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Partial best approximations and the absolute Cesaro summability of multiple Fourier series

The article is devoted to the problem of absolute Cesaro summability of multiple trigonometric Fourier series. Taking a central place in the theory of Fourier series this problem was developed quite widely in the one-dimensional case and the fundamental results of this theory are set forth in the famous monographs by N.K. Bari, A. Zigmund, R. Edwards, B.S. Kashin and A.A. Saakyan [1–4]. In the case of multiple series, the corresponding theory is not well developed. The multidimensional case has own specifics and the analogy with the one-dimensional case does not always be unambiguous and obvious. In this article, we obtain sufficient conditions for the absolute summability of multiple Fourier series of the function $f \in L_q(I_s)$ in terms of partial best approximations of this function. Four theorems are proved and four different sufficient conditions for the $|C;\bar{\beta}|_\lambda$-summability of the Fourier series of the function f are obtained. In the first theorem, a sufficient condition for the absolute $|C;\bar{\beta}|_\lambda$- summability of the Fourier series of the function f is obtained in terms of the partial best approximation of this function which consists of s conditions, in the case when $\beta_1 = ... = \beta_s = \frac{1}{q'}$. Other sufficient conditions are obtained for double Fourier series. Sufficient conditions for the $|C;\beta_1;\beta_2|_\lambda$-summability of the Fourier series of the function $f \in L_q(I_2)$ are obtained in the cases $\beta_1 = \frac{1}{q'}$, $-1 < \beta_2 < \frac{1}{q'}$ (in the second theorem), $\frac{1}{q'} < \beta_1 < +\infty$, $\beta_2 = \frac{1}{q'}$ (in the third theorem), $-1 < \beta_1 < \frac{1}{q'}$, $\frac{1}{q'} < \beta_2 < +\infty$ (in the fourth theorem).

Keywords: trigonometric series, Fourier series, Lebesgue space, partial best approximation of a function, absolute summability of the series.

Introduction

Let R^s be a s-dimensional Euclidean space of points $\bar{x} = (x_1, x_2, ..., x_s)$ with real coordinates; $I_s = \{\bar{x} \in R^s : 0 \le x_j \le 2\pi, \quad j = 1, 2, ..., s\}$ is a s-dimensional cube.

We put $\gamma_i(nx) = \begin{cases} \cos nx, & i = 1, \\ \sin nx, & i = 2 \end{cases}.$

We will consider the following multiple series

$$\sum_{\bar{n} \ge 1} B_{\bar{n}}(\bar{x}) = \sum_{n_1=1}^{\infty} \dots \sum_{n_s=1}^{\infty} B_{n_1,...,n_s}(x_1, ..., x_s), \tag{1}$$

where $\bar{n} \ge \bar{\alpha} = (\alpha_1, ..., \alpha_s)$ means $n_j \ge \alpha_j$ for all $j = 1, 2, ..., s$;

$$B_{\bar{n}}(\bar{x}) = \sum_{\bar{1} \le \bar{i} \le \bar{2}} a_{\bar{n}}^{(\bar{i})} \cdot \prod_{\nu=1}^{s} \gamma_{i_\nu}(n_\nu x_\nu).$$

Assume that $A_n^{(\beta)} = \frac{(\beta+1)(\beta+2)...(\beta+n)}{n!}, \beta \in R$, where n is natural number.

The sum

$$\sigma_{\bar{n}}^{(\bar{\beta})}(\bar{x}) = \sum_{\bar{1} \le \bar{k} \le \bar{n}} \prod_{j=1}^{s} A_{n_j - k_j}^{(\beta_j - 1)} \left(A_{n_j}^{(\beta_j)}\right)^{-1} B_{\bar{k}}(\bar{x})$$

is called $(C;\bar{\beta}) \equiv (C;\beta_1, ..., \beta_s)$ average of the series (1).

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For a given number $b_{\bar{n}}$ we define the mixed difference as follows:

$$\Delta b_{\bar{n}} = \sum_{\bar{0}\le\bar{\varepsilon}\le\bar{1}} (-1)^{s-\sum\limits_{i=1}^{s}\varepsilon_i} \cdot b_{\bar{n}-\bar{1}+\bar{\varepsilon}}.$$

Here $\bar{\varepsilon} = (\varepsilon_1, ..., \varepsilon_s)$. The series (1) is called $\left|C;\bar{\beta}\right|_\lambda$ - summable (or absolutely Cesaro summable), $\lambda \ge 1$, at the point $\bar{x} \in I_s$, if [5]:

$$\sum_{\bar{n}\ge\bar{1}} \left|\Delta\sigma_{\bar{n}}^{(\bar{\beta})}(\bar{x})\right|^\lambda \cdot \prod_{j=1}^{s} n_j^{\lambda-1} < +\infty.$$

Further, we put $\rho_{\bar{k}} = \sqrt{\sum\limits_{\bar{1}\le\bar{i}\le\bar{2}} \left|a_{\bar{k}}^{(\bar{i})}\right|^2}$.

By $L_q(I_s)$ we denote the space of all Lebesgue measurable, 2π-periodic in each variable functions $f(\bar{x})$, for which

$$\|f\|_q = \left(\int\limits_{I_s} |f(\bar{x})|^q d\bar{x}\right)^{\frac{1}{q}} < +\infty, \quad 1 \le q < +\infty.$$

Let

$$E_{n_1,...,n_s}(f)_q = \inf_{T_{n_1,...,n_s}} \|f(\cdot) - T_{n_1,...,n_s}(\cdot)\|_q$$

be the full best approximation [6] of the function f by trigonometric polynomials of order not exceeding n_j in variables $x_j (j = 1, .., s)$. Let us also consider the partial best approximation of the function f which is determined by the formula [6]:

$$E_{n,\infty}^{(j)}(f)_q = \inf_{T_n} \left\|f(x_1, ..., x_s) - T_n(x_1, ..., x_{j-1}, (x_j), x_{j+1}, ..., x_s)\right\|_q,$$

where $T_n(x_1, ..., x_{j-1}, (x_j), x_{j+1}, ..., x_s)$ is a trigonometric polynomial in the variable x_j of order not exceeding n with coefficients from the space $L_q(I_{s-1})$.

It is known that [6; 44]:

$$E_{n_1,...,n_s}(f)_q \le C \cdot \sum_{j=1}^{s} E_{n_j,\infty}^{(j)}(f)_q, \quad 1 \le q < \infty.$$

Conditions for the absolute summability of series (1) in the case $\lambda = 1$, $s = 2$, $0 < \beta_j < \frac{1}{2}$, $j = 1, 2$ were investigated by I.E. Zhak and M.F. Timan [7], and the questions of $|C;\bar{\beta}|_\lambda$ summability of the Fourier series of the function $f \in L_2(I_s)$ were studied by Yu.A. Ponomarenko, M.F. Timan [5], and in the one-dimensional case these questions were studied by I. Szalay [8]. Questions of the absolute summability of multiple trigonometric series were also investigated in [9–18].

Results

Theorem 1. Let $1 < q \le 2$, $1 \le \lambda \le q$, $\beta_1 = ... = \beta_s = \frac{1}{q'}$, $\frac{1}{q} + \frac{1}{q'} = 1$. If $f \in L_q(I_s)$ and

$$\sum_{n=3}^{\infty} n^{\frac{s\lambda}{q}(2-q)-1} (\ln n)^{\frac{s\lambda}{q}} \left(E_{n,\infty}^{(j)}(f)_q\right)^\lambda < +\infty, \quad j = 1, ..., s,$$

then the Fourier series of the function $f \in L_q(I_s)$ will be $|C;\bar{\beta}|_\lambda$ -summable almost everywhere on I_2.

Proof. We prove the theorem for $s = 2$ (the methods are similar for the higher dimensions). In [9] it was proved that if $1 < q \le 2$, $1 \le \lambda \le q$, $\beta_1 = \beta_2 = \frac{1}{q'}$, $\frac{1}{q} + \frac{1}{q'} = 1$ and

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}} \sum_{k_2=2^{n_2}+1}^{2^{n_2+1}} \rho_{k_1k_2}^q \cdot \ln k_1 \ln k_2 \right)^{\frac{\lambda}{q}} < +\infty,$$

then a double series of the form (1) will be $|C;\beta_1;\beta_2|_\lambda$-summable almost everywhere on I_2.

Let's estimate the last series. For this we use the following inequality

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot\ln k_1\ln k_2\right)^{\frac{\lambda}{q}}\leq$$

$$\leq\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{n_1}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot\ln k_1\ln k_2\right)^{\frac{\lambda}{q}}+$$

$$+\sum_{n_2=0}^{\infty}\sum_{n_1=0}^{n_2}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot\ln k_1\ln k_2\right)^{\frac{\lambda}{q}}. \tag{2}$$

To estimate the first term, we apply Holder's inequalities to the sum over n_2 for $\theta=\frac{q}{\lambda}$, $\frac{1}{\theta}+\frac{1}{\theta'}=1$. We get

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{n_1}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot\ln k_1\ln k_2\right)^{\frac{\lambda}{q}}\leq$$

$$\leq C\cdot\sum_{n_1=0}^{\infty}2^{n_1(2-q)\frac{\lambda}{q}}(n_1+1)^{\frac{\lambda}{q}}\cdot\sum_{n_2=0}^{n_1}2^{n_2(2-q)\frac{\lambda}{q}}(n_2+1)^{\frac{\lambda}{q}}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}(k_1k_2)^{q-2}\rho_{k_1k_2}^{q}\right)^{\frac{\lambda}{q}}\leq$$

$$\leq C\cdot\sum_{n_1=0}^{\infty}2^{n_1(2-q)\frac{\lambda}{q}}(n_1+1)^{\frac{\lambda}{q}}\cdot\left(\sum_{n_2=0}^{n_1}\left(2^{n_2(2-q)\frac{\lambda}{q}}(n_2+1)^{\frac{\lambda}{q}}\right)^{\theta'}\right)^{\frac{1}{\theta'}}\times$$

$$\times\left(\sum_{n_2=0}^{n_1}\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}(k_1k_2)^{q-2}\rho_{k_1k_2}^{q}\right)^{\frac{\lambda}{q}}\leq$$

$$\leq C\cdot\sum_{n_1=0}^{\infty}2^{n_1(2-q)\frac{\lambda}{q}}(n_1+1)^{\frac{\lambda}{q}}\cdot\left(\sum_{n_2=0}^{n_1}2^{n_2(2-q)\frac{\lambda}{q-\lambda}}(n_2+1)^{\frac{\lambda}{q-\lambda}}\right)^{\frac{q-\lambda}{q}}\times$$

$$\times\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2}^{2^{n_1+1}}(k_1k_2)^{q-2}\rho_{k_1k_2}^{q}\right)^{\frac{\lambda}{q}}. \tag{3}$$

Now let's estimate the following sum:

$$\sum_{n_2=0}^{n_1}2^{n_2(2-q)\frac{\lambda}{q-\lambda}}(n_2+1)^{\frac{\lambda}{q-\lambda}}\leq(n_1+1)^{\frac{\lambda}{q-\lambda}}\sum_{n_2=0}^{n_1}2^{n_2(2-q)\frac{\lambda}{q-\lambda}}\leq C(n_1+1)^{\frac{\lambda}{q-\lambda}}2^{n_1(2-q)\frac{\lambda}{q-\lambda}}.$$

Using this inequality from (3), we obtain

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{n_1}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot\ln k_1\ln k_2\right)^{\frac{\lambda}{q}}\leq$$

$$\leq C\cdot\sum_{n_1=0}^{\infty}2^{2n_1(2-q)\frac{\lambda}{q}}(n_1+1)^{2\frac{\lambda}{q}}\cdot\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2}^{2^{n_1+1}}(k_1k_2)^{q-2}\rho_{k_1k_2}^{q}\right)^{\frac{\lambda}{q}}.$$

It is similarly proved

$$\sum_{n_2=0}^{\infty}\sum_{n_1=0}^{n_2}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot\ln k_1\ln k_2\right)^{\frac{\lambda}{q}}\leq$$

$$\leq C\cdot\sum_{n_2=0}^{\infty}2^{2n_2(2-q)\frac{\lambda}{q}}(n_2+1)^{2\frac{\lambda}{q}}\cdot\left(\sum_{k_1=2^{n_2}+1}^{2^{n_2+1}}\sum_{k_2=2}^{2^{n_2+1}}(k_1k_2)^{q-2}\rho_{k_1k_2}^{q}\right)^{\frac{\lambda}{q}}.$$

Taking into account the last two inequalities from (2), we obtain

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot\ln k_1\ln k_2\right)^{\frac{\lambda}{q}}\leq$$

$$\leq C\cdot\sum_{n=0}^{\infty}2^{2n(2-q)\frac{\lambda}{q}}(n+1)^{2\frac{\lambda}{q}}\cdot\left(\sum_{k_1=1}^{2^{n+1}}\sum_{k_2=2^{n}+1}^{2^{n+1}}(k_1k_2)^{q-2}\rho_{k_1k_2}^{q}\right)^{\frac{\lambda}{q}}. \quad (4)$$

Further, applying the Hardy-Littlewood theorem [19] and using the monotonicity of the logarithmic function and the best approximation from (4), we have

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot\ln k_1\ln k_2\right)^{\frac{\lambda}{q}}\leq$$

$$\leq C\cdot\sum_{n=0}^{\infty}2^{2n(2-q)\frac{\lambda}{q}}(n+1)^{2\frac{\lambda}{q}}\cdot E_{2^n,2^n}^{\lambda}(f)_q<C\cdot E_{1,1}^{\lambda}(f)_q+C\cdot E_{2,2}^{\lambda}(f)_q+$$

$$+C\cdot\sum_{n=2}^{\infty}(2^n)^{\frac{2\lambda}{q}(2-q)-1}(n-1)^{\frac{2\lambda}{q}}\cdot E_{2^n,2^n}^{\lambda}(f)_q\cdot\sum_{m=2^{n-1}+1}^{2^n}1\leq$$

$$\leq C\cdot E_{1,1}^{\lambda}(f)_q+C\cdot E_{2,2}^{\lambda}(f)_q+C\sum_{m=3}^{\infty}m^{\frac{2\lambda}{q}(2-q)-1}(\ln m)^{\frac{2\lambda}{q}}\cdot E_{m,m}^{\lambda}(f)_q.$$

The theorem is proved.

Theorem 2. Let $1<q\leq 2$, $1\leq\lambda\leq q$, $\beta_1=\frac{1}{q'}$, $-1<\beta_2<\frac{1}{q'}$, $\frac{1}{q}+\frac{1}{q'}=1$. If $f\in L_q(I_2)$ and

$$\sum_{n=3}^{\infty}n^{\frac{\lambda}{q}(3-q(1+\beta_2))-1}(\ln n)^{\frac{\lambda}{q}}\left(E_{n,\infty}^{(j)}(f)_q\right)^{\lambda}<+\infty,\quad j=1,2,$$

then the Fourier series of function $f\in L_q(I_2)$ will be $|C;\beta_1;\beta_2|_\lambda$ -summable almost everywhere on I_2.

Proof. Since $1<q\leq 2$, $1\leq\lambda\leq q$, $\beta_1=\frac{1}{q'}$, $-1<\beta_2<\frac{1}{q'}$, $\frac{1}{q}+\frac{1}{q'}=1$, then for $|C;\beta_1;\beta_2|_\lambda$- summability almost everywhere on I_2 of a double series of the form (1) is sufficient that [12]

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot\ln k_1\cdot k_2^{q(1-\beta_2)-1}\right)^{\frac{\lambda}{q}}<+\infty.$$

It is known that

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot\ln k_1\cdot k_2^{q(1-\beta_2)-1}\right)^{\frac{\lambda}{q}}\leq$$

$$\leq\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{n_1}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot\ln k_1\cdot k_2^{q(1-\beta_2)-1}\right)^{\frac{\lambda}{q}}+$$

$$+\sum_{n_2=0}^{\infty}\sum_{n_1=0}^{n_2}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot\ln k_1\cdot k_2^{q(1-\beta_2)-1}\right)^{\frac{\lambda}{q}}. \quad (5)$$

Let us estimate the first term. To do this, we apply Holder's inequalities to the sum over n_2 for $\theta = \frac{q}{\lambda}$, $\frac{1}{\theta} + \frac{1}{\theta'} = 1$. We get

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{n_1}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot\ln k_1\cdot k_2^{q(1-\beta_2)-1}\right)^{\frac{\lambda}{q}} \leq$$

$$\leq C\cdot\sum_{n_1=0}^{\infty}2^{n_1(2-q)\frac{\lambda}{q}}(n_1+1)^{\frac{\lambda}{q}}\cdot\sum_{n_2=0}^{n_1}2^{n_2(1-q\beta_2)\frac{\lambda}{q}}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}(k_1k_2)^{q-2}\rho_{k_1k_2}^{q}\right)^{\frac{\lambda}{q}} \leq$$

$$\leq C\cdot\sum_{n_1=0}^{\infty}2^{n_1(3-q(1+\beta_2))\frac{\lambda}{q}}(n_1+1)^{\frac{\lambda}{q}}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2}^{2^{n_1+1}}(k_1k_2)^{q-2}\rho_{k_1k_2}^{q}\right)^{\frac{\lambda}{q}}.$$

For the second term (5), we similarly obtain

$$\sum_{n_2=0}^{\infty}\sum_{n_1=0}^{n_2}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot\ln k_1\cdot k_2^{q(1-\beta_2)-1}\right)^{\frac{\lambda}{q}} \leq$$

$$\leq C\cdot\sum_{n_2=0}^{\infty}2^{n_2(3-q(1+\beta_2))\frac{\lambda}{q}}(n_2+1)^{\frac{\lambda}{q}}\cdot\left(\sum_{k_2=2^{n_1}+1}^{2^{n_2+1}}\sum_{k_1=2}^{2^{n_2+1}}(k_1k_2)^{q-2}\rho_{k_1k_2}^{q}\right)^{\frac{\lambda}{q}}.$$

Therefore

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot\ln k_1\cdot k_2^{q(1-\beta_2)-1}\right)^{\frac{\lambda}{q}} \leq$$

$$\leq C\cdot\sum_{n=0}^{\infty}2^{n(3-q(1+\beta_2))\frac{\lambda}{q}}(n+1)^{\frac{\lambda}{q}}\cdot\left(\sum_{k_2=2^{n}+1}^{2^{n+1}}\sum_{k_1=2}^{2^{n+1}}(k_1k_2)^{q-2}\rho_{k_1k_2}^{q}\right)^{\frac{\lambda}{q}}. \quad (6)$$

Now, applying the Hardy-Littlewood theorem [19] and using the monotonicity of the logarithmic function and the best approximation from (6), we obtain

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot\ln k_1\cdot k_2^{q(1-\beta_2)-1}\right)^{\frac{\lambda}{q}} \leq$$

$$\leq C\cdot E_{1,1}^{\lambda}(f)_q + C\cdot E_{2,2}^{\lambda}(f)_q + C\sum_{m=3}^{\infty}m^{\frac{\lambda}{q}(3-q(1+\beta_2))-1}(\ln m)^{\frac{\lambda}{q}}\cdot E_{m,m}^{\lambda}(f)_q.$$

The theorem is proved.

Theorem 3. Let $1 < q \leq 2$, $1 \leq \lambda \leq q$, $\frac{1}{q'} < \beta_1 < +\infty$, $\beta_2 = \frac{1}{q'}$, $\frac{1}{q} + \frac{1}{q'} = 1$. If $f \in L_q(I_2)$ and

$$\sum_{n=3}^{\infty}n^{2\frac{\lambda}{q}(2-q)-1}(\ln n)^{\frac{\lambda}{q}}\left(E_{n,\infty}^{(j)}(f)_q\right)^{\lambda} < +\infty, \quad j = 1, 2,$$

then the Fourier series of function $f \in L_q(I_2)$ will be $|C;\beta_1;\beta_2|_\lambda$ -summable almost everywhere on I_2.

Proof. It was proved in [12] that, if $1 < q \leq 2$, $1 \leq \lambda \leq q$, $\frac{1}{q'} < \beta_1 < +\infty$, $\beta_2 = \frac{1}{q'}$, $\frac{1}{q} + \frac{1}{q'} = 1$ and

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot\ln k_2\right)^{\frac{\lambda}{q}} < +\infty,$$

then a double series of the form (1) is $|C;\beta_1;\beta_2|_\lambda$- summable almost everywhere on I_2.

Similarly to the previous theorem, we obtain

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot\ln k_2\right)^{\frac{\lambda}{q}}\le$$

$$\le C\cdot\sum_{n=0}^{\infty}2^{2n(2-q)\frac{\lambda}{q}}(n+1)^{\frac{\lambda}{q}}\cdot\left(\sum_{k_2=2^{n}+1}^{2^{n+1}}\sum_{k_1=2}^{2^{n+1}}(k_1k_2)^{q-2}\rho_{k_1k_2}^{q}\right)^{\frac{\lambda}{q}}.$$

Hence, applying the Hardy-Littlewood theorem [19] and using the monotonicity of the logarithmic function and the best approximation, we have:

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot\ln k_2\right)^{\frac{\lambda}{q}}\le$$

$$\le C\cdot E_{1,1}^{\lambda}(f)_q+C\cdot E_{2,2}^{\lambda}(f)_q+C\sum_{m=3}^{\infty}m^{\frac{2\lambda}{q}(2-q)-1}(\ln m)^{\frac{\lambda}{q}}\cdot E_{m,m}^{\lambda}(f)_q.$$

The theorem is proved.

Theorem 4. Let $1<q\le 2$, $1\le\lambda\le q$, $-1<\beta_1<\frac{1}{q'}$, $\frac{1}{q'}<\beta_2<+\infty$, $\frac{1}{q}+\frac{1}{q'}=1$. If $f\in L_q(I_2)$ and

$$\sum_{n=2}^{\infty}n^{\frac{\lambda}{q}(3-q(1+\beta_1))-1}\left(E_{n,\infty}^{(j)}(f)_q\right)^{\lambda}<+\infty,\quad j=1,2,$$

then the Fourier series of function $f\in L_q(I_2)$ will be $|C;\beta_1;\beta_2|_\lambda$ -summable almost everywhere on I_2.

Proof. It was proved in [12] that, if $1<q\le 2$, $1\le\lambda\le q$, $-1<\beta_1<\frac{1}{q'}$, $\frac{1}{q'}<\beta_2<+\infty$, $\frac{1}{q}+\frac{1}{q'}=1$ and

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot {k_1}^{q(1-\beta_1)-1}\right)^{\frac{\lambda}{q}}<+\infty,$$

then a double series of the form (1) is $|C;\beta_1;\beta_2|_\lambda$-summable almost everywhere on I_2.

For the last converging series, similarly to Theorem 2, we obtain

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot {k_1}^{q(1-\beta_1)-1}\right)^{\frac{\lambda}{q}}\le$$

$$\le C\cdot\sum_{n=0}^{\infty}2^{n(3-q(1+\beta_1))\frac{\lambda}{q}}\cdot\left(\sum_{k_2=2^{n}+1}^{2^{n+1}}\sum_{k_1=2}^{2^{n+1}}(k_1k_2)^{q-2}\rho_{k_1k_2}^{q}\right)^{\frac{\lambda}{q}}.$$

Hence, applying the Hardy-Littlewood theorem [19] and the monotonicity of the best approximation, we have:

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\sum_{k_1=2^{n_1}+1}^{2^{n_1+1}}\sum_{k_2=2^{n_2}+1}^{2^{n_2+1}}\rho_{k_1k_2}^{q}\cdot {k_1}^{q(1-\beta_1)-1}\right)^{\frac{\lambda}{q}}\le$$

$$\le C\cdot E_{1,1}^{\lambda}(f)_q+C\sum_{m=2}^{\infty}m^{\frac{\lambda}{q}(3-q(1+\beta_1))-1}\cdot E_{m,m}^{\lambda}(f)_q.$$

The theorem is proved.

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Дербес ең жақсы жуықтаулар және еселі Фурье қатарының абсолютті чезаролық қосындылануы

Мақала еселі тригонометриялық Фурье қатарының Чезаро бойынша абсолютті қосындылану сұрағына арналған. Фурье қатарлары теориясында ерекше орны бар бұл сұрақ бір өлшемді жағдайда жеткілікті кең зерттелген және бұл теорияның іргелі нәтижелері Н.К. Бари, А. Зигмунд, Р. Эдвардс, Б.С. Кашин және А.А. Саакянның [1–4] белгілі монографияларында келтірілген. Еселі қатарлар жағдайына сәйкес теория соншалықты күшті жасалмаған. Еселі жағдайдың өз ерекшеліктері бар және бір өлшемді жағдаймен ұқсастық әрқашан бірмәнді емес және айқын бола бермейді. Мақалада $f \in L_q(I_s)$ функциясының еселі Фурье қатарының абсолютті қосындылануының жеткілікті шарттары осы функцияның дербес ең жақсы жуықтаулары тілінде алынған. Төрт теорема дәлелденіп, $f \in L_q(I_s)$ функциясының Фурье қатарының $|C;\bar{\beta}|_\lambda$ қосындылануының әртүрлі төрт жеткілікті шарты нақтыланған. Бірінші теоремада $f \in L_q(I_s)$ функциясының Фурье қатарының $|C;\bar{\beta}|_\lambda$ қосындылануы осы функцияның дербес ең жақсы жуықтаулары тіліндегі s шарттан тұратын жеткілікті шарты $\beta_1 = ... = \beta_s = \frac{1}{q'}$ жағдайында алынған. Басқа жеткілікті шарттар екі еселі Фурье қатарлары үшін алынды. $f \in L_q(I_2)$ функциясының Фурье қатарының $|C;\beta_1;\beta_2|_\lambda$ қосындылану шарттары мына жағдайларда алынды: $\beta_1 = \frac{1}{q'}$, $-1 < \beta_2 < \frac{1}{q'}$ (екінші теоремада), $\frac{1}{q'} < \beta_1 < +\infty$, $\beta_2 = \frac{1}{q'}$ (үшінші теоремада), $-1 < \beta_1 < \frac{1}{q'}$, $\frac{1}{q'} < \beta_2 < +\infty$ (төртінші теоремада).

Кілт сөздер: тригонометриялық қатар, Фурье қатары, Лебег кеңістігі, функцияның дербес ең жақсы жуықтауы, қатардың абсолютті қосындылануы.

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Частные наилучшие приближения и абсолютная чезаровская суммируемость кратных рядов Фурье

Статья посвящена вопросу абсолютной суммируемости по Чезаро кратных тригонометрических рядов Фурье. Этот вопрос, занимая центральное место в теории рядов Фурье, в одномерном случае разработан достаточно широко и фундаментальные результаты этой теории изложены в известных монографиях Н.К. Бари, А. Зигмунда, Р. Эдвардса, Б.С. Кашина и А.А. Саакяна [1–4]. В случае кратных рядов соответствующая теория разработана не столь сильно. Многомерный случай имеет свою специфику, и аналогия с одномерным случаем далеко не всегда однозначна и очевидна. В статье получены достаточные условия абсолютной суммируемости кратных рядов Фурье функции $f \in L_q(I_s)$ в терминах частных наилучших приближений данной функции. Доказаны четыре теоремы и получены четыре разных достаточных условий $|C;\bar{\beta}|_\lambda$ суммируемости ряда Фурье функции $f \in L_q(I_s)$. В первой теореме получено достаточное условие абсолютной $|C;\bar{\beta}|_\lambda$ суммируемости ряда Фурье функции $f \in L_q(I_s)$ в терминах частного наилучшего приближения этой функции, которое состоит из s условий, в случае когда $\beta_1 = ... = \beta_s = \frac{1}{q'}$. Другие достаточные условия получены для двойных рядов Фурье. Достаточные условия $|C;\beta_1;\beta_2|_\lambda$ суммируемости ряда Фурье функции $f \in L_q(I_2)$ получены в случаях $\beta_1 = \frac{1}{q'}$, $-1 < \beta_2 < \frac{1}{q'}$ (во второй теореме), $\frac{1}{q'} < \beta_1 < +\infty$, $\beta_2 = \frac{1}{q'}$ (в третьей теореме), $-1 < \beta_1 < \frac{1}{q'}$, $\frac{1}{q'} < \beta_2 < +\infty$ (в четвертой теореме).

Ключевые слова: тригонометрический ряд, ряд Фурье, пространство Лебега, частное наилучшее приближение функции, абсолютная суммируемость ряда.

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