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Integro-differential equations with bounded operators in Banach spaces

The paper investigates integro-differential equations in Banach spaces with operators, which are a composition of convolution and differentiation operators. Depending on the order of action of these two operators, we talk about integro-differential operators of the Riemann—Liouville type, when the convolution operator acts first, and integro-differential operators of the Gerasimov type otherwise. Special cases of the operators under consideration are the fractional derivatives of Riemann—Liouville and Gerasimov, respectively. The classes of integro-differential operators under study also include those in which the convolution has an integral kernel without singularities. The conditions of the unique solvability of the Cauchy type problem for a linear integro-differential equation of the Riemann—Liouville type and the Cauchy problem for a linear integro-differential equation of the Gerasimov type with a bounded operator at the unknown function are obtained. These results are used in the study of similar equations with a degenerate operator at an integro-differential operator under the condition of relative boundedness of the pair of operators from the equation. Abstract results are applied to the study of initial boundary value problems for partial differential equations with an integro-differential operator, the convolution in which is given by the Mittag-Leffler function multiplied by a power function.

Keywords: integro-differential equation, integro-differential operator, convolution, Cauchy problem, Cauchy type problem, degenerate evolution integro-differential equation, initial boundary value problem.

Introduction

In recent decades, the importance of fractional integro-differential calculus has grown markedly in solving both theoretical and applied problems in many areas of mathematical modeling: In continuum mechanics, in mathematical biology, in finance theory, etc. [1–4]. At the same time, over the past few years, works have appeared containing the construction of new fractional derivatives, which in most cases are compositions of a convolution operator and the operator of an integer order differentiation, but unlike classical fractional derivatives, the kernel in the convolution operator has no singularities [5, 6].

This paper considerers abstract integro-differential operators of the form of composition of a convolution and an integer order differentiation and equations in Banach spaces with them. Using the methods of the Laplace transform theory, we investigate the initial problems for such equations are formulated and the issues of the unique solvability of such problems are investigated. If $m-1<\alpha\leq m\in\mathbb{N}$, the kernel in the convolution is a power function $s^{m-\alpha}/\Gamma(\alpha)$ at the differentiation operator of the order m, the integro-differential operator is the Riemann—Liouville or Gerasimov fractional derivative, depending on the order of action of the convolution and the integer order differentiation. In other cases, we obtain other integro-differential operators of Riemann—Liouville or Gerasimov type. Note also that the kernel in the convolution is supposed to be operator-valued. This makes it possible to study some systems of equations within the framework of the studied equations in Banach spaces, for example, with fractional derivatives of various orders.

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The first section contains the Cauchy type problem for the linear equation in a Banach space with an integro-differential operator of Riemann—Liouville type, when the convolution operator acts on the function first, and with a bounded operator at the unknown function. A unique solvability theorem was proved for the problem, the solution is presented in the form of a sum of the Dunford–Taylor integrals. In the second section, the Cauchy problem is studied for the equation with an integro-differential operator of Gerasimov type, when the convolution operator acts after the differentiation operator. We show that there exists a unique solution to such problem, and present the solution in the similar form as in the previous section. In the third and fourth sections, initial problems for analogous linear equations with a degenerate operator at an integro-differential operator are studied under the condition of relative boudedness of the pair of operators from the equation. The last section contains an application of abstract results to initial boundary value problems with an integro-differential operator of Atangana–Baleanu type [6] with singular kernel (with the Mittag-Leffler function multiplied by a negative power as the kernel of the convolution) with respect to time and with some differential operators in spatial variables.

Note that, by similar methods, various fractional differential equations in Banach spaces, including degenerate ones, were researched in the works [7–10], see the references therein also. In this sense, it is necessary to mention the monograph by J. Prüss [11] on evolution integral equations in Banach spaces.

1 Integro-differential equation of Riemann—Liouville type

Let $\mathcal{X}$ be a Banach space, $\mathcal{L}(\mathcal{X})$ be the Banach space of all linear bounded operators on $\mathcal{X}$, $A \in \mathcal{L}(\mathcal{X})$, $\mathbb{R}_+ = \{a \in \mathbb{R} : a > 0\}$, $\overline{\mathbb{R}}_+ := \{0\} \cup \mathbb{R}_+$, $K \in C(\mathbb{R}_+; \mathcal{L}(\mathcal{X}))$. Define the convolution

$$(J^K x)(t) := \int_0^t K(t-s)x(s)ds$$

and integro-differential operator of the Riemann—Liouville type

$$(D^{m,K}x)(t) := D^m (J^K x)(t) := D^m \int_0^t K(t-s)x(s)ds,$$

where D^m is a usual derivative of the order m. Consider the Cauchy type problem

$$(J^K x)^{(k)}(0) = x_k \in \mathcal{X}, \quad k = 0, 1, \dots, m-1, \tag{1}$$

for the equation

$$(D^{m,K}x)(t) = Ax(t), \ t > 0. \tag{2}$$

A solution of problem (1), (2) is called a function $x : \mathbb{R}_+ \to \mathcal{X}$, such that $J^K x \in C^{m-1}(\overline{\mathbb{R}}_+; \mathcal{X}) \cap C^m(\mathbb{R}_+; \mathcal{X})$, conditions (1) and equality (2) for $t \in \mathbb{R}_+$ are satisfied.

For a function $h : \mathbb{R}_+ \to \mathcal{X}$ we denote its Laplace transform by $\widehat{h}$, or $\mathfrak{L}[h]$, if the expression for h is too long.

Suppose that $\widehat{K}$ is a single-valued analytic operator-function in the region

$$\Omega_{R_0} := \{\mu \in \mathbb{C} : |\mu| > R_0, \, |\arg \mu| < \pi\}$$

for some $R_0 > 0$ and define the operators

$$X_k(t) = \frac{1}{2\pi i} \int\limits_{\gamma} (\lambda^m \widehat{K}(\lambda) - A)^{-1} \lambda^{m-1-k} e^{\lambda t} d\lambda, \quad t > 0, \quad k = 0, 1, \dots, m-1,$$

where $\gamma := \gamma_R \cup \gamma_{R,+} \cup \gamma_{R,-}$ is a positively oriented contour, $\gamma_R := \{Re^{i\varphi} : \varphi \in (-\pi, \pi)\}$, $\gamma_{R,+} := \{re^{i\pi} : r \in [R, \infty)\}$, $\gamma_{R,-} := \{re^{-i\pi} : r \in [R, \infty)\}$, $R > R_0$.

Theorem 1. Let $A \in \mathcal{L}(\mathcal{X})$, $K \in C(\mathbb{R}_+; \mathcal{L}(\mathcal{X}))$, there exist $\widehat{K}$, which be single-valued analytic operator-function in Ω_{R_0} for some $R_0 > 0$, and

$$\exists \chi > 0 \quad \exists c > 0 \quad \forall \lambda \in \Omega_{R_0} \quad \|\widehat{K}(\lambda)^{-1}\|^{-1}_{\mathcal{L}(\mathcal{X})} > c|\lambda|^{\chi-1}. \tag{3}$$

Suppose that for all $\lambda \in \Omega_{R_0}$ there exists $\widehat{K}(\lambda)^{-1} \in \mathcal{L}(\mathcal{X})$. Then for all $x_0, x_1, \dots, x_{m-1} \in \mathcal{X}$ there exists an unique solution to problem (1), (2). It has the form

$$x(t) = \sum_{k=0}^{m-1} X_k(t) x_k.$$

Proof. Due to condition (3) there exists $\delta \geq R_0 > 0$ such that for all $\lambda \in \Omega_\delta$ $\|\lambda^{-m}\widehat{K}(\lambda)^{-1}\|_{\mathcal{L}(\mathcal{X})} < c^{-1}|\lambda|^{1-\chi-m} < (2\|A\|_{\mathcal{L}(\mathcal{X})})^{-1}$. Hence, there exists the inverse operator $(\lambda^m \widehat{K}(\lambda) - A)^{-1}$ and

$$\|(\lambda^m \widehat{K}(\lambda) - A)^{-1}\|_{\mathcal{L}(\mathcal{X})} = \sum_{n=0}^{\infty} |\lambda|^{-m(n+1)} \|\widehat{K}(\lambda)^{-1}\|^{n+1}_{\mathcal{L}(\mathcal{X})} \|A\|^n_{\mathcal{L}(\mathcal{X})} <$$

$$< \sum_{n=0}^{\infty} c^{-n-1} |\lambda|^{(1-\chi-m)(n+1)} \|A\|^n_{\mathcal{L}(\mathcal{X})} < \frac{2}{c|\lambda|^{\chi-1+m}}.$$

Here we obtain the inequality $\|(I - \lambda^{-m} A\widehat{K}(\lambda)^{-1})^{-1}\|_{\mathcal{L}(\mathcal{X})} < 2$ also. Besides,

$$\|(\lambda^m \widehat{K}(\lambda) - A)^{-1}\lambda^{m-1-k}\|_{\mathcal{L}(\mathcal{X})} = \|(I - \lambda^{-m}\widehat{K}(\lambda)^{-1}A)^{-1}\widehat{K}(\lambda)^{-1}\lambda^{-1-k}\|_{\mathcal{L}(\mathcal{X})} < 2c^{-1}|\lambda|^{-k-\chi}$$

and there exists the Laplace transform $\widehat{X}_k$ for $k = 1, 2, \dots, m-1$ and for $k = 0$, if $\chi > 1$. For $k = 0$, $\chi \in (0, 1)$ we have by the definition

$$\|X_0(t)\| \leq \frac{2R^{1-\chi}e^{Rt}}{c} + \frac{2}{\pi c}\int_R^\infty r^{-\chi}e^{-rt}dr = \frac{2R^{1-\chi}e^{Rt}}{c} + \frac{2\Gamma(1-\chi)t^{\chi-1}}{\pi c} \leq Ct^{\chi-1}e^{Rt};$$

for $k = 0$, $\chi = 1$, choosing $R > 1$, obtain

$$\|X_0(t)\| \leq \frac{2e^{Rt}}{c} + \frac{2}{\pi c}\int_R^\infty r^{-1/2}e^{-rt}dr = \frac{2e^{Rt}}{c} + \frac{2\Gamma(1/2)t^{-1/2}}{\pi c} \leq Ct^{-1/2}e^{Rt}, \ t > 0.$$

There exists the Laplace transform $\widehat{X}_0$.

Take $R > \delta$ in the definition of γ. We have for $l \in \{0, 1, \dots, m-1\}$

$$\widehat{J^K X_l}(\lambda) = \widehat{K}(\lambda)\widehat{X}_l(\lambda) = \widehat{K}(\lambda)(\lambda^m \widehat{K}(\lambda) - A)^{-1}\lambda^{m-1-l} = \lambda^{-1-l}(I - \lambda^{-m}A\widehat{K}(\lambda)^{-1})^{-1},$$

consequently,

$$J^K X_l(t) = \frac{1}{2\pi i}\int_\gamma \lambda^{-1-l}(I - \lambda^{-m}A\widehat{K}(\lambda)^{-1})^{-1}e^{\lambda t}d\lambda, \quad t > 0,$$

$$(J^K X_l)^{(k)}(t) = \frac{1}{2\pi i}\int\limits_{\gamma} \lambda^{k-1-l}(I - \lambda^{-m}A\widehat{K}(\lambda)^{-1})^{-1}e^{\lambda t}d\lambda, \quad t > 0,$$

for $k, l = 0, 1, \dots, m-1$. For every $k = 0, 1, \dots, l-1$

$$\|\lambda^{k-1-l}(I - \lambda^{-m}A\widehat{K}(\lambda)^{-1})^{-1}\|_{\mathcal{L}(\mathcal{Z})} < \frac{2}{|\lambda|^2},$$

hence, $(J^K X_l)^{(k)}(0) = 0$. For $k = l$

$$(J^K X_l)^{(l)}(t) = \frac{1}{2\pi i}\int\limits_{\gamma} \lambda^{-1}(I - \lambda^{-m}A\widehat{K}(\lambda)^{-1})^{-1}e^{\lambda t}d\lambda =$$

$$= \frac{1}{2\pi i}\int\limits_{\gamma} \lambda^{-1}\sum_{n=0}^{\infty}\lambda^{-mn}(A\widehat{K}(\lambda)^{-1})^n e^{\lambda t}d\lambda = I + \frac{1}{2\pi i}\int\limits_{\gamma} \lambda^{-1}\sum_{n=1}^{\infty}\lambda^{-mn}(A\widehat{K}(\lambda)^{-1})^n e^{\lambda t}d\lambda,$$

$$\left\|\lambda^{-1}\sum_{n=1}^{\infty}\lambda^{-mn}(A\widehat{K}(\lambda)^{-1})^n\right\|_{\mathcal{L}(\mathcal{X})} \le |\lambda|^{-1}\sum_{n=1}^{\infty}c^{-n}|\lambda|^{(1-\chi-m)n}\|A\|^n_{\mathcal{L}(\mathcal{X})} =$$

$$= \frac{c^{-1}|\lambda|^{1-\chi-m}\|A\|_{\mathcal{L}(\mathcal{X})}}{|\lambda|(1 - c^{-1}|\lambda|^{1-\chi-m}\|A\|_{\mathcal{L}(\mathcal{X})})} < \frac{2\|A\|_{\mathcal{L}(\mathcal{X})}}{c|\lambda|^{m+\chi}}.$$

Therefore, $(J^K X_l)^{(l)}(0) = I$.

Now let $k = l+1, l+2, \dots, m-1$, then

$$(J^K X_l)^{(k)}(t) = \frac{1}{2\pi i}\int\limits_{\gamma} \lambda^{k-1-l-m}(I - \lambda^{-m}A\widehat{K}(\lambda)^{-1})^{-1}A\widehat{K}(\lambda)^{-1}e^{\lambda t}d\lambda, \quad t > 0,$$

$$\|\lambda^{k-1-l-m}(I - \lambda^{-m}A\widehat{K}(\lambda)^{-1})^{-1}A\widehat{K}(\lambda)^{-1}\|_{\mathcal{L}(\mathcal{X})} \le \frac{2\|A\|_{\mathcal{L}(\mathcal{X})}}{c|\lambda|^{\chi+1}}$$

due to (3). Hence, $(J^K X_l)^{(k)}(0) = 0$ and all conditions (1) are satisfied.

We have

$$D^m(J^K X_l)(t) = \frac{1}{2\pi i}\int\limits_{\gamma} \lambda^{m-1-l}(I - \lambda^{-m}A\widehat{K}(\lambda)^{-1})^{-1}e^{\lambda t}d\lambda =$$

$$= \frac{1}{2\pi i}\int\limits_{\gamma} \lambda^{2m-1-l}\widehat{K}(\lambda)(\lambda^m\widehat{K}(\lambda) - A)^{-1}e^{\lambda t}d\lambda = AX_l(t), \quad t > 0,$$

hence, equality (2) holds.

If there exist two solutions y_1 and y_2 to problem (1), (2), then $y := y_1 - y_2$ is a solution to the same problem with $x_0 = x_1 = \dots = x_{m-1} = 0$. Define y on $(T, +\infty)$ at some $T > 0$ by zero. Then there exists $\widehat{y}$, and due to (1), (2) $(\lambda^m\widehat{K}(\lambda) - A)\widehat{y}(\lambda) = 0$ for $\mathrm{Re}\lambda > 0$. Under the conditions of this theorem $\widehat{y}(\lambda) \equiv 0$, therefore, $y(t) = 0$ for $t \in (0, T)$. Since we can choose an arbitrary $T > 0$, then $y(t) = 0$ and $y_1(t) = y_2(t)$ for all $t > 0$.

Consider the inhomogeneous equation

$$(D^{m,K}x)(t) = Ax(t) + f(t), \; t \in (0, T], \tag{4}$$

with $f:(0,T]\to\mathcal{X}$.

Lemma 1. Let $A\in\mathcal{L}(\mathcal{X})$, $K\in C(\mathbb{R}_+;\mathcal{L}(\mathcal{X}))$, there exist $\widehat{K}$, which be single-valued analytic operator-function in Ω_{R_0} for some $R_0>0$, and condition (3) hold. Suppose that for all $\lambda\in\Omega_{R_0}$ there exists $\widehat{K}(\lambda)^{-1}\in\mathcal{L}(\mathcal{X})$, $f\in C((0,T];\mathcal{X})\cap L_1(0,T;\mathcal{X})$. Then there exists an unique solution to problem (1), (4) with $x_0=x_1=\ldots=x_{m-1}=0$. It has the form

$$x_f(t)=\int\limits_0^t X_{m-1}(t-s)f(s)ds.$$

Proof. We have $\widehat{x}_f(\lambda)=\widehat{X}_{m-1}(\lambda)\widehat{f}(\lambda)=(\lambda^m\widehat{K}(\lambda)-A)^{-1}\widehat{f}(\lambda)$, therefore,

$$\widehat{J^Kx_f}(\lambda)=\widehat{K}(\lambda)(\lambda^m\widehat{K}(\lambda)-A)^{-1}\widehat{f}(\lambda),\quad J^Kx_f(t)=\int\limits_0^t X(t-s)f(s)ds,$$

where

$$X(t)=\frac{1}{2\pi i}\int\limits_\gamma \widehat{K}(\lambda)(\lambda^m\widehat{K}(\lambda)-A)^{-1}e^{\lambda t}d\lambda=\frac{1}{2\pi i}\int\limits_\gamma \lambda^{-m}(I-\lambda^{-m}A\widehat{K}(\lambda)^{-1})^{-1}e^{\lambda t}d\lambda.$$

Hence, $\|X^{(k)}(t)\|_{\mathcal{L}(\mathcal{X})}\le Ct^{m-k-1}$ for all $t\in(0,T]$, $k=0,1,\ldots,m-1$; $X^{(k)}(0)=0$, $k=0,1,\ldots,m-2$, and

$$(J^Kx_f)^{(k)}(t)=\int\limits_0^t X^{(k)}(t-s)f(s)ds,\quad k=0,1,\ldots,m-1,$$

$$\|(J^Kx_f)^{(k)}(t)\|_{\mathcal{L}(\mathcal{X})}\le C_1\int\limits_0^t\|f(s)\|_{\mathcal{L}(\mathcal{X})}ds,\quad (J^Kx_f)^{(k)}(0)=0,\quad k=0,1,\ldots,m-1.$$

Finally,

$$\mathfrak{L}[(J^Kx_f)^{(m)}]=\lambda^m\widehat{K}(\lambda)(\lambda^m\widehat{K}(\lambda)-A)^{-1}\widehat{f}(\lambda)=A(\lambda^m\widehat{K}(\lambda)-A)^{-1}\widehat{f}(\lambda)+\widehat{f}(\lambda),$$

therefore, equality (4) is fulfilled. Hence, x_f is a solution to problem (1), (4). The uniqueness of a solution can be proved in the same way, as for the homogeneous equation.

The assertions follow immediately from Theorem 1 and Lemma 1 due to the linearity of equation (4).

Theorem 2. Let $A\in\mathcal{L}(\mathcal{X})$, $K\in C(\mathbb{R}_+;\mathcal{L}(\mathcal{X}))$, there exist $\widehat{K}$, which be single-valued analytic operator-function in Ω_{R_0} for some $R_0>0$, and condition (3) hold. Suppose that for all $\lambda\in\Omega_{R_0}$ there exists $\widehat{K}(\lambda)^{-1}\in\mathcal{L}(\mathcal{X})$, $f\in C((0,T];\mathcal{X})\cap L_1(0,T;\mathcal{X})$. Then for all $x_0,x_1,\ldots,x_{m-1}\in\mathcal{X}$ there exists an unique solution of problem (1), (4). It has the form

$$x(t)=\sum_{k=0}^{m-1}X_k(t)x_k+\int\limits_0^t X_{m-1}(t-s)f(s)ds.$$

2 Integro-differential equation of Gerasimov type

Consider the integro-differential operator of Gerasimov type

$$(D^{K,m}x)(t):=J^K(D^mx)(t):=\int_0^t K(t-s)x^{(m)}(s)ds.$$

Consider the Cauchy problem

$$x^{(k)}(0) = x_k \in \mathcal{X}, \ \ k = 0, 1, \dots, m-1, \tag{5}$$

for the equation

$$(D^{K,m}x)(t) = Ax(t), \ t \geq 0. \tag{6}$$

A solution to problem (5), (6) is called a function $x \in C^{m-1}(\overline{\mathbb{R}}_+; \mathcal{X}) \cap C^m(\mathbb{R}_+; \mathcal{X})$, such that $J^K x^{(m)} \in C(\overline{\mathbb{R}}_+; \mathcal{X})$, conditions (5) and equality (6) for $t \in \overline{\mathbb{R}}_+$ are satisfied.

Theorem 3. Let $A \in \mathcal{L}(\mathcal{X})$, $K \in C(\mathbb{R}_+; \mathcal{L}(\mathcal{X}))$, there exist $\widehat{K}$, which be single-valued analytic operator-function in Ω_{R_0} for some $R_0 > 0$, and condition (3) hold. Suppose that for all $\lambda \in \Omega_{R_0}$ there exists $\widehat{K}(\lambda)^{-1} \in \mathcal{L}(\mathcal{X})$. Then for all $x_0, x_1, \dots, x_{m-1} \in \mathcal{X}$ there exists an unique solution to problem (5), (6). It has the form

$$x(t) = \sum_{k=0}^{m-1} Y_k(t)x_k,$$

where

$$Y_k(t) = \frac{1}{2\pi i}\int_\gamma (\lambda^m \widehat{K}(\lambda) - A)^{-1}\widehat{K}(\lambda)\lambda^{m-1-k}e^{\lambda t}d\lambda, \quad k = 0, 1, \dots, m-1.$$

The contour γ is defined as in the previous section.

Proof. We have

$$\|(\lambda^m \widehat{K}(\lambda) - A)^{-1}\widehat{K}(\lambda)\lambda^{m-1-k}\|_{\mathcal{L}(\mathcal{X})} = \|(I - \lambda^{-m}\widehat{K}(\lambda)^{-1}A)^{-1}\lambda^{-1-k}\|_{\mathcal{L}(\mathcal{X})} < 2|\lambda|^{-k-1}.$$

So, there exists the Laplace transform $\widehat{Y}_k$ for $k = 1, 2, \dots, m-1$. For $k = 0$

$$Y_0(t) = I + \frac{1}{2\pi i}\int_\gamma \lambda^{-1}\sum_{n=1}^{\infty}\lambda^{-mn}(\widehat{K}(\lambda)^{-1}A)^n e^{\lambda t}d\lambda,$$

$$\left\|\lambda^{-1}\sum_{n=1}^{\infty}\lambda^{-m}(\widehat{K}(\lambda)^{-1}A)^n\right\|_{\mathcal{L}(\mathcal{X})} \leq |\lambda|^{-1}\sum_{n=1}^{\infty}c^{-n}|\lambda|^{(1-\chi-m)n}\|A\|^n_{\mathcal{L}(\mathcal{X})} =$$

$$= \frac{c^{-1}|\lambda|^{1-\chi-m}\|A\|_{\mathcal{L}(\mathcal{X})}}{|\lambda|(1 - c^{-1}|\lambda|^{1-\chi-m}\|A\|_{\mathcal{L}(\mathcal{X})})} < \frac{2\|A\|_{\mathcal{L}(\mathcal{X})}}{c|\lambda|^{\chi+m}}.$$

Thus, there exists the Laplace transform $\widehat{X}_0$.

For large enough $R > 0$ in the definition of γ, $k, l \in \{0, 1, \dots, m-1\}$

$$Y_l^{(k)}(t) = \frac{1}{2\pi i}\int_\gamma \lambda^{k-1-l}(I - \lambda^{-m}\widehat{K}(\lambda)^{-1}A)^{-1}e^{\lambda t}d\lambda, \quad t > 0.$$

By repeating the reasoning from Theorem 1, we get the fulfillment of conditions (5) with arbitrary $x_l \in \mathcal{X}$, $x_k = 0$ for every $k \in \{0, 1, \dots, m-1\} \setminus \{l\}$.

Further, we have

$$\widehat{J^K Y_l^{(m)}}(\lambda) = \widehat{K}(\lambda)\widehat{Y_l^{(m)}}(\lambda) = \widehat{K}(\lambda)(\lambda^m \widehat{Y}_l(\lambda) - \lambda^{m-l-1}I),$$

$$J^K Y_l^{(m)}(t) = \frac{1}{2\pi i}\int_\gamma \lambda^{m-1-l}\widehat{K}(\lambda)[(I - \lambda^{-m}\widehat{K}(\lambda)^{-1}A)^{-1} - I]e^{\lambda t}d\lambda =$$

$$= A\frac{1}{2\pi i}\int\limits_{\gamma}\lambda^{-1-l}(I-\lambda^{-m}\widehat{K}(\lambda)^{-1}A)^{-1}e^{\lambda t}d\lambda = AY_l(t), \quad t>0.$$

The uniqueness of a solution can be proved in the same way as in Theorem 1.

Consider the inhomogeneous equation with $f:[0,T]\to\mathcal{X}$

$$(D^{K,m}x)(t) = Ax(t)+f(t),\ t\in[0,T]. \tag{7}$$

Lemma 2. Let $A\in\mathcal{L}(\mathcal{X})$, $K\in C(\mathbb{R}_+;\mathcal{L}(\mathcal{X}))$, there exist $\widehat{K}$, which be single-valued analytic operator-function in Ω_{R_0} for some $R_0>0$, and condition (3) hold. Suppose that for all $\lambda\in\Omega_{R_0}$ there exists $\widehat{K}(\lambda)^{-1}\in\mathcal{L}(\mathcal{X})$, $f\in C([0,T];\mathcal{X})$. Then there exists an unique solution to problem (5), (7) with $x_0=x_1=\ldots=x_{m-1}=0$. It has the form

$$x_f(t) = \int\limits_0^t X_{m-1}(t-s)f(s)ds.$$

Proof. For $k=0,1,\ldots,m-2$ we have $X_{m-1}^{(k)}(0)=0$, hence,

$$x_f^{(k)}(t) = \int\limits_0^t X_{m-1}^{(k)}(t-s)f(s)ds, \quad k=0,1,\ldots,m-1.$$

Since

$$\|\lambda^k(\lambda^m\widehat{K}(\lambda)-A)^{-1}\|_{\mathcal{L}(\mathcal{X})} = \|\lambda^{k-m}\widehat{K}(\lambda)^{-1}(I-\lambda^{-m}A\widehat{K}(\lambda)^{-1})^{-1}\|_{\mathcal{L}(\mathcal{X})} < \frac{2}{c|\lambda|^{m-k-1+\chi}},$$

we have $\|X_{m-1}^{(k)}(t)\|_{\mathcal{L}(\mathcal{X})}\le Ct^{m-k-2+\chi}$, consequently,

$$\|x_f^{(k)}(t)\|_{\mathcal{L}(\mathcal{X})}\le C_1\|f\|_{C([0,T];\mathcal{X})}t^{m-k-1+\chi}, \quad x_f^{(k)}(0)=0, \quad k=0,1,\ldots,m-1.$$

Further,

$$\mathfrak{L}[J^K x_f^{(m)}] = \lambda^m\widehat{K}(\lambda)(\lambda^m\widehat{K}(\lambda)-A)^{-1}\widehat{f}(\lambda) = A(\lambda^m\widehat{K}(\lambda)-A)^{-1}\widehat{f}(\lambda)+\widehat{f}(\lambda),$$

hence, x_f is a solution to problem (5), (7). The proof of the uniqueness of a solution is the same as in Theorem 1.

Theorem 4. Let $A\in\mathcal{L}(\mathcal{X})$, $K\in C(\mathbb{R}_+;\mathcal{L}(\mathcal{X}))$, there exist $\widehat{K}$, which be single-valued analytic operator-function in Ω_{R_0} for some $R_0>0$, and condition (3) hold. Suppose that for all $\lambda\in\Omega_{R_0}$ there exists $\widehat{K}(\lambda)^{-1}\in\mathcal{L}(\mathcal{X})$, $f\in C([0,T];\mathcal{X})$. Then for all $x_0,x_1,\ldots,x_{m-1}\in\mathcal{X}$ there exists an unique solution to problem (5), (7). It has the form

$$x(t) = \sum_{k=0}^{m-1}Y_k(t)x_k + \int\limits_0^t X_{m-1}(t-s)f(s)ds.$$

Example 1. Take $m-1<\alpha\le m\in\mathbb{N}$, $K_\alpha(s) := \frac{s^{\alpha-1}}{\Gamma(\alpha)}I$, then $J^{K_\alpha} := J^\alpha$ is the operator of the fractional Riemann—Liouville integration of the order α, $D^{m,K_{m-\alpha}} := {}^{RL}D^\alpha$ is the operator of the fractional Riemann—Liouville differentiation of the order α, $D^{K_{m-\alpha},m} := {}^{GC}D^\alpha$ is the operator of the fractional Gerasimov—Caputo differentiation of the order α.

Example 2. Take $\mathcal{X} = \mathbb{R}^2$, $a_{ij}, b_{ij} \in \mathbb{R}$, $m_{ij} - 1 < \alpha_{ij} < m_{ij} \in \mathbb{N}$, $i, j = 1, 2$, $m := \max\limits_{i,j=1,2} m_{ij}$,

$$A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad K(s) := \begin{pmatrix} b_{11}\frac{s^{m-\alpha_{11}-1}}{\Gamma(m-\alpha_{11})} & b_{12}\frac{s^{m-\alpha_{12}-1}}{\Gamma(m-\alpha_{12})} \\ b_{21}\frac{s^{m-\alpha_{21}-1}}{\Gamma(m-\alpha_{21})} & b_{22}\frac{s^{m-\alpha_{22}-1}}{\Gamma(m-\alpha_{22})} \end{pmatrix}$$

then

$$D^{m,K} = \begin{pmatrix} b_{11}{}^{RL}D^{\alpha_{11}} & b_{12}{}^{RL}D^{\alpha_{12}} \\ b_{21}{}^{RL}D^{\alpha_{21}} & b_{22}{}^{RL}D^{\alpha_{22}} \end{pmatrix},$$

for Gerasimov—Caputo derivatives similar construction case is possible in the general, if $m_{11} = m_{12} = m_{21} = m_{22}$. So, equation (2) has the form of the system of equations

$$b_{11}{}^{RL}D^{\alpha_{11}}x_1(t) + b_{12}{}^{RL}D^{\alpha_{12}}x_2(t) = a_{11}x_1(t) + a_{12}x_2(t),$$
$$b_{21}{}^{RL}D^{\alpha_{21}}x_1(t) + b_{22}{}^{RL}D^{\alpha_{22}}x_2(t) = a_{21}x_1(t) + a_{22}x_2(t).$$

Note that

$$\widehat{K}(\lambda) := \begin{pmatrix} b_{11}\lambda^{\alpha_{11}-m} & b_{12}\lambda^{\alpha_{12}-m} \\ b_{21}\lambda^{\alpha_{21}-m} & b_{22}\lambda^{\alpha_{22}-m} \end{pmatrix},$$

therefore, condition (3) is fulfilled with some $\chi \in (0, \alpha + 1 - m)$, and the condition of reversibility of $\widehat{K}(\lambda)$ for large enough $|\lambda|$ is not too restrictive. Indeed, $\widehat{K}(\lambda)$ is invertible, only if the matrix, consisting of b_{ij}, does not contain zero rows and zero columns, and $\alpha_{11}\alpha_{22} \neq \alpha_{12}\alpha_{21}$, or $b_{11}b_{22} \neq b_{12}b_{21}$ in the case $\alpha_{11}\alpha_{22} = \alpha_{12}\alpha_{21}$.

3 Degenerate equation of Riemann—Liouville type

Assume that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, $L \in \mathcal{L}(\mathcal{X};\mathcal{Y})$, i.e., it is a linear bounded operator from $\mathcal{X}$ to $\mathcal{Y}$, $M \in \mathcal{C}l(\mathcal{X};\mathcal{Y})$, i.e., it is a linear closed operator with a dense domain D_M in $\mathcal{X}$, acting to $\mathcal{Y}$. Introduce the denotations $\rho^L(M) := \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathcal{Y};\mathcal{X})\}$, $R^L_\mu(M) := (\mu L - M)^{-1}L$, $L^L_\mu := L(\mu L - M)^{-1}$. We will suppose that $\ker L \neq \{0\}$, in other words, the operator L is degenerate.

An operator M is called (L, σ)-*bounded*, if

$$\exists a > 0 \quad \forall \mu \in \mathbb{C} \quad (|\mu| > a) \Rightarrow (\mu \in \rho^L(M)).$$

In [12; 89, 90], it was shown that if an operator M is (L, σ)-bounded, $\gamma_r := \{\mu \in \mathbb{C} : |\mu| = r > a\}$, then the operators

$$P = \frac{1}{2\pi i}\int\limits_{\gamma_r} R^L_\mu(M)\,d\mu \in \mathcal{L}(\mathcal{X}), \quad Q = \frac{1}{2\pi i}\int\limits_{\gamma_r} L^L_\mu(M)\,d\mu \in \mathcal{L}(\mathcal{Y})$$

are projections. Put $\mathcal{X}^0 := \ker P$, $\mathcal{X}^1 := \mathrm{im}P$, $\mathcal{Y}^0 := \ker Q$, $\mathcal{Y}^1 := \mathrm{im}Q$. Denote by L_k (M_k) the restriction of the operator L (M) on $\mathcal{X}^k$ ($D_{M_k} = D_M \cap \mathcal{X}^k$), $k = 0, 1$.

Theorem 5 [12; 91]. Let an operator M be (L, σ)-bounded. Then

(i) $M_1 \in \mathcal{L}(\mathcal{X}^1;\mathcal{Y}^1)$, $M_0 \in \mathcal{C}l(\mathcal{X}^0;\mathcal{Y}^0)$, $L_k \in \mathcal{L}(\mathcal{X}^k;\mathcal{Y}^k)$, $k = 0, 1$;

(ii) there exist operators $M_0^{-1} \in \mathcal{L}(\mathcal{Y}^0;\mathcal{X}^0)$, $L_1^{-1} \in \mathcal{L}(\mathcal{Y}^1;\mathcal{X}^1)$.

Denote $G := M_0^{-1}L_0$. For $p \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ operator M is called (L, p)-bounded, if it is (L, σ)-bounded, $G^p \neq 0$, $G^{p+1} = 0$.

Consider the initial problem

$$D^{k,K}(Px)(0) = x_k, \quad k = 0, 1, \ldots, m - 1, \tag{8}$$

for a linear inhomogeneous integro-differential equation of Riemann—Liouville type

$$LD^{m,K}x(t) = Mx(t) + g(t), \quad t \in (0,T], \tag{9}$$

in which $g \in C((0,T];\mathcal{Y})$. This equation is called degenerate, since it contains degenerate operator L at the integro-differential operator.

A solution to problem (8), (9) is called a function $x : (0,T] \to D_M$, for which $Mx \in C((0,T];\mathcal{Y})$, $J^K Px \in C^{m-1}([0,T];\mathcal{Y})$, $J^K x \in C^m((0,T];\mathcal{Y})$, equality (9) is valid for all $t \in (0,T]$ and conditions (8) are true.

Lemma 3. Let $K \in C(\mathbb{R}_+;\mathcal{L}(\mathcal{X}))$, $H \in \mathcal{L}(\mathcal{X})$ be a nilpotent operator with a power $p \in \mathbb{N}_0$, a function $h : (0,T] \to \mathcal{X}$ be such that for $l = 0,1,\dots,p$ $(D^{m,K}H)^l h \in C((0,T];\mathcal{X})$, $D^{m,K}(D^{m,K}H)^l h \in C((0,T];\mathcal{X})$. Then there exists a unique solution to the equation

$$D^{m,K}Hx(t) = x(t) + h(t). \tag{10}$$

It has the form

$$x(t) = -\sum_{l=0}^{p}(D^{m,K}H)^l h(t). \tag{11}$$

Proof. Let $z = z(t)$ be a solution of (10). Act by the operator H on the both parts of (10) and obtain the equality $HD^{m,K}Hz(t) = Hz(t) + Hh(t)$. Under the theorem conditions there exists a continuous derivative $D^{m,K}$ for the the right-hand side of this equality. Acting by $D^{m,K}$ on the both parts of this equality, we will get

$$(D^{m,K}H)^2 z = D^{m,K}Hz + D_t^{m,K}Hh = z + h + D^{m,K}Hh.$$

Continuing such arguing, we obtain that

$$z + \sum_{l=0}^{p}(D^{m,K}H)^l h = (D^{m,K}H)^{p+1}z = (D^{m,K})^{p+1}H^{p+1}z \equiv 0$$

due to the continuity and nilpotency of the operator H. The existence of a solution can be checked by the substitution of (11) into (10).

The difference of two solutions is a solution of equation (10) with $h \equiv 0$, then (11) implies that the difference is identically equal to zero.

Define

$$U_k(t) = \frac{1}{2\pi i}\int_{\gamma}(\lambda^m \widehat{K}(\lambda) - L_1^{-1}M_1)^{-1}\lambda^{m-1-k}e^{\lambda t}d\lambda, \quad t > 0, \quad k = 0,1,\dots,m-1.$$

Theorem 6. Let an operator M be (L,p)-bounded, $K \in C(\mathbb{R}_+;\mathcal{L}(\mathcal{X}))$, there exist $\widehat{K}$, which be single-valued analytic operator-function in Ω_{R_0} for some $R_0 > 0$, and condition (3) hold. Suppose that for all $\lambda \in \Omega_{R_0}$ there exists $\widehat{K}(\lambda)^{-1} \in \mathcal{L}(\mathcal{X})$, $g \in C((0,T];\mathcal{Y}) \cap L_1(0,T;\mathcal{Y})$, $(D^{m,K}G)^l M_0^{-1}(I-Q)g$, $D^{m,K}(D^{m,K}G)^l M_0^{-1}(I-Q)g \in C((0,T];\mathcal{X})$ for $l = 0,1,\dots,p$, $x_k \in \mathcal{X}^1$, $k = 0,1,\dots,m-1$. Then there exists a unique solution to problem (8), (9), it has the form

$$x(t) = \sum_{k=0}^{m-1}U_k(t)x_k + \int_0^t U_{m-1}(t-s)L_1^{-1}Qg(s)ds - \sum_{l=0}^{p}(D^{m,K}G)^l M_0^{-1}(I-Q)g(t).$$

Proof. Acting on the both sides of (9) by $L_1^{-1}Q \in \mathcal{L}(\mathcal{Y}^1;\mathcal{X}^1)$, obtain

$$D^{m,K}v(t) = L_1^{-1}M_1 v(t) + L_1^{-1}Qg(t), \tag{12}$$

where $v(t) = Px(t)$. Act by the operator $M_0^{-1}(I-Q) \in \mathcal{L}(\mathcal{Y}^0;\mathcal{X}^0)$ on (9) and get

$$D^{m,K}Gw(t) = w(t) + M_0^{-1}(I-Q)g(t), \tag{13}$$

$w(t) = (I-P)x(t)$. Here we use the evident equalities $LP = QL$, $MP = QM$ and Theorem 5. Conditions (8) can be rewritten in the form

$$D^{k,K}v(0) = x_k, \ k = 0,1,\dots,m-1. \tag{14}$$

By Theorem 2, problem (12), (14) has an unique solution, and it has the form

$$v(t) = \sum_{k=0}^{m-1} U_k(t)x_k + \int_0^t U_{m-1}(t-s)L_1^{-1}Qg(s)ds.$$

Due to Lemma 3, equation (13) has an unique solution

$$w(t) = -\sum_{l=0}^{p} (D^{m,K}G)^l M_0^{-1}(I-Q)g(t).$$

Remark 1. It is not difficult to make sure that for $p = 0$ we have $L_0 = 0$, hence, initial conditions (8) are equivalent to the conditions

$$D^{m,K}Lx(0) = y_k, \quad k = 0,1,\dots,m-1, \tag{15}$$

where $y_k = Lx_k$, or $x_k = L_1^{-1}y_k$, $k = 0,1,\dots,m-1$.

Remark 2. It follows from the proof of Theorem 6 that if we consider the Cauchy type problem

$$D^{m,K}x(0) = x_k, \quad k = 0,1,\dots,m-1,$$

for equation (9), we obtain the necessity of conditions

$$(I-P)x_k = -\sum_{l=0}^{p} (D^{m,K}G)^l M_0^{-1}(I-Q)g(0), \quad k = 0,1,\dots,m-1,$$

for the problem solvability.

4 Degenerate equation of Gerasimov type

Now consider the initial problem

$$(Px)^{(k)}(0) = x_k, \quad k = 0,1,\dots,m-1, \tag{16}$$

for a degenerate linear inhomogeneous integro-differential equation of Gerasimov type

$$LD^{K,m}x(t) = Mx(t) + g(t), \quad t \in [0,T], \tag{17}$$

in which $g \in C([0,T];\mathcal{Y})$.

A solution to problem (16), (17) is called a function $x : [0,T] \to D_M$, for which $Mx \in C([0,T];\mathcal{Y})$, $Px \in C^{m-1}([0,T];\mathcal{Y})$, $LJ^K x^{(m)} \in C([0,T];\mathcal{Y})$, equality (17) is valid for all $t \in [0,T]$ and conditions (16) are fulfilled.

Analogously to Lemma 3 the next assertion can be proved.

Lemma 4. Let $K \in C(\mathbb{R}_+;\mathcal{L}(\mathcal{X}))$, $H \in \mathcal{L}(\mathcal{X})$ be a nilpotent operator with a power $p \in \mathbb{N}_0$, a function $h : [0,T] \to \mathcal{X}$ be such that for $l = 0,1,\dots,p$ $(D^{K,m}H)^l h \in C([0,T];\mathcal{X})$, $D^{K,m}(D^{K,m}H)^l h \in C([0,T];\mathcal{X})$. Then there exists an unique solution to the equation

$$D^{K,m}Hx(t) = x(t) + h(t).$$

And it has the form

$$x(t) = -\sum_{l=0}^{p}(D^{K,m}H)^l h(t).$$

Define

$$V_k(t) = \frac{1}{2\pi i}\int_{\gamma}(\lambda^m\widehat{K}(\lambda) - L_1^{-1}M_1)^{-1}\widehat{K}(\lambda)\lambda^{m-1-k}e^{\lambda t}d\lambda, \quad t > 0, \quad k = 0,1,\dots,m-1,$$

Theorem 7. Let an operator M be (L,p)-bounded, $K \in C(\mathbb{R}_+;\mathcal{L}(\mathcal{X}))$, there exist $\widehat{K}$, which be single-valued analytic operator-function in Ω_{R_0} for some $R_0 > 0$, and condition (3) hold. Suppose that for all $\lambda \in \Omega_{R_0}$ there exists $\widehat{K}(\lambda)^{-1} \in \mathcal{L}(\mathcal{X})$, $g \in C([0,T];\mathcal{Y})$, $(D^{K,m}G)^l M_0^{-1}(I-Q)g \in C([0,T];\mathcal{X})$, $D^{K,m}(D^{K,m}G)^l M_0^{-1}(I-Q)g \in C([0,T];\mathcal{X})$ for $l = 0,1,\dots,p$, $x_k \in \mathcal{X}^1$, $k = 0,1,\dots,m-1$. Then there exists an unique solution to problem (16), (17), it has the form

$$x(t) = \sum_{k=0}^{m-1}V_k(t)x_k + \int_0^t U_{m-1}(t-s)L_1^{-1}Qg(s)ds - \sum_{l=0}^{p}(D^{K,m}G)^l M_0^{-1}(I-Q)g(t).$$

Proof. As in the proof of Theorem 6, reduce the problem to the system

$$D^{K,m}v(t) = L_1^{-1}Mv(t) + L_1^{-1}Qg(t), \quad D^{K,m}Gw(t) = w(t) + M_0^{-1}(I-Q)g(t),$$

where $v(t) = Px(t)$, $w(t) = (I-P)x(t)$, endowed by the initial conditions

$$v^{(k)}(0) = x_k, \ k = 0,1,\dots,m-1.$$

By Theorem 4 and Lemma 4 we get the required.

Remark 3. For $p = 0$ initial conditions (16) are equivalent to the conditions

$$D^{m,K}Lx(0) = y_k, \quad k = 0,1,\dots,m-1,$$

where $y_k = Lx_k$, $k = 0,1,\dots,m-1$.

Remark 4. For the Cauchy problem

$$x^{(k)}(0) = x_k, \quad k = 0,1,\dots,m-1,$$

to equation (17) the conditions

$$(I-P)x_k = -\sum_{l=0}^{p}(D^{K,m}G)^l M_0^{-1}(I-Q)g(0), \quad k = 0,1,\dots,m-1,$$

are necessary for the problem solvability.

5 Application to initial boundary value problems

Take $a \in \mathbb{R}$, $\alpha > 0$, $\beta \in (0,1)$, $K(s) = s^{-\beta} E_{\alpha,1-\beta}(as^\alpha) I$, then

$$\widehat{K}(\lambda) = \frac{\lambda^{\alpha+\beta-1}}{\lambda^\alpha - a} I$$

satisfies condition (3) with $\chi \in (0,\beta)$, and it is invertible for all $|\lambda| > a^{1/\alpha}$. Here $E_{\alpha,\delta}$ is the Mittag-Leffler function. Note that the kernel $K(s)$ is singular at zero.

Let $P_\varrho(\lambda) = \sum\limits_{j=0}^{\varrho} c_j \lambda^j$, $Q_\varrho(\lambda) = \sum\limits_{j=0}^{\varrho} d_j \lambda^j$, $c_j, d_j \in \mathbb{C}$, $j = 0, 1, \ldots, \varrho \in \mathbb{N}$, $c_\varrho \neq 0$. Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded region with a smooth boundary $\partial\Omega$,

$$(\Lambda u)(s) := \sum_{|q| \le 2r} a_q(s) \frac{\partial^{|q|} u(s)}{\partial s_1^{q_1} \partial s_2^{q_2} \ldots \partial s_d^{q_d}}, \quad a_q \in C^\infty(\overline{\Omega}),$$

$$(B_l u)(s) := \sum_{|q| \le r_l} b_{lq}(s) \frac{\partial^{|q|} u(s)}{\partial s_1^{q_1} \partial s_2^{q_2} \ldots \partial s_d^{q_d}}, \quad b_{lq} \in C^\infty(\partial\Omega),\ l = 1, 2, \ldots, r,$$

$q = (q_1, q_2, \ldots, q_d) \in \mathbb{N}_0^d$, $|q| = q_1 + \cdots + q_d$, the operator pencil $\Lambda, B_1, B_2, \ldots, B_r$ is regularly elliptical [13]. Define an operator $\Lambda_1 \in \mathcal{C}l(L_2(\Omega))$, acting on the domain

$$D_{\Lambda_1} = H^{2r}_{\{B_l\}}(\Omega) := \{v \in H^{2r}(\Omega) : B_l v(s) = 0,\ l = 1, 2, \ldots, r,\ s \in \partial\Omega\}$$

by the rule $\Lambda_1 u := \Lambda u$. Let Λ_1 be a self-adjoint operator, then the spectrum $\sigma(\Lambda_1)$ of the operator Λ_1 is real, discrete, with finite multiplicity [13]. Suppose, in addition, that the spectrum $\sigma(\Lambda_1)$ is bounded from the right and does not contain zero, denote by $\{\varphi_k : k \in \mathbb{N}\}$ an orthonormal in $L_2(\Omega)$ system of eigenfunctions of the operator Λ_1, numbered in the order of non-increasing of the corresponding eigenvalues $\{\lambda_k : k \in \mathbb{N}\}$, taking into account their multiplicity.

Consider the initial boundary value problem

$$\frac{\partial^k}{\partial t^k} \int_0^t (t-s)^{-\beta} E_{\alpha,1-\beta}(a(t-s)^\alpha) u(\xi,s) ds|_{t=0} = u_k(\xi),\ k = 0, 1, \ldots, m-1,\ \xi \in \Omega, \tag{18}$$

$$B_l \Lambda^k u(\xi,t) = 0, \quad k = 0, 1, \ldots, \varrho - 1, \quad l = 1, 2, \ldots, r, \quad (\xi,t) \in \partial\Omega \times (0,T], \tag{19}$$

$$P_\varrho(\Lambda) \frac{\partial^m}{\partial t^m} \int_0^t (t-s)^{-\beta} E_{\alpha,1-\beta}(a(t-s)^\alpha) u(\xi,s) ds = Q_\varrho(\Lambda) u(\xi,t) + h(\xi,t) \tag{20}$$

in $\Omega \times (0,T]$. Here

$$J^K u(\xi,t) = \int_0^t (t-s)^{-\beta} E_{\alpha,1-\beta}(a(t-s)^\alpha) u(\xi,s) ds$$

is the Atangana—Baleanu type integral [6], but with a singular kernel, $h : \Omega \times [0,T] \to \mathbb{R}$. Take $\mathcal{X} = \{v \in H^{2r\varrho}(\Omega) : B_l \Lambda^k v(s) = 0,\ k = 0, 1, \ldots, \varrho - 1,\ l = 1, 2, \ldots, r,\ s \in \partial\Omega\}$, $\mathcal{Y} = L_2(\Omega)$, $L = P_\varrho(\Lambda)$, $M = Q_\varrho(\Lambda) \in \mathcal{L}(\mathcal{X};\mathcal{Y})$.

Let $P_\varrho(\lambda_k) \neq 0$ for all $k \in \mathbb{N}$, then there exists an inverse operator $L^{-1} \in \mathcal{L}(\mathcal{Y};\mathcal{X})$ and problem (18)–(20) is representable as problem (1), (4), where $A = L^{-1}M \in \mathcal{L}(\mathcal{Z})$, $x_k = u_k(\cdot)$, $k = 0, 1, \ldots, m-1$,

$f(t) = L^{-1}h(\cdot, t)$. By Theorem 2 there exists a unique solution to problem (18)–(20) for any $u_k \in \mathcal{X}$, $k = 0, 1, \ldots, m-1$, if $h \in C((0,T]; L_2(\Omega)) \cap L_1(0,T;\mathcal{X})$.

Now assume that $P_\varrho(\lambda_k) = 0$ for some $k \in \mathbb{N}$. If the polynomials P_ϱ and Q_ϱ have no common roots on the set $\{\lambda_k\}$, the operator M is $(L, 0)$-bounded (see [14]), the projectors have the form

$$P = \sum_{P_\varrho(\lambda_k)\neq 0} \langle \cdot, \varphi_k \rangle \varphi_k, \quad Q = \sum_{P_\varrho(\lambda_k)\neq 0} \langle \cdot, \varphi_k \rangle \varphi_k,$$

where $\langle \cdot, \varphi_k \rangle$ is the inner product in $L_2(\Omega)$. The initial conditions, taking into account Remark 1, can be given in the form

$$P_\varrho(\Lambda)\frac{\partial^k}{\partial t^k}\int_0^t (t-s)^{-\beta}E_{\alpha,1-\beta}(a(t-s)^\alpha)u(\xi,s)ds|_{t=0} = y_k(s),\ k = 0, 1, \ldots, m-1,\ s \in \Omega. \tag{21}$$

Then problem (19)–(21) is represented as (9), (15) with the spaces $\mathcal{X}$, $\mathcal{Y}$ and the operators L, M selected above. Theorem 6 implies the unique solvability of problem (19)–(21), if $h \in C([0,T]; L_2(\Omega))$ and $y_k \in L_2(\Omega)$, $k = 0, 1, \ldots, m-1$, such that $\langle y_k, \varphi_l \rangle = 0$ for all $l \in \mathbb{N}$, for which $P_\varrho(\lambda_l) = 0$.

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Банах кеңістіктеріндегі шектелген операторлары бар интегро-дифференциалдық теңдеулер

Мақалада және дифференциалдау және үйірткі операторларының құрамдары болып табылатын операторлармен Банах кеңістігіндегі интегралдық-дифференциалдық теңдеулер зерттелген. Осы екі оператордың әрекет ету ретіне байланысты үйірткі операторы бірінші әрекет еткенде Риман-Лиувиль типіндегі интегро-дифференциалдық операторлар, ал басқаша Герасимов типті интегро-дифференциалдық операторлар туралы айтылады. Қарастырылып отырған операторлардың дербес жағдайлары сәйкесінше Риман-Лиувиль және Герасимов бөлшек туындылары болып табылады. Зерттелетін интегро-дифференциалдық операторлардың кластарына үйірткісі сингулярлықсыз интегралдық ядросы барлар да кіреді. Риман-Лиувилль типті сызықтық интегро-дифференциалдық теңдеу үшін Коши типтес есептің және ізделінді функция үшін шектелген операторы бар Герасимов типті сызықтық интегро-дифференциалдық теңдеу үшін Коши есебінің бірегей шешімін табу шарттары алынды. Бұл нәтижелер теңдеуден операторлар жұбының салыстырмалы шектелуі шартында интегро-дифференциалдық оператор үшін өзгеше операторы бар ұқсас теңдеулерді зерттеуде қолданылды. Абстрактілі нәтижелер Миттаг-Леффлер функциясымен берілген, яғни ерекшеліктері жоқ, интегро-дифференциалдық үйірткі операторы бар дербес туындылы теңдеулер үшін бастапқы-шектік есептерді зерттеуде пайдаланылды.

Кілт сөздер: интегро-дифференциалдық теңдеу, интегро-дифференциалдық оператор, үйірткі, Коши есебі, Коши типтес есеп, өзгеше интегро-дифференциалдық теңдеу, бастапқы-шекаралық есеп.

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Интегро-дифференциальные уравнения с ограниченными операторами в банаховых пространствах

В статье исследованы интегрально-дифференциальные уравнения в банаховых пространствах с операторами, представляющими собой композицию операторов свертки и дифференцирования. В зависимости от порядка действия этих двух операторов говорится об интегро-дифференциальных операторах типа Римана-Лиувилля, когда первым действует оператор свертки, и интегро-дифференциальных операторах типа Герасимова в противном случае. Частными случаями рассматриваемых операторов являются дробные производные Римана—Лиувилля и Герасимова соответственно. В исследуемые классы интегро-дифференциальных операторов входят и такие, в которых свертка имеет интегральное ядро без сингулярностей. Получены условия однозначной разрешимости задачи типа Коши для линейного интегро-дифференциального уравнения типа Римана-Лиувилля и задачи Коши для линейного интегро-дифференциального уравнения типа Герасимова с ограниченным оператором при искомой функции. Эти результаты использованы при исследовании аналогичных уравнений с вырожденным оператором при интегро-дифференциальном операторе при условии относительной ограниченности пары операторов из уравнения. Абстрактные результаты использованы при исследовании начально-краевых задач для уравнений в частных производных с интегро-дифференциальным оператором, свертка в котором задается функцией Миттаг-Леффлера, то есть не имеет особенностей.

Ключевые слова: интегро-дифференциальное уравнение, интегро-дифференциальный оператор, свертка, задача Коши, задача типа Коши, вырожденное интегро-дифференциальное уравнение, начально-краевая задача.