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Asymptotics solutions of a singularly perturbed integro-differential fractional order derivative equation with rapidly oscillating coefficients

In this paper, the regularization method of S.A.Lomov is generalized to the singularly perturbed integro-differential fractional-order derivative equation with rapidly oscillating coefficients. The main goal of the work is to reveal the influence of the oscillating components on the structure of the asymptotics of the solution to this problem. The case of the absence of resonance is considered, i.e. the case when an integer linear combination of a rapidly oscillating inhomogeneity does not coincide with a point in the spectrum of the limiting operator at all points of the considered time interval. The case of coincidence of the frequency of a rapidly oscillating inhomogeneity with a point in the spectrum of the limiting operator is called the resonance case. This case is supposed to be studied in our subsequent works. More complex cases of resonance (for example, point resonance) require more careful analysis and are not considered in this work.

Keywords: singularly perturbed, fractional order derivation, integro-differential equation, iterative problems, solvability of iterative problems.

Introduction

An initial problem is considered for a singularly perturbed integro-differential equation:

$$L_\varepsilon z(t,\varepsilon) \equiv \varepsilon z^{(\alpha)} - a(t)z - \int_{t_0}^{t} K(t,s)z(s,\varepsilon)ds = h_1(t) + h_2(t)\sin\frac{\beta(t)}{\varepsilon},$$

$$z(t_0,\varepsilon) = y^0, \quad t\in[t_0,T], \quad t_0>0 \tag{1}$$

for a scalar unknown function $z(t,\varepsilon)$, in which $a(t),\ h_1(t),\ h_2(t),\ \beta'(t)>0,\ (\forall t\in[t_0,T])$ are known functions, $0<\alpha<1$, z^0 constant number, $\varepsilon>0$ is a small parameter. The problem is posed of constructing a regularized [1–2] asymptotic solution to problem (1). Previously, systems for ordinary differential equations [3–6] and integro-differential equations with rapidly oscillating coefficients [7–11] were considered.

By definition of the fractional derivative [12], the fractional derivative $z^{(\alpha)}$ in terms of integer derivatives is denoted in the following form $t^{(1-\alpha)}\frac{dz}{dt}$. Accordingly, we rewrite the original fractional order equation (1) in the following form:

$$L_\varepsilon z(t,\varepsilon) \equiv \varepsilon t^{(1-\alpha)}\frac{dz}{dt} - a(t)z - \int_{t_0}^{t} K(t,s)z(s,\varepsilon)ds = h_1(t) + h_2(t)\sin\frac{\beta(t)}{\varepsilon},\ z(t_0,\varepsilon)=z^0,\ t\in[t_0,T]. \tag{2}$$

In problem (2), the frequency of the rapidly oscillating sine is $\beta'(t)$. In what follows, the function $\lambda_1(t)=a(t)$ is called the spectrum of problem (2), and functions $\lambda_2(t)=-i\beta'(t),\ \lambda_3(t)=+i\beta'(t)$ spectrum of a rapidly oscillating sine.

Problem (1) will be considered under the following conditions:

1) $a(t),\beta(t),h_1(t),h_2(t)\in C[t_0,T],\ K(t,s)\in C^\infty(t_0\le s\le t\le T)$;

2) $a(t)<0\ \ \forall t\in[t_0,T]$.

We will develop an algorithm for constructing a regularized asymptotic solution [6] of problem (1).

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Regularization of the problem (2)

Denote by $\sigma_j = \sigma_j(\varepsilon)$ independent of magnitude $\sigma_1 = e^{-\frac{i}{\varepsilon}\beta(t_0)}$, $\sigma_2 = e^{+\frac{i}{\varepsilon}\beta(t_0)}$, and introduce the regularized variables:

$$\tau_1 = \frac{1}{\varepsilon}\int_{t_0}^{t} \theta^{(\alpha-1)}\lambda_1(\theta)d\theta \equiv \frac{\psi_1(t)}{\varepsilon}, \quad \tau_j = \frac{1}{\varepsilon}\int_{t_0}^{t} \lambda_j(\theta)d\theta \equiv \frac{\psi_j(t)}{\varepsilon}, j = 2, 3, \tag{3}$$

and instead of problem (2), consider the problem

$$L_\varepsilon \tilde{z}(t,\tau,\sigma,\varepsilon) \equiv \varepsilon t^{(1-\alpha)}\frac{\partial \tilde{z}}{\partial t} + \lambda_1(t)\frac{\partial \tilde{z}}{\partial \tau_1} + t^{(1-\alpha)}\sum_{j=2}^{3}\lambda_j(t)\frac{\partial \tilde{z}}{\partial \tau_j} - \lambda_1(t)\tilde{z} - \int\limits_{t_0}^{t} K(t,s)\tilde{z}(s,\frac{\psi(s)}{\varepsilon},\sigma,\varepsilon)ds =$$

$$= h_1(t) - \frac{1}{2i}h_2(t)\left(e^{\tau_2}\sigma_1 - e^{\tau_3}\sigma_2\right), \ \tilde{z}(t,\tau,\sigma,\varepsilon)|_{t=t_0,\tau=0} = z^0, \ t \in [t_0, T]. \tag{4}$$

for the function $\tilde{z} = \tilde{z}(t,\tau,\sigma,\varepsilon)$, where is indicated (according (3)): $\tau = (\tau_1, \tau_2, \tau_3)$, $\psi = (\psi_1, \psi_2, \psi_3)$. It is clear that if $\tilde{z} = \tilde{z}(t,\tau,\sigma,\varepsilon)$ is a solution of the problem (4), then the function is $\tilde{z} = \tilde{z}\left(t, \frac{\psi(t)}{\varepsilon}, \sigma, \varepsilon\right)$ an exact solution to problem (3), therefore, problem (4) is extended with respect to problem (2). However, it cannot be considered fully regularized, since it does not regularize the integral

$$J\tilde{z} \equiv J\left(\tilde{z}(t,\tau,\sigma,\varepsilon)|_{t=s,\tau=\psi(s)/\varepsilon}\right) = \int\limits_{t_0}^{t} K(t,s)\tilde{z}(s,\frac{\psi(s)}{\varepsilon},\varepsilon)ds.$$

For its regularization, we introduce the class M_ε asymptotically invariant with respect to the operator $J\tilde{z}$ (see [1; 62]). Consider first the space U of vector functions $z(t,\tau,\sigma)$, representable by the sums

$$z(t,\tau,\sigma) = z_0(t,\sigma) + \sum_{i=1}^{3} z_i(t,\sigma)e^{\tau_i}, \quad z_i(t,\sigma) \in C^\infty\left([t_0,T],\mathbb{C}\right), i = \overline{0,3}. \tag{5}$$

In addition, the elements of space U depend on bounded in $\varepsilon > 0$ terms of constants $\sigma_1 = \sigma_1(\varepsilon)$ and $\sigma_2 = \sigma_2(\varepsilon)$ which do not affect the development of the algorithm described below, therefore, in the record of element (5) of this space U, we omit the dependence on $\sigma = (\sigma_1, \sigma_2)$ for brevity. We show that the class $M_\varepsilon = U|_{\tau=\psi(t)/\varepsilon}$ is asymptotically invariant with respect to the operator J.

For the space U, we take the space of functions $z(t,\tau,\sigma)$, represented by sums

$$J\tilde{z}(t,\tau,\varepsilon) \equiv \int\limits_{t_0}^{t} K(t,s)z_0(s)ds + \int\limits_{t_0}^{t} K(t,s)z_1(s)e^{\frac{1}{\varepsilon}\int\limits_{t_0}^{s}\theta^{(\alpha-1)}\lambda_1(\theta)d\theta}ds +$$

$$+\sum_{i=2}^{3}\int\limits_{t_0}^{t} K(t,s)z_i(s)e^{\frac{1}{\varepsilon}\int\limits_{t_0}^{s}\lambda_i(\theta)d\theta}ds.$$

Integrating by parts, we write the image of the operator J on the element (5) of the space U as a series

$$J\tilde{z}(t,\tau,\varepsilon) = \int\limits_{t_0}^{t} K(t,s)z_0(s)ds +$$

$$+\sum_{i=1}^{3}\sum_{\nu=0}^{\infty}(-1)^\nu\varepsilon^{\nu+1}\left[\left(I_i^\nu\left(K(t,s)z_i(s)\right)\right)_{s=t}e^{\tau_i} - \left(I_i^\nu\left(K(t,s)z_i(s(s)\right)\right)_{s=t_0}\right],$$

where are indicated:

$$I_1^0 = \frac{1}{s^{(\alpha-1)}\lambda_1(s)}\cdot, I_1^\nu = \frac{1}{s^{(\alpha-1)}\lambda_i(s)}\frac{\partial}{\partial s}I_1^{\nu-1},$$

$$I_i^0 = \frac{1}{\lambda_i(s)}\cdot, I_i^\nu = \frac{1}{\lambda_i(s)}\frac{\partial}{\partial s}I_i^{\nu-1},\ \ i=2,3.$$

It is easy to show (see, for example, [13; 291–294] that this series converges asymptotically for $\varepsilon \to +0$ (uniformly in $t \in [t_0, T]$). This means that the class M_ε is asymptotically invariant (for $\varepsilon \to +0$) with respect to the operator J.

We introduce operators $R_\nu : U \to U$, acting on each element $z(t,\tau) \in U$ of the form (5) according to the law:

$$R_0 z(t,\tau) = \int\limits_{t_0}^{t} K(t,s) z_0(s) ds, \tag{6_0}$$

$$R_1 z(t,\tau) = \sum_{i=1}^{3} \Big[\big(I_i^0 \left(K(t,s)z_i(s)\right)\big)_{s=t} e^{\tau_i} - \big(I_i^0 \left(K(t,s)z_i(s)\right)\big)_{s=t_0} \Big], \tag{6_1}$$

$$R_{\nu+1} z\left(t,\tau\right) = \sum_{i=1}^{3}\sum_{\nu=0}^{\infty} (-1)^\nu \varepsilon^{\nu+1} \left[\left(I_i^\nu \left(K(t,s)z_i(s)\right)\right)_{s=t} e^{\tau_i} - \left(I_i^\nu \left(K(t,s)z_i(s)\right)\right)_{s=t_0} \right], \nu \geq 1. \tag{$6_{\nu+1}$}$$

Now, let $\tilde{z}(t,\tau,\varepsilon)$ be an arbitrary continuous function on $(t,\tau) \in G = [t_0,T] \times \{\tau : Re\tau_1 < 0,\ Re\tau_j \leq 0,$ $j = 2,3\}$, with asymptotic expansion

$$\tilde{z}(t,\tau,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k z_k(t,\tau), y_k(t,\tau) \in U \tag{7}$$

converging as $\varepsilon \to +0$ (uniformly in $(t,\tau) \in G$). Then, the image $J\tilde{z}\left(t,\tau,\varepsilon\right)$ of this function is decomposed into an asymptotic series

$$J\tilde{z}(t,\tau,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k J z_k(t,\tau) = \sum_{r=0}^{\infty} \varepsilon^r \sum_{s=0}^{r} R_{r-s} z_s(t,\tau)|_{\tau=\psi(t)/\varepsilon}.$$

This equality is the basis for introducing an extension of an operator J on series of the form (7):

$$\tilde{J}\tilde{z} \equiv \tilde{J}\left(\sum_{k=0}^{\infty} \varepsilon^k z_k(t,\tau)\right) = \sum_{r=0}^{\infty} \varepsilon^r \left(\sum_{k=0}^{r} R_{r-k} z_k(t,\tau)\right).$$

Although the operator $\tilde{J}$ is formally defined, its utility is obvious, since in practice it is usual to construct the N-th approximation of the asymptotic solution of the problem (3), in which impose only N-th partial sums of the series (6), which have not a formal, but a true meaning. Now, one can write a problem that is completely regularized with respect to the original problem (3):

$$L_\varepsilon \tilde{z}(t,\tau,\sigma,\varepsilon) \equiv \varepsilon t^{(1-\alpha)} \frac{\partial \tilde{z}}{\partial t} + \lambda_1(t) \frac{\partial \tilde{z}}{\partial \tau_1} + t^{(1-\alpha)} \sum_{j=2}^{3} \lambda_j(t) \frac{\partial \tilde{z}}{\partial \tau_j} - \lambda_1(t)\tilde{z} - \tilde{J}\tilde{z} =$$

$$= h_1(t) - \frac{1}{2i} h_2(t) \left(e^{\tau_2} - e^{\tau_3}\right),\ \tilde{z}(t_0,0,\sigma,\varepsilon) = z^0,\ t \in [t_0,T]. \tag{8}$$

Iterative problems and their solvability in the space U

Substituting the series (7) into (8) and equating the coefficients of the same powers of ε, we obtain the following iterative problems:

$$L z_0(t,\tau,\sigma) \equiv \lambda_1(t) \frac{\partial z_0}{\partial \tau_1} + t^{(1-\alpha)} \sum_{j=2}^{3} \lambda_j(t) \frac{\partial z_0}{\partial \tau_j} - \lambda_1(t) z_0 - R_0 z_0 =$$

$$= h_1(t) - \frac{1}{2i} h_2(t) \left(e^{\tau_2} - e^{\tau_3}\right),\ \ z_0(t_0,0) = z^0; \tag{9_0}$$

$$Lz_1(t,\tau,\sigma) = -t^{(1-\alpha)}\frac{\partial z_0}{\partial t} + R_1 z_0, \quad z_1(t_0,0) = 0; \tag{9_1}$$

$$Lz_2(t,\tau,\sigma) = -t^{(1-\alpha)}\frac{\partial z_1}{\partial t} + R_1 z_1 + R_2 z_0, \quad z_0(t_0,0) = 0; \tag{9_2}$$

....

$$L\,z_k(t,\tau,\sigma) = -t^{(1-\alpha)}\frac{\partial z_{k-1}}{\partial t} + R_k z_0 + \ldots +$$

$$+ \ldots + R_1 z_{k-1}, \; z_k(t_0,0) = 0, \quad k \geq 1. \tag{9_k}$$

Each iterative problem (9_k) has the form

$$L\,z(t,\tau,\sigma) \equiv \lambda_1(t)\frac{\partial z}{\partial \tau_1} + t^{(1-\alpha)}\sum_{j=2}^{3}\lambda_j(t)\frac{\partial z}{\partial \tau_j} - \lambda_1(t)z - R_0 z = H(t,\tau,\sigma), \;\; z(t_0,0) = z^* \tag{10}$$

where $H(t,\tau,\sigma) = H_0(t,\sigma) + \sum_{i=1}^{3} H_i(t,\sigma)e^{\tau_i}$ is the known function of space U, y_* is the known function of the complex space $\mathbf{C}$, and the operator R_0 has the form (see (6_0))

$$R_0 z \equiv R_0\left(z_0(t) + \sum_{j=1}^{3} z_j(t)e^{\tau_j} +\right) \triangleq \int_{t_0}^{1} K(t,s)z_0(s)ds.$$

We introduce scalar (for each $t \in [t_0,T]$) product in space U:

$$< u, w > \equiv < u_0(t) + \sum_{j=1}^{3} u_j(t)e^{\tau_j}, w_0(t) + \sum_{j=1}^{3} w_j(t)e^{\tau_j} > \equiv$$

$$\equiv (u_0(t), w_0(t)) + \sum_{j=1}^{3}\,(u_j(t), w_j(t))$$

where we denote by $(*,*)$ the usual scalar product in the complex space $\mathbf{C}$: $(u,v) = u\cdot\bar{v}$. Let us prove the following statement.

Theorem 1. Let conditions (1), (2) be fulfilled and the right-hand side $H(t,\tau,\sigma) = H_0(t,\sigma) + + \sum_{j=1}^{3} H_j(t,\sigma)e^{\tau_j}$ of equation (10) belongs to the space U. Then the equation (10) is solvable in U, if and only if

$$< H(t,\tau), e^{\tau_1} > \equiv 0, \forall t \in [t_0, T]. \tag{11}$$

Proof. We will determine the solution of equation (10) as an element (5) of the space U :

$$z(t,\tau,\sigma) = z_0(t,\sigma) + \sum_{j=1}^{3} z_j(t,\sigma)e^{\tau_j}. \tag{12}$$

Substituting (12) into equation (10), and equating here the free terms and coefficients separately for identical exponents, we obtain the following equations of equations:

$$\lambda_1(t)z_0(t,\sigma) - \int_{t_0}^{t} K(t,s)z_0(s,\sigma)ds = H_0(t,\sigma), \tag{13}$$

$$0\cdot z_1(t,\sigma) = H_1(t,\sigma), \tag{13_1}$$

$$\left[t^{(1-\alpha)}\lambda_j(t) - \lambda_1(t)\right] z_j(t,\sigma) = H_j(t,\sigma), \; j = \overline{2,3}. \tag{13_j}$$

Since the $\lambda_1(t) \neq 0$, the equation (13) can be written as

$$z_0(t,\sigma) = \int_{t_0}^{t} \left(-\lambda_1^{-1}(t)K(t,s)\right) z_0(s,\sigma)ds - \lambda_1^{-1}(t)H_0(t,\sigma). \qquad (13_0)$$

Due to the smoothness of the kernel $\left(-\lambda_1^{-1}(t)K(t,s)\right)$ and heterogeneity $-\lambda_1^{-1}(t)H_0(t,\sigma)$, this Volterra integral equation has a unique solution $z_0(t,\sigma) \in C^\infty([t_0,T],\mathbf{C})$. The equations (13_2) and (13_3) also have unique solutions

$$z_j(t,\sigma) = [\lambda_j(t) - \lambda_1(t)]^{-1} H_j(t,\sigma) \in C^\infty([t_0,T],\mathbf{C}), j = 2,3 \qquad (14)$$

since $\lambda_2(t), \lambda_3(t)$ not equal to $\lambda_1(t)$.

The equation (13_1) is solvable in space $C^\infty([t_0,T],\mathbf{C})$ if and only if $(H_1(t,\tau), e^{\tau_1}) \equiv 0 \, \forall t \in [t_0,T]$ hold. It is not difficult to see that these identities coincide with identities (10). Thus, condition (10) is necessary and sufficient for the solvability of equations (9) in the space U. Theorem 1 is proved.

Remark 1. If identity (10) holds, then under conditions (1), (2), equation (9) has the following solution in the space U :

$$z(t,\tau,\sigma) = z_0(t,\sigma) + \alpha_1(t,\sigma)e^{\tau_1} + \sum_{j=2}^{3} \left[t^{(1-\alpha)}\lambda_j(t) - \lambda_1(t)\right]^{-1} H_j(t,\sigma)e^{\tau_j} \qquad (15)$$

where $\alpha_1(t,\sigma) \in C^\infty([t_0,T],\mathbf{C})$ are arbitrary function, $z_0(t,\sigma)$ is the solution of an integral equation (13_0).

The unique solvability of the general iterative problem in the space U. Residual term theorem

Let us proceed to the description of the conditions for the unique solvability of equation (10) in space U. Along with problem (10), we consider the equation

$$Lz(t,\tau) = -t^{(1-\alpha)}\frac{\partial z}{\partial t} + R_1 z + Q(t,\tau), \qquad (16)$$

where $z = z(t,\tau)$ is the solution (16) of the equation (10), $Q(z,\tau) \in U$ is the well-known function of the space U. The right part of this equation:

$$G(t,\tau) \equiv -t^{(1-\alpha)}\frac{\partial z}{\partial t} + R_1 z + Q(t,\tau) =$$

$$= -t^{(1-\alpha)}\frac{\partial}{\partial t}\left(z_0(t) + \sum_{j=1}^{3} z_j(t)e^{\tau_j}\right) + R_1\left(z_0(t) + \sum_{j=1}^{3} z_j(t)e^{\tau_j}\right) + Q(t,\tau)$$

may not belong to space U, if $z = z(t,\tau) \in U$. Indeed, taking into account the form (14) of the function $z = z(t,\tau) \in U$, we consider in $G(t,\tau)$, for example, the terms

$$Z(t,\tau) \equiv \frac{g(t)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left[z_0(t) + \sum_{j=1}^{3} z_j(t)e^{\tau_j} + \sum_{2\le|m|\le N_H}^{*} z^m(t)e^{(m,\tau)}\right] =$$

$$= \frac{g(t)}{2} z_0(t)\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right) + \sum_{j=1}^{3}\frac{g(t)}{2} z_j(t)\left(e^{\tau_j+\tau_2}\sigma_1 + e^{\tau_j+\tau_3}\sigma_2\right) +$$

$$+ \frac{g(t)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\sum_{2\le|m|\le N_H}^{*} z^m(t)e^{(m,\tau)}.$$

Here, for instance, terms with exponents

$$e^{\tau_2+\tau_3} = e^{(m,\tau)}|_{m=(0,1,1)},\ e^{\tau_2+(m,\tau)}\ (if\ m_1 = 0, m_2+1 = m_3),$$

$$e^{\tau_3+(m,\tau)}\ (if\ m_1 = 0,\ m_3+1 = m_2),\ e^{\tau_2+(m,\tau)}\ (if\ m_1 = 0, m_2 = m_3), \qquad (*)$$

$$e^{\tau_3+(m,\tau)}\,(if\ m_1=0,\, m_2=m_3)\,,\, e^{\tau_2+(m,\tau)}\,(if\ m_1=1,\, m_2=m_3)\,,$$
$$e^{\tau_3+(m,\tau)}\,(if\ m_1=1,\, m_2=m_3)$$

do not belong to space U, since multi-indexes

$$(0,n,n)\ \in\Gamma_0,\ (0,n+1,n)\in\Gamma_1,\ (0,n,n+1)\in\Gamma_2\,\forall n\in N$$

are resonant. Then, according to the well-known theory (see, [6; 234]), we embed these terms in the space U according to the following rule (see $(*)$):

$$\widehat{e^{\tau_2+\tau_3}}=e^0=1,\ \widehat{e^{\tau_2+(m,\tau)}}=e^0=1\,(if\ m_1=0,m_2+1=m_3)\,,$$
$$\widehat{e^{\tau_3+(m,\tau)}}=e^0=1\,(if\ m_1=0,\ m_3+1=m_2)\,,$$
$$\widehat{e^{\tau_2+(m,\tau)}}=e^{\tau_2}\,(if\ m_1=0,m_2=m_3)\,,\widehat{e^{\tau_3+(m,\tau)}}=e^{\tau_3}\,(if\ m_1=0,\ m_2=m_3)\,,$$
$$\widehat{e^{\tau_2+(m,\tau)}}\,(if\ m_1=1,\ m_2=m_3)=e^{\tau_1},\ \widehat{e^{\tau_3+(m,\tau)}}=e^{\tau_1}\,(if\ m_1=1,\ m_2=m_3)\,.$$

In other words, terms with resonant exponentials $e^{(m,\tau)}$ replaced by members with exponents $e^0, e^{\tau_1}, e^{\tau_2}, e^{\tau_3}$ according to the following rule:

$$\widehat{e^{(m,\tau)}}|_{m\in\Gamma_0}=e^0=1,\ \widehat{e^{(m,\tau)}}|_{m\in\Gamma_1}=e^{\tau_1},\ \widehat{e^{(m,\tau)}}|_{m\in\Gamma_2}=e^{\tau_2},\ \widehat{e^{(m,\tau)}}|_{m\in\Gamma_3}=e^{\tau_3}.$$

After embedding, the right-hand side of equation (15) will look like

$$\widehat{G}(t,\tau)=-t^{(1-\alpha)}\frac{\partial}{\partial t}\left[z_0(t)+\sum_{j=1}^{3}z_j(t)e^{\tau_j}+\sum_{2\le|m|\le N_H}^{*}z^m(t)e^{(m,\tau)}+\sum_{j=0}^{3}\sum_{m^j\in\Gamma_j}z^{m^j}(t)\,e^{\tau_j}\right]+Q(t,\tau)$$

As indicated in [6], the embedding $G(t,\tau)\to\widehat{G}(t,\tau)$ will not affect the accuracy of the construction of asymptotic solutions of problem (2), since $G(t,\tau)$ at $\tau=\frac{\psi(t)}{\varepsilon}$ coincides with $\widehat{G}(t,\tau)$.

Theorem 2. Let conditions (1), (2) be fulfilled and the right-hand side $H(t,\tau)=H_0(t)+\sum_{j=1}^{3}H_j(t)e^{\tau_j}+$ $+\sum_{2\le|m|\le N_H}^{*}H^m(t)e^{(m,\tau)}\in U$ of equation (10) satisfy condition (11). Then problem (10) under additional conditions

$$<\widehat{G}(t,\tau),e^{\tau_1}>\equiv 0\ \forall t\in[t_0,T] \tag{17}$$

where $Q(t,\tau)=Q_0(t)+\sum_{k=1}^{3}Q_k(t)e^{\tau_i}+\sum_{2\le|m|\le N_z}^{*}Q^m(t)e^{(m,\tau)}$ *is the known function of space* U*, is uniquely solvable in* U*.*

Proof. Since the right-hand side of equation (10) satisfies condition (11), this equation has a solution in space U in the form (14), where $\alpha_1(t)\in C^\infty([t_0,T],\mathbf{C})$ is arbitrary function. Submit (14) to the initial condition $y(t_0,0)=y^*$. We get $\alpha_1(t_0,t)=y_*$, where denoted

$$z_*=z^*+A^{-1}(t_0)H_0(t_0)-\frac{H_2(t_0)}{t_0{}^{(1-\alpha)}\lambda_2(t_0)-\lambda_1(t_0)}-\frac{H_3(t_0)}{t_0{}^{(1-\alpha)}\lambda_3(t_0)-\lambda_1(t_0)}-$$
$$-\sum_{2\le|m|\le N_H}^{*}[(m,\lambda(t_0))-A(t_0)]^{-1}H^m(t_0).$$

Now, we subordinate the solution (15) to the orthogonality condition (17). We write $G(t,\tau)$ in more detail the right side of equation (10):

$$G(t,\tau)\equiv -t^{(1-\alpha)}\frac{\partial}{\partial t}\left[z_0(t)+\alpha_1(t)e^{\tau_1}+h_{21}(t)e^{\tau_2}+h_{31}(t)e^{\tau_3}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}\right]+$$

$$+\frac{g(t)}{2}\left(e^{\tau_2}\sigma_1+e^{\tau_3}\sigma_2\right)\left[z_0(t)+\alpha_1(t)e^{\tau_1}+h_{21}(t)e^{\tau_2}+h_{31}(t)e^{\tau_3}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}\right]+$$

$$+R_1\left[z_0(t)+\alpha_1(t)e^{\tau_1}+h_{21}(t)e^{\tau_2}+h_{31}(t)e^{\tau_3}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}\right]+Q(t,\tau).$$

Embedding this function into space U, we will have

$$\hat{G}(t,\tau)\equiv-t^{(1-\alpha)}\frac{\partial}{\partial t}\left[z_0(t)+\alpha_1(t)e^{\tau_1}+h_{21}(t)e^{\tau_2}+h_{31}(t)e^{\tau_3}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}\right]+$$

$$+\left\{\frac{g(t)}{2}z_0(t)e^{\tau_2}\sigma_1+\frac{g(t)}{2}z_0(t)e^{\tau_3}\sigma_2+\sum_{j=1}^{3}\frac{g(t)}{2}z_j(t)e^{\tau_j+\tau_2}\sigma_1+\sum_{j=1}^{3}\frac{g(x)}{2}z_j(t)e^{\tau_j+\tau_3}\sigma_2+\right.$$

$$\left.+\sum_{2\le|m|\le N_H}^{*}\frac{g(t)}{2}z^m(t)e^{(m,\tau)+\tau_2}\sigma_1+\sum_{2\le|m|\le N_H}^{*}\frac{g(t)}{2}z^m(t)e^{(m,\tau)+\tau_3}\sigma_2\right\}^{\wedge}+$$

$$+R_1\left[z_0(t)+\alpha_1(t)e^{\tau_1}+h_{21}(t)e^{\tau_2}+h_{31}(t)e^{\tau_3}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}\right]+Q(t,\tau)=$$

$$=-t^{(1-\alpha)}\frac{\partial}{\partial t}\left[z_0(t)+\alpha_1(t)e^{\tau_1}+h_{21}(t)e^{\tau_2}+h_{31}(t)e^{\tau_3}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}\right]+$$

$$+\frac{g(t)}{2}\{z_0(t)e^{\tau_2}\sigma_1+z_0(t)e^{\tau_3}\sigma_2+\underline{\alpha_1(t)e^{\tau_1+\tau_2}\sigma_1}+h_{21}(t)e^{2\tau_2}\sigma_1+h_{31}(t)e^{\tau_3+\tau_2}\sigma_1+$$

$$+\underline{\alpha_1(t)e^{\tau_1+\tau_3}\sigma_2}+h_{21}(t)e^{\tau_2+\tau_3}\sigma_2+h_{31}(t)e^{2\tau_3}\sigma_2+$$

$$\left.+\underline{\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)+\tau_2}\sigma_1}+\underline{\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)+\tau_3}\sigma_2}\right\}^{\wedge}+$$

$$+R_1\left[z_0(t)+\alpha_1(t)e^{\tau_1}+h_{21}(t)e^{\tau_2}+h_{31}(t)e^{\tau_3}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}\right]+Q(t,\tau).$$

The embedding operation acts only on resonant exponentials, leaving the coefficients unchanged at these exponents. Given that the expression

$$R_1\left[z_0(t)+\alpha_1(t)e^{\tau_1}+h_{21}(t)e^{\tau_2}+h_{31}(t)e^{\tau_3}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}\right]$$

linearly depends on $\alpha_1(t)$ (see formula (5_1)), we also conclude that after the embedding operation the function $\hat{G}(t,\tau)$ will linearly depend on the scalar function $\alpha_1(t)$. Given that in condition (16) scalar multiplication by functions e^{τ_1}, containing only the exponent e^{τ_1}, in the expression for $\widehat{G}(t,\tau)$ it is necessary to keep only the term with the exponent e^{τ_1}. Then condition (17) takes the form

$$<-t^{(1-\alpha)}\frac{\partial}{\partial t}\left(\alpha_1(t)e^{\tau_1}\right)+\left(\sum_{|m^1|=2:\,m^1\in\Gamma_1}^{N}w^{m^1}\left(\alpha_1\left(t\right),t\right)\right)e^{\tau_1}+Q_1(t)e^{\tau_1},e^{\tau_1}>=0\quad\forall t\in[t_0,T]$$

where $w^{m^1}\left(\alpha_1\left(t\right),t\right)$ are some functions linearly dependent on $\alpha_1(t)$. Performing scalar multiplication here, we obtain a linear ordinary differential equation (relative t) for a function $\alpha_1(t)$. Given the initial condition

$\alpha_1(t_0) = y_*$, found above, we find uniquely the function $\alpha_1(t) \in C^\infty[t_0, T]$ and therefore, we will uniquely construct a solution to equation (9) in the space U. The theorem is proved.

As mentioned above, the right-hand sides of iterative problems (8_k) (if solved sequentially) may not belong to space U. Then, according to [6; 234], the right-hand sides of these problems must be embedded into U, according to the above rule. As a result, we obtain the following problems:

$$L\, z_0(t,\tau,\sigma) \equiv \lambda_1(t)\frac{\partial z_0}{\partial \tau_1} + t^{(1-\alpha)}\sum_{j=2}^{3}\lambda_j(t)\frac{\partial z_0}{\partial \tau_j} - \lambda_1(t)z_0 - R_0 z_0 = h(t), \quad z_0(t_0,0) = z^0; \tag{$\bar{9}_1$}$$

$$Lz_1(t,\tau) = -t^{(1-\alpha)}\frac{\partial z_0}{\partial t} + \left[\frac{g(t)}{2}(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2)z_0\right]^{\wedge} + R_1 z_0, \quad z_1(t_0,0) = 0; \tag{$\bar{8}_1$}$$

$$Lz_2(t,\tau) = -t^{(1-\alpha)}\frac{\partial z_1}{\partial t} + \left[\frac{g(t)}{2}(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2)z_1\right]^{\wedge} + R_1 z_1 + R_2 z_0, \quad z_2(t_0,0) = 0; \tag{$\bar{8}_2$}$$

....

$$Lz_k(t,\tau) = -t^{(1-\alpha)}\frac{\partial z_{k-1}}{\partial t} + \left[\frac{g(t)}{2}(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2)z_{k-1}\right]^{\wedge} + R_k z_0 + \ldots + R_1 z_{k-1},$$

$$z_k(t_0,0) = 0,\ k \geq 1. \tag{$\bar{8}_k$}$$

(images of linear operators $\frac{\partial}{\partial t}$ and R_ν do not need to be embedding in space U, since these operators operate from U to U). Such a change will not affect the construction of the asymptotic solution of the original problem (1) (or the equivalent problem (2)), so on the restriction $\tau = \frac{\psi(t)}{\varepsilon}$ series of problems $(\bar{8}_k)$ will coincide with a series of problems (8_k) (see [6; 234–235].

Applying Theorems 1 and 2 to iterative problems (8_k) (in this case, the right-hand sides $H^{(k)}(t,\tau)$ of these problems are embedded in the space U, i.e. $H^{(k)}(t,\tau)$ we replace with $\hat{H}^{(k)}(t,\tau) \in U$), we find uniquely their solutions in space U and construct series (6). Just as in [13, 14], we prove the following statement.

Theorem 3. Suppose that conditions 1), 2) are satisfied for equation (2). Then, when $\varepsilon \in (0,\varepsilon_0](\varepsilon_0 > 0$ is sufficiently small), equation (2) has a unique solution $z(t,\varepsilon) \in C^1([t_0,T], \mathbf{C})$, in this case, the estimate

$$||z(t,\varepsilon) - z_{\varepsilon N}(t)||_{C[t_0,T]} \leq c_N \varepsilon^{N+1}, \ \ N = 0,1,2,\ldots$$

holds true, where $z_{\varepsilon N}(t)$ is the restriction (for $\tau = \frac{\psi(t)}{\varepsilon}$) of the N – partial sum of series (6) (with coefficients $z_k(t,\tau) \in U$, satisfying the iteration problems (8_k)), and the constant $c_N > 0$ does not depend on $\varepsilon \in (0,\varepsilon_0]$.

Construction of the solution of the first iteration problem

Using Theorem 1, we will try to find a solution to the first iteration problem $(\bar{8}_0)$. Since the right side $h(t)$ of the equation $(\bar{8}_0)$ satisfies condition (10), this equation has (according to (15)) a solution in the space U in the form

$$z_0(t,\tau) = z_0^{(0)}(t) + \alpha_1^{(0)}(t)e^{\tau_1} \tag{18}$$

where $\alpha_1^{(0)}(t) \in C^\infty([t_0,T], \mathbf{C})$ are arbitrary function, $y_0^{(0)}(t)$ is the solution of the integral equation

$$z_0^{(0)}(t) = \int_{t_0}^{t} \left(-\lambda_1^{-1}(t)K(t,s)\right) z_0^{(0)}(s)ds - \lambda_1^{-1}(t)h(t). \tag{19}$$

Subordinating (18) to the initial condition $z_0(t_0,0) = z^0$, we have

$$z_0^{(0)}(t_0) + \alpha_1^{(0)}(t_0) = z^0 \ \ \Leftrightarrow \ \ \alpha_1^{(0)}(t_0) = z^0 - z_0^{(0)}(t_0) \Leftrightarrow \alpha_1^{(0)}(t_0) = z^0 + \lambda_1^{-1}(t_0)h(t_0).$$

To fully compute the function $\alpha_1^{(0)}(t)$, we proceed to the next iteration problem $(\bar{8}_1)$. Substituting into it the solution (18) of the equation $(\bar{8}_0)$, we arrive at the following equation:

$$Lz_1(t,\tau) = -t^{(1-\alpha)}\frac{\partial}{\partial t}z_0^{(0)}(t) - t^{(1-\alpha)}\frac{\partial}{\partial t}\left(\alpha_1^{(0)}(t)\right)e^{\tau_1} + \left[\frac{g(t)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left(z_0^{(0)}(t) + \alpha_1^{(0)}(t)e^{\tau_1}\right)\right]^{\wedge} +$$

$$+\frac{K(t,t)\alpha_1^{(0)}(t)}{t^{(1-\alpha)}\lambda_1(t)}e^{\tau_1}-\frac{K(t,t_0)\alpha_1^{(0)}(t_0)}{t_0{}^{(1-\alpha)}\lambda_1(t_0)}+\sum_{j=2}^{3}\left[\frac{K(t,t)z_j^{(0)}(t)}{\lambda_j(t)}e^{\tau_j}-\frac{K(t,t_0)z_j^{(0)}(t_0)}{\lambda_j(t_0)}\right]$$

(here, we used the expression (6_1) for $R_1z(t,\tau)$ and took into account that when $z(t,\tau)=z_0(t,\tau)$ the sum (6_1) contains only terms with e^{τ_1}).

Let us calculate

$$M=\left[\frac{g(t)}{2}\left(e^{\tau_2}\sigma_1+e^{\tau_3}\sigma_2\right)\left(z_0^{(0)}(t)+\alpha_1^{(0)}(t)e^{\tau_1}\right)\right]^{\wedge}=$$

$$=\frac{g(t)}{2}\left\{\sigma_1 z_0^{(0)}(t)e^{\tau_2}+\sigma_2 z_0^{(0)}(t)e^{\tau_3}+\sigma_1\alpha_1^{(0)}(t)e^{\tau_2+\tau_1}+\sigma_2\alpha_1^{(0)}(t)e^{\tau_3+\tau_1}\right\}^{\wedge}.$$

Let us analyze the exponents of the second dimension included here for their resonance:

$$e^{\tau_2+\tau_1}|_{\tau=\psi(t)/\varepsilon}=e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\left(-i\beta'(\theta)+A(\theta)\right)d\theta},\qquad e^{\tau_2+\tau_1}|_{\tau=\psi(t)/\varepsilon}=e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\left(-i\beta'(\theta)+A(\theta)\right)d\theta},$$

$$-i\beta'+A=\left[\begin{array}{l}0,\\ A,\\ -i\beta',\\ +i\beta',\end{array}\right.\Leftrightarrow\emptyset,\qquad -i\beta'+A=\left[\begin{array}{l}0,\\ A,\\ -i\beta',\\ +i\beta',\end{array}\right.\Leftrightarrow\emptyset.$$

Thus, exponents $e^{\tau_2+\tau_1}$ and $e^{\tau_3+\tau_1}$ are not resonant. Then, for solvability, equation (18) it is necessary and sufficient that the condition

$$-t^{(1-\alpha)}\frac{\partial}{\partial t}\left(\alpha_1^{(0)}(t)\right)+\frac{K(t,t)\alpha_1^{(0)}(t)}{t^{(1-\alpha)}\lambda_1(t)}=0$$

is satisfied. Attaching the initial condition

$$\alpha_1^{(0)}(t_0)=z^0+\lambda_1^{-1}(t_0)h(t_0)$$

to this equation, we find

$$\alpha_1^{(0)}(t)=\alpha_1^{(0)}(t_0)e^{\int\limits_{t_0}^{t}\left(\frac{K(\theta,\theta)}{\theta^{2(1-\alpha)}\lambda_1(\theta)}\right)d\theta}$$

and therefore, we uniquely calculate the solution (18) of the problem $(\bar{9}_0)$ in the space U. Moreover, the main term of the asymptotic of the solution to problem (2) has the form

$$z_{\varepsilon 0}(t)=z_0^{(0)}(t)+\alpha_1^{(0)}(t_0)e^{\int\limits_{t_0}^{t}\left(\frac{K(\theta,\theta)}{\theta^{2(1-\alpha)}\lambda_1(\theta)}\right)d\theta+\frac{1}{\varepsilon}\int\limits_{t_0}^{t}\lambda_1(\theta)d\theta}\qquad(20)$$

where $\alpha_1^{(0)}(t_0)=z^0+\lambda_1^{-1}(t_0)h(t_0)$, $z_0^{(0)}(t)$ is the solution of the integrated equation (19). From expression (20) for $z_{\varepsilon 0}(t)$ it is clear that $z_{\varepsilon 0}(t)$ is independent of rapidly oscillating terms. However, already in the next approximation, their influence on the asymptotic solution of problem (2) is revealed.

References

1 Ломов С.А. Введение в общую теорию сингулярных возмущений / С.А. Ломов. – М.: Наука, 1981. – 400 с.

2 Ломов С.А. Основы математической теории пограничного слоя / С.А. Ломов, И.С. Ломов. – М.: Изд-во Моск. гос. ун-та, 2011.

3 Kalimbetov B.T. Regularized asymptotics of solutions for systems of singularly perturbed diffe-rential equations of fractional order / B.T. Kalimbetov // Intern. Jour. Fuzzy Math. Archive. – 2019. – P. 67–74. DOI: http://dx.doi.org/10.22457/ijfma.v16n1a9.

4 Kalimbetov B.T. On the Question of asymptotic integration of singularly perturbed fractional-order problems / B.T. Kalimbetov // Asian Jour. of Fuzzy and Appl. Math. – 2019. – 6 (3). – P. 44–49. DOI: https://doi.org/10.24203/ ajfam.v6:3.5600.

5 Kalimbetov B.T. Internal boundary layer in a singularly perturbed problem of fractional derivative / B.T. Kalimbetov, A.N. Temirbekov, B.I. Yeskaraeva // Bulletin of the Karaganda University. Mathematics series. – 100 (4). – P. 92–100. DOI 10.31489/2020M4/92-100

6 Kalimbetov B.T. On the asymptotic solutions of singularly perturbed differential systems of fractional order / B.T. Kalimbetov, E. Abylkasymova, G. Beissenova // Jour. of Math. and Comp. Science. – 24. – P. 165–172. doi: 10.22436/jmcs.024.02.07.

7 Kalimbetov B.T. Integro-differentiated singularly perturbed equations with fast oscillating coefficients / B.T. Kalimbetov, V.F. Safonov // Bulletin of the Karaganda University. Mathematics series. – 2019. – 94(2). – P. 33–47. DOI 10.31489/2019M2/33-47

8 Bobodzhanov A.A. Integro-differential problem about para-metric amplification and its asymptotical integration / A.A. Bobodzhanov, B.T. Kalimbetov, V.F. Safonov // International Journal of Applied Mathematics. – 2020. – 33 (2). — P. 331–353. doi.org/10.12732/ijam.v33i2.12.

9 Kalimbetov B.T. Asymptotic solutions of scalar integro-differential equations with partial derivatives and with fast oscillating coefficients / B.T. Kalimbetov, A.N. Temirbekov, A.S. Tulep // EJPAM. – 2020. – 13 (2). – P. 287–302. doi.org/10.29020/nybg.ejpam.v13i2.3664.

10 Kalimbetov B.T. Regularization method for singularly perturbed integro-differential equations with rapidly oscillating coefficients and rapidly changing kernels / B.T. Kalimbetov, V.F. Safonov // Axioms. – 2020. – 9 (4). – 131. https://doi.org/10.3390/axioms9040131.

11 Kalimbetov B.T. Asymptotic solutions of scalar integro-differential equations with partial derivatives and with rapidly oscillating coefficients / B.T. Kalimbetov, Kh.F. Etmishev // Bulletin of the Karaganda University. Mathematics series. – 2020. – 97 (1). – P. 52–67. DOI 10.31489/2020M1/52-67.

12 Khalil R. A new definition of fractional derivative / A. Khalil, M.Al Horani, A. Yousef, M. Sababheh // Jour. Comput. Appl. Math. – 2014. – 264. – P. 65–70.

13 Сафонов В.Ф. Курс высшей математики. Сингулярно-возмущенные задачи и метод регуляризации: учеб. пос. / В.Ф. Сафонов, А.А. Бободжанов. – М.: Изд. дом МЭИ, 2012. – 414 с.

14 Бободжанов А.А. Метод регуляризации для сингулярно-возмущенных задач с быстро осциллирующими коэффициентами / А.А. Бободжанов, Б.Т. Калимбетов, В.Ф. Сафонов. – Шымкент: Алем, 2020. – 312 с.

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Жылдам осцилляцияланатын коэффициентті бөлшек ретті туындылы сингуляр ауытқыған интегро-дифференциалдық теңдеудің асимптотикасы

Мақалада С.А.Ломовтың регуляризация әдісі жылдам осцилляцияланатын коэффициенттері бар бөлшек-ретті туындылы интегро-дифференциалдық теңдеуі жалпыланған. Жұмыстың басты мақсаты — осцилляцияланатын компоненттердің есептік шешімінің асимптотикасының структурасына әсерін зерттеу болып табылады. Резонанстың болмауы жағдайы қарастырылған, яғни, жылдам тербелетін біртектіліксіздіктің бүтін сызықтық комбинациясы берілген уақыт интервалының барлық нүктелеріндегі шекті операторының спектрінің мәндеріне сәйкес келмейтін жағдай зерттелген. Шекті оператор спектрімен жылдам тербелетін біртектіліксіздіктің жиілігінің сәйкес келу жағдайы резонанстық жағдай деп аталады. Бұл жағдайдың зерттелуі келесі еңбекте жоспарланған. Резонанстың күрделі жағдайлары (мысалы, тепе-теңдік резонансы) мұқият талдауды қажет етеді және бұл жұмыста қарастырылмаған.

Кілт сөздер: сингуляр ауытқу, бөлшек ретті туындылы интегро-дифференциалдық теңдеу, итерациялық есептер, итерацион есептердің шешімділігі.

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Асимптотика решений сингулярно-возмущенного интегро-дифференциального уравнения дробного порядка с быстро осциллирующими коэффициентами

В статье метод регуляризации С.А. Ломова обобщен на сингулярно-возмущенное интегро-дифференциальное уравнение дробного производного с быстро осциллирующими коэффициентами. Основная цель работы — выявить влияние осциллирующих составляющих на структуру асимптотики решения этой задачи. Рассмотрен случай отсутствия резонанса, т.е. случай, когда целочисленная линейная комбинация быстро осциллирующей неоднородности не совпадает с точкой спектра предельного оператора на всех точках рассматриваемого отрезка времени. Случай совпадения частоты быстро осциллирующей неоднородности с точкой спектра предельного оператора называется резонансным. Данный случай предполагается изучить в наших последующих работах. Более сложные случаи резонанса (например, точечный резонанс) требуют тщательного подхода и в данной работе не будут рассматриваться.

Ключевые слова: сингулярно-возмущенное, интегро-дифференциальное уравнение производного дробного порядка, итерационные задачи, разрешимость итерационных задач.

References

1 Lomov, S.A. (1981). *Vvedenie v obshchuiu teoriiu singuliarnykh vozmushchenii [Introduction to the general theory of singular perturbations]*. Moscow: Nauka [in Russian].

2 Lomov, S.A. & Lomov, I.S. (2011). *Osnovy matematicheskoi teorii pogranichnogo sloia [Founda-tions of mathematical theory of boundary layer]*. Moscow: Izdatelstvo Moskovskogo gosudarstvennogo universiteta [in Russian].

3 Kalimbetov, B.T. (2019). Regularized asymptotics of solutions for systems of singularly perturbed differential equations of fractional order. *Intern. Jour. Fuzzy Math. Archive*, 67–74. DOI: http:// dx.doi.org/ 10.22457/ijfma.v16n1a9.

4 Kalimbetov, B.T. (2019). On the Question of asymptotic integration of singularly perturbed fractional-order problems. *Asian Jour. of Fuzzy and Appl. Math., 6*(3), 44–49. DOI: https://doi. org/10.24203/ ajfam.v6:3.5600.

5 Kalimbetov, B.T., Temirbekov, A.N., & Yeskaraev,a B.I. (2020). Internal boundary layer in a singularly perturbed problem of fractional derivative. *Bulletin of KarSU, Series Mathematics, 100*(4), 92–100. DOI 10.31489/2020M4/92-100

6 Kalimbetov, B.T., Abylkasymova, E., & Beissenova, G. On the asymptotic solutions of singularly perturbed differential systems of fractional order. *Jour. of Math. and Comp. Science, 24*, 165–172. doi: 10.22436/jmcs. 024.02.07.

7 Kalimbetov, B.T., & Safonov, V.F. (2019). Integro-differentiated singularly perturbed equations with fast oscillating coefficients. *Bulletin of KarU, series Mathematics, 94*(2), 33–47. DOI 10.31489/2019M2/33-47

8 Bobodzhanov, A.A., Kalimbetov, B.T., & Safonov, V.F. (2020). Integro-differential problem about parametric amplification and its asymptotical integration. *International Journal of Applied Mathematics, 33*(2), 331–353. doi.org/10.12732/ijam.v33i2.12.

9 Kalimbetov, B.T., Temirbekov, A.N., & Tulep, A.S. (2020). Asymptotic solutions of scalar integro-differential equations with partial derivatives and with fast oscillating coefficients. *EJPAM, 13*(2), 287–302. doi.org/10.29020/nybg.ejpam.v13i2.3664.

10 Kalimbetov, B.T., & Safonov, V.F. (2020). Regularization method for singularly perturbed integro-differential equations with rapidly oscillating coefficients and rapidly changing kernels. *Axioms, 9*(4), 131. https://doi.org/10.3390/axioms9040131.

11 Kalimbetov, B.T. & Etmishev, Kh.F. (2020). Asymptotic solutions of scalar integro-differential equations with partial derivatives and with rapidly oscillating coefficients. *Bulletin of KarU, series Mathematics, 97*(1), 52–67. DOI 10.31489/2020M1/52-67.

12 Khalil, R., Al Horani, M., Yousef, A., & Sababheh, M. (2014). A new definition of fractional derivative. *Jour. Comput. Appl. Math., 264*, 65–70.

13 Safonov, V.F. & Bobodzhanov, A.A. (2012). *Kurs vysshei matematiki. Singuliarno-vozmushchennye zadachi i metod reguliarizatsii [The course of higher mathematics. Singularly perturbed problems and regularization method]*. Moscow: Publishing House MPEI [in Russian].

14 Bobodzhanov, A.A., Kalimbetov, B.T., & Safonov, V.F. (2020). *Metod reguliarizatsii dlia singuliarno-vozmushchennykh zadach s bystro ostsilliruiushchimi koeffitsientami [Regularization method for singularly perturbed problems with rapidly oscillating coefficients]*. Shymkent: Alem [in Russian].