. *Part 7, Work collection, Trudy Matematicheskogo Instituta imeni V.A. Steklova — Proceedings of the Steklov Institute of Mathematics, 150*, 265–305 [in Russian].

20 Lorenzi, L., & Rhandi, A. (2012). On Schrödinger type operators with unbounded coefficients: generation and heat kernel estimates. *Journal of Evolution Equations, 15*, 53–88.

21 Metafune, G., Pallara, D., Rabier, P., & Schnaubelt, R. (2010). Uniqueness for elliptic operators on with unbounded coefficients. *Forum Math., 22*(3), 583–601.

22 Muratbekov, M., & Otelbaev, M. (2016). On the existence of a resolvent and separability for a class of singular hyperbolic type differential operators on an unbounded domain. *Eurasian Math. J., 7*, 1, 50–67.

23 Muratbekov, M.B., & Muratbekov, M.M. (2021). On the compactness of the resolvent of a Schrodinger type singular operator with a negative parameter. *Chaos, Solitons and Fractals, 151*, 111–248.

24 Muratbekov, M.B., & Igissinov, S. (2022). Estimates of Eigenvalues of a Semiperiodic Dirichlet Problem for a Class of Degenerate Elliptic Equations. *Symmetry, 14*, 692.

25 Muratbekov, M.B., & Muratbekov, M.M. (2019). On the bounded invertibility of a Schrodinger operator with a negative parameter in the space $L_2(R^n)$. *Bulletin of the Karaganda University-Mathematics, 93*, 1, 36–47.

26 Muratbekov, M.B., & Suleimbekova, A.O. (2021). On the existence of the resolvent and separability of a class of the Korteweg-de Vriese type linear singular operators. *Bulletin of the Karaganda University-Mathematics, 101*, 1, 87–97.