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On Boundary Value Problems for a Mixed Type Fractional Differential Equation with Caputo Operator

This article is devoted to study the boundary value problems of the first and second kind with respect to the spatial variable for a mixed inhomogeneous differential equation of parabolic-hyperbolic type with a fractional Caputo operator in a rectangular domain. In the study of such boundary value problems, we abandoned the boundary value condition with respect to the first argument and instead it is used additional gluing condition. In this case, in the justification of the unique solvability of the problems, the conditions on the boundary domain are removed. This allowed us to weaken the criterion for the unique solvability of boundary value problems under consideration. The solution is constructed in the form of Fourier series with eigenfunctions corresponding to homogeneous spectral problems. Estimates for the convergence of Fourier series are obtained as a regular solution of this mixed equation.

Keywords: Mixed differential equation, fractional order, Caputo operator, non model equation, Fourier series, gluing conditions, unique solvability.

Introduction

The theory of boundary value problems for differential equations of mixed parabolic-hyperbolic and elliptic-hyperbolic types, by virtue of its applied and theoretical significance, in recent years has become one of the most important branches of the theory of partial differential equations. In 1940, F. I. Frankl discovered applications of the Tricomi problem for the Chaplygin equation in transonic gas dynamics. Later the new applications of mixed-type equations have been found in the theory of Laval nozzles, in plasma theory, and in other branches of physics and mechanics.

Mathematicians began to study more often by the method of Fourier series the unique solvability and stability of the solution of the Dirichlet and Tricomi problems for a mixed type model differential equations of the second order

$$0 = \begin{cases} u_t - u_{xx} + b^2 u, \ t \geq 0, \\ u_{tt} - u_{xx} + b^2 u, \ t < 0 \end{cases}$$

in rectangle domain $\Omega = \{(t, x) : -p < t < q, \, 0 < x < l\}$. We note that in [1-3] for this kind of equations in the rectangular domain with two gluing conditions and with a condition over the entire boundary domain were studied. In studying the unique solvability of Dirichlet and Tricomi problems for this kind of mixed equations there is a condition to the measure of the boundary domain. Our approach interfered with the global solvability of the considering problem in an arbitrary rectangle. The method of Fourier series is also widely used in the works of other authors in the study of local and nonlocal boundary value problems for differential and integro-differential equations (see, for example, works [4–9]). The problem of the correct choice of boundary value conditions for a wide class of singular partial differential equations are solved in [10]. The aggregated theorems of existence and uniqueness of classical solutions can be proved with continuously depending of experimental definite function. In [11] a nonlocal problem for the fourth order system of loaded partial differential equations is considered and the questions of a existence unique solution of the considered problem and ways of its construction are investigated.

The fractional differential and integral operators have applications in many fields of mathematical physics, engineering, neurobiology, economics, control theory and combustion science [12, 13]. Therefore, this kind of differential and integral operators plays an important role in the theory of linear and nonlinear analysis. It is also

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known that in control theory is obtained a dynamic system, describing by the aid of fractional order differential equations [14]. There are classical methods of solving some kind of fractional differential equations with Riemann-Liouville operator, with the Caputo operator, or with the Erdeli-Kober operator. The Cauchy problems for the diffusion-wave equation with fractional differentiation operators in the sense of Riemann-Liouville and Caputo were investigated in [15–19]. Such kind of problems are of great importance in the construction of mathematical models of diffusion processes. Interesting results were obtained in works [20–22] for the fractional partial differential equations.

In this paper in the rectangular domain the unique solvability of the problem with boundary conditions first and second kind with respect to the spatial variable is established for an inhomogeneous parabolic-hyperbolic equation with fractional Caputo operator. In studying this boundary value problem application of three gluing conditions allowed us to solve the problem in arbitrary rectangular domain. This work is a further development of work [23] for the case of an inhomogeneous equation.

So, in rectangle domain $\Omega = \{(t,\, x) : -p < t < q,\, 0 < x < l\}$ we consider a differential equation of mixed parabolic-hyperbolic type

$$\begin{cases} {}_cD_{0t}^{\alpha} U - U_{xx} + \lambda^2 U = f\,(t,\, x), & t \geq 0, \\ U_{tt} - U_{xx} + \lambda^2 U = f\,(t,\, x), & t < 0, \end{cases} \tag{1}$$

where $\lambda \geq 0$, $l > 0$, $p > 0$, $q > 0$ are known real numbers, $f\,(t,\, x)$ is known function and ${}_cD_{0t}^{\alpha}$ is fractional order operator in the sense of Caputo:

$${}_cD_{0t}^{\alpha}\, g\,(t) = \begin{cases} \frac{1}{\Gamma\,(1-\alpha)} \int\limits_0^t (t-z)^{-\alpha} g'\,(z)\,dz, & 0 < \alpha < 1, \\ \frac{d}{dt}\, g\,(t), & \alpha = 1, \end{cases} \tag{2}$$

$\Gamma\,(1-\alpha)$ is Euler gamma function.

Note that the Caputo operator (2) can be represented as follows:

$${}_cD_{0t}^{\alpha}\, U\,(t,\, x) = I_{0t}^{1-\alpha} \frac{\partial U\,(t,\, x)}{\partial t}, \tag{3}$$

where

$$I_{at}^{\alpha}\, g_0\,(t) = \frac{1}{\Gamma\,(\alpha)} \int\limits_a^t (t-z)^{\alpha-1}\, g_0\,(z)\,dz \tag{4}$$

is fractional order Riemann-Liouville integral.

We introduce the notations: $J = \{(t,\, x) : t = 0,\, 0 < x < l\}$, $\Omega = \Omega_1 \cup \Omega_2 \cup J$,

$$\Omega_1 = \Omega \cap \{(t,\, x) : t > 0,\, x > 0\}\,,\; \Omega_2 = \Omega \cap \{(t,\, x) : t < 0,\, x < 0\}\,.$$

In the domain Ω we consider the following problem:

Problem 1. It is required to find a function $U\,(t,\, x)$ with the following properties:

$$U\,(t,\, x) \in C\,(\overline{\Omega}) \cap C^1\big(\Omega_2 \cup J\big), \quad t^{1-\alpha}\, U_t(t,\, x),\; t^{2-\alpha}\, U_{tt}(t,\, x) \in C\,\big(\Omega_1 \cup J\big);$$

$$U_{tt} \in C\,(\Omega_2 \cup J),\;\; U_{xx} \in C\,(\Omega_1 \cup \Omega_2),\;\; {}_cD_{0t}^{\alpha} U \in C\,(\Omega_1 \cup J)$$

and satisfies the equation (1) in the domains Ω_j $(j = 1, 2)$; on the line J satisfies the gluing conditions

$$U\,(+0,\, x) = U\,(-0,\, x), \quad (0,\, x) \in \bar{J}, \tag{5}$$

$$\lim_{t\to+0} t^{1-\alpha}\, U_t(t,\, x) = \lim_{t\to-0} U_t(t,\, x), \quad (0,\, x) \in J, \tag{6}$$

$$\lim_{t\to+0} t^{2-\alpha}\, U_{tt}(t,\, x) = \lim_{t\to-0} U_{tt}\,(t,\, x), \quad (0,\, x) \in J; \tag{7}$$

satisfies the following boundary value conditions

$$U\,(t,\, 0) = 0, \quad U\,(t,\, l) = 0, \quad -p \leq t \leq q. \tag{8}$$

So, we note that in studying this problem we use three gluing conditions (5)–(7).

Solutions of the equation (1), satisfying zero boundary value conditions (8), are sought in the form of the Fourier series

$$U\;(t,\,x)=\sum_{n=1}^{\infty}u_n\,(t)\;\vartheta_n\,(x)\,, \tag{9}$$

where

$$u_n\;(t)=\int_0^l U\;(t,\,x)\;\vartheta_n\,(x)\;d\,x,$$

$$\vartheta_n\,(x)=\sqrt{\frac{2}{l}}\sin\,\rho_n x,\;\;\rho_n=\sqrt{\mu_n^2+\lambda^2},\;\;\mu_n=\frac{\pi n}{l},\,n\in\mathbb{N}.$$

It is known that a system of functions $\{\vartheta_n\,(x)\}_{n=1}^{\infty}$ form a complete system of orthonormal functions in the space $L_2\,[0,l]$. We also expand the function $f\,(t,\,x)$ in a Fourier series by eigenvalue functions $\vartheta_n\,(x)$:

$$f\;(t,\,x)=\sum_{n=1}^{\infty}f_n\,(t)\;\vartheta_n\,(x)\,, \tag{10}$$

where

$$f_n\,(t)=\int_0^l f\;(t,\,x)\;\vartheta_n\,(x)\;d\,x.$$

Uniqueness of the solution of the problem 1

Theorem 1. If there exists a solution of the problem 1, then this solution is unique.

Proof. Let $f\,(t,\,x)=0$ be in $\overline{\Omega}$. We prove that a homogeneous problem U_I has only a trivial solution. We consider the function

$$\theta_m\,(t)=\int_0^l U\,(t,\,x)\,\vartheta_m(x)\,d\,x. \tag{11}$$

Then for the homogeneous equation (1) we obtain

$${}_cD_{0t}^{\alpha}\,\theta_m(t)=\sqrt{\frac{2}{l}}\int_0^l\;[{}_cD_{0t}^{\alpha}\,U\,(t,\,x)]\;\sin\,\rho_m\,x\,d\,x=\sqrt{\frac{2}{l}}\int_0^l\;U_{x\,x}(\,t,\,x)\;\sin\,\rho_m\,x\,d\,x-$$

$$-\sqrt{\frac{2}{l}}\int_0^l\lambda^2\,U\,(\,t,\,x)\;\sin\,\rho_m\,x\,d\,x,\quad 0<t<q,$$

$$\theta_m''(t)=\sqrt{\frac{2}{l}}\int_0^l\;U_{t\,t}(t,\,x)\;\sin\,\rho_m\,x\,d\,x=\sqrt{\frac{2}{l}}\int_0^l\;U_{x\,x}(t,\,x)\;\sin\,\rho_m x\,d\,x-$$

$$-\lambda^2\sqrt{\frac{2}{l}}\int_0^l U\,(t,\,x)\;\sin\,\rho_m\,x\,d\,x,\quad -p<t<0.$$

Hence, integrating by parts two times over x and taking the conditions (5)-(7) into account, we obtain

$${}_cD_{0\,t}^{\alpha}\theta_m(t)+\rho_m^2\theta_m(t)=0,\quad 0<t<q, \tag{12}$$

$$\theta_m''(t)+\rho_m^2\theta_m(t)=0,\quad -p<t<0, \tag{13}$$

Applying the gluing conditions (5)-(7) to (11), we obtain

$$\theta_m(+0)=\int_0^l U\,(+0,\,x)\,\vartheta_m(x)\,d\,x=$$

$$= \int_0^l U(-0, x)\,\vartheta_m(x)\,dx = \theta_m(-0), \tag{14}$$

$$\lim_{t\to+0} t^{1-\alpha}\theta'_m(t) = \lim_{t\to+0} \int_0^l t^{1-\alpha} U_t(t, x)\,\vartheta_m(x)\,dx =$$

$$= \lim_{t\to-0} \int_0^l U_t(t, x)\,\vartheta_m(x)\,dx = \lim_{t\to-0} \theta'_m(t), \tag{15}$$

$$\lim_{t\to+0} t^{2-\alpha}\theta''_m(t) = \lim_{t\to+0} \int_0^l t^{2-\alpha} U_{tt}(t, x)\,\vartheta_m(x)\,dx =$$

$$= \lim_{t\to-0} \int_0^l U_{tt}(t, x)\,\vartheta_m(x)\,dx = \lim_{t\to-0} \theta''_m(t). \tag{16}$$

By virtue of (3) and (4), the countable systems of differential equations (12) and (13), respectively, have general solutions

$$\theta_m(t) = c_m\, E_{\frac{1}{\alpha}}(-\rho_m^2 t^\alpha,\ 1), \quad 0 < t < q, \tag{17}$$

$$\theta_m(t) = a_m \cos \rho_m t + b_m \sin \rho_m t, \quad -p < t < 0, \tag{18}$$

where $a_m,\ b_m,\ c_m$ are arbitrary constants and $E_{\frac{1}{\alpha}}(z,\, 1)$ is the Mittag-Leffler function with the form:

$$E_{\alpha,\sigma}(z) \equiv E_{\frac{1}{\alpha}}(z,\, \sigma) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(\alpha\, i + \sigma)}, \quad \sigma > 0.$$

Substituting (17) and (18) into (14)-(16) and taking into account the property of the Mittag-Leffler function

$$E_{\frac{1}{\alpha}}(z) = 1 + z\, E_{\frac{1}{\alpha}}(z,\ \alpha + 1),$$

we obtain

$$c_m = a_m, \quad b_m = -\frac{\rho_m}{\Gamma(\alpha)} c_m, \quad \left[\frac{(1-\alpha)}{\Gamma(\alpha)} + 1\right] c_m = 0.$$

Hence, we find that $c_m = a_m = b_m = 0$. Consequently,

$$\theta_m(t) = \int_0^l U(t, x)\ \vartheta_m(x)\ dx = 0, \ \ t \in [-p, q].$$

Therefore, by virtue of completeness of the systems of eigenfunctions $\{\vartheta_m(x)\}_{n=1}^{\infty}$ in the space $L_2[0, l]$, implies $U(t, x) = 0$ almost everywhere on $[0, l]$ for all $t \in [-p, q]$.

Since, by virtue of the first condition of the problem 1, the function $U(t, x)$ is continuous in $\overline{\Omega}$. Therefore, the solution of the problem 1 is unique. The Theorem 1 was proved.

Justification of the existence of a solution of the problem 1

Substituting the expansions (9) and (10) into equation (1), we obtain

$$\sum_{n=1}^{\infty} \vartheta_n(x)\ {}_cD_{0t}^{\alpha} u_n(t) = \sum_{n=1}^{\infty} f_n(t)\ \vartheta_n(x) - \sum_{n=1}^{\infty} \rho_n^2 u_n(t)\ \vartheta_n(x),\ (t, x) \in \Omega_1,$$

$$\sum_{n=1}^{\infty} u''_n(t)\ \vartheta_n(x) = \sum_{n=1}^{\infty} f_n(t)\ \vartheta_n(x) - \sum_{n=1}^{\infty} \rho_n^2 u_n(t)\ \vartheta_n(x),\ (t, x) \in \Omega_2.$$

Hence, taking into account the fact that the system of eigenvalue functions $\{\vartheta_n(x)\}_{n=1}^{\infty}$ form a complete system of orthonormal functions in the space $L_2[0,l]$, we arrive at countable systems of differential equations

$$ {}_cD_{0t}^{\alpha}[u_n(t)] + \rho_n^2 u_n(t) = f_n(t), \ \ 0 \le t \le q, \tag{19}$$

$$ u''_n(t) + \rho_n^2 u_n(t) = f_n(t), \ \ -p \le t \le 0, \ n \in \mathbb{N}. \tag{20}$$

By virtue of (3), the equation (19) takes the form

$$ I_{0t}^{1-\alpha} u'_n(t) + \rho_n^2 u_n(t) = f_n(t), \ 0 \le t \le q. \tag{21}$$

Applying the operator $I_{0t}^{\alpha}[\cdot]$ to both sides of equation (21) and taking

$$ I_{0t}^{\alpha} I_{0t}^{1-\alpha} u'_n(t) = I_{0t}^{1} u'_n(t) = u_n(t) - u_n(0) $$

into account we bring this equation to the Volterra integral equation of the second kind with respect to the unknown function $u_n(t)$:

$$ u_n(t) + \frac{\rho_n^2}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u_n(\tau)\, d\tau = $$

$$ = u_n(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f_n(\tau)\, d\tau, \ \ 0 \le t \le q. \tag{22}$$

Taking

$$ \frac{1}{\Gamma(\beta)} \int_0^z t^{\sigma-1} E_{\alpha,\sigma}(\lambda t^{\alpha})(z-t)^{\beta-1}\, dt = z^{\sigma+\beta-1} E_{\alpha,\sigma+\beta}(\lambda z^{\alpha}), $$

$$ \frac{1}{\Gamma(\sigma)} + z E_{\alpha,\alpha+\sigma}(z) = E_{\alpha,\sigma}(z), \ \ \sigma > 0, \ \beta > 0 $$

into account we obtain the solution of the equation (22) in the form

$$ u_n(t) = c_n E_{\frac{1}{\alpha}}(-\rho_n^2 t^{\alpha}, 1) + $$

$$ + \int_0^t (t-\tau)^{\alpha-1} E_{\frac{1}{\alpha}}\left(-\rho_n^2 (\tau-t)^{\alpha}, \alpha\right) f_n(\tau)\, d\tau, \ 0 \le t \le q, \tag{23}$$

where c_n is arbitrary constant.

Solving the equation (20) by the Lagrange method, we obtain the representation

$$ u_n(t) = a_n \cos \rho_n t + b_n \sin \rho_n t + \frac{1}{\rho_n} \int_t^0 f_n(\tau) \sin \rho_n(\tau - t)\, d\tau, \ \ -p \le t \le 0, \tag{24}$$

where $a_n, \ b_n$ are arbitrary constants.

Applying the gluing conditions (14)-(16) for $m = n$ to representations (23) and (24), we derive unknown coefficients $a_n, \ b_n, \ c_n$:

$$ c_n = a_n = \frac{f_n(0)}{\rho_n^2}, \ \ b_n = 0. \tag{25}$$

Substituting (25) into (23) and into (24), we obtain the following representations

$$ u_n(t) = \frac{f_n(0)}{\rho_n^2} E_{\frac{1}{\alpha}}(-\rho_n^2 t^{\alpha}, \ 1) + $$

$$ + \int_0^t (t-\tau)^{\alpha-1} E_{\frac{1}{\alpha}}\left(-\rho_n^2 (\tau-t)^{\alpha}, \alpha\right) f_n(\tau)\, d\tau, \ 0 \le t \le q, \tag{26}$$

$$u_n(t) = \frac{f_n(0)}{\rho_n^2} \cos\rho_n t + \frac{1}{\rho_n} \int_t^0 f_n(\tau) \sin\rho_n(\tau - t)\, d\tau, \ -p \leq t \leq 0. \tag{27}$$

Substituting (26) and (27) into the Fourier series (9), we formally represent the solution of the problem 1 in the form of the following Fourier series

$$U(t, x) = \sum_{n=1}^{\infty} \vartheta_n(x) \left[\frac{f_n(0)}{\rho_n^2} E_{\frac{1}{\alpha}}(-\rho_n^2 t^\alpha, 1) + \right.$$

$$\left. + \int_0^t (t-\tau)^{\alpha-1} E_{\frac{1}{\alpha}}\left(-\rho_n^2 (t-\tau)^\alpha, \alpha\right) f_n(\tau)\, d\tau \right], \quad (t, x) \in \Omega_1, \tag{28}$$

$$U(t, x) = \sum_{n=1}^{\infty} \vartheta_n(x) \left[\frac{f_n(0)}{\rho_n^2} \cos\rho_n t + \frac{1}{\rho_n} \int_t^0 f_n(\tau) \sin\rho_n(\tau - t)\, d\tau \right], \ (t, x) \in \Omega_2. \tag{29}$$

Theorem 2. Let the functions $f(t, x)$, $f_x(t, x)$, $f_{xx}(t, x)$, $f_t(t, x)$ be continuous in $\overline{\Omega}$ and $f_{tt}(t, x) \in \in L_2(\Omega)$, $f(t, 0) = f(t, l) = 0$, $-p \leq t \leq q$. Then a regular solution of the problem 1 exists and is defined in the form of series (28) and (29).

Proof. In proving the convergence of the series (28) and (29) with the properties of the problem 1, an important role are played the applications of the Cauchy-Schwartz inequality and Bessel inequality. First, estimate the following functions

$$\frac{1}{\rho_n^2} = \frac{1}{\mu_n^2 + \lambda^2} \leq \frac{1}{\mu_n^2} = \left(\frac{l}{\pi}\right)^2 \frac{1}{n^2}, \ \rho_n = \sqrt{\mu_n^2 + \lambda^2}, \ \mu_n = \frac{\pi n}{l}, \ n \in \mathbb{N};$$

$$\left| E_{\frac{1}{\alpha}}\left(-\rho_n^2 (t-\tau)^\alpha, \alpha\right) \right| \leq \frac{C}{1 + \rho_n^2 (t-\tau)^\alpha} \leq \frac{C}{\rho_n^2 (t-\tau)^\alpha} \leq \frac{(t-\tau)^{1-\alpha} C}{\rho_n^2 (t-\tau)} \leq$$

$$\leq \frac{(t-\tau)^{-\alpha} C}{\rho_n^2} \leq \frac{M_0}{\rho_n^2}, \ M_0 \geq (t-\tau)^{-\alpha} C, \ 0 < C = \text{const};$$

$$\left| \int_0^t (t-\tau)^{\alpha-1} d\tau \right| = \left| \int_0^t (t-\tau)^{\alpha-1} d(t-\tau) \right| = \left| \frac{(t-\tau)^\alpha}{\alpha} \right|_{\tau=0}^{\tau=t} \leq \frac{q^\alpha}{\alpha}.$$

Then applying the Cauchy-Schwartz inequality and Bessel inequality to the series (28), we obtain the following estimate

$$|U(t, x)| \leq \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \left[\left| \frac{f_n(0)}{\rho_n^2} \right| \cdot \left| E_{\frac{1}{\alpha}}\left(-\rho_n^2 t^\alpha, 1\right) \right| + \right.$$

$$\left. + \int_0^t (t-\tau)^{\alpha-1} \left| E_{\frac{1}{\alpha}}\left(-\rho_n^2 (t-\tau)^\alpha, \alpha\right) \right| \cdot |f_n(\tau)|\, d\tau \right] \leq$$

$$\leq \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \frac{|f_n(0)|}{\rho_n^2} \cdot \frac{M_0}{\rho_n^2} + \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} M_0 \frac{\max\limits_{0 \leq t \leq q} |f_n(t)|}{\rho_n^2} \left| \int_0^t (t-\tau)^{\alpha-1} d\tau \right| \leq$$

$$\leq M_0 \sqrt{\frac{2}{l}} \sqrt{\sum_{n=1}^{\infty} \frac{1}{\mu_n^8}} \sqrt{\sum_{n=1}^{\infty} \left| \int_0^l f(0, x)\, \vartheta_n(x)\, dx \right|^2} +$$

$$+ M_0 \sqrt{\frac{2}{l}} \frac{q^\alpha}{\alpha} \sqrt{\sum_{n=1}^{\infty} \frac{1}{\mu_n^4}} \sqrt{\sum_{n=1}^{\infty} \max_{0 \leq t \leq q} \left| \int_0^l f(t, x)\, \vartheta_n(x)\, dx \right|^2} \leq$$

$$\leq 2\sqrt{\frac{2}{l}}\, M_0 M_1 M_2 M_3 \max_{0\leq t\leq q} \| f(t,x)\|_{L_2[0;l]} < \infty,\ (t,x)\in\Omega_1, \tag{30}$$

where

$$M_1 = \max\left\{\frac{q^{\alpha}}{\alpha};\ 1\right\};\ \ M_2 = \max\left\{\left(\frac{l}{\pi}\right)^2;\ \left(\frac{l}{\pi}\right)^4\right\};\ \ M_3 = \sqrt{\sum_{n=1}^{\infty}\frac{1}{n^4}}.$$

Similarly to estimate (30), for series (29) we obtain the estimate

$$|U(t,x)| \leq \sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\frac{|f_n(0)|}{\rho_n^2} + p\sum_{n=1}^{\infty} M_0\,\frac{\max\limits_{-p\leq t\leq 0}|f_n(t)|}{\rho_n} \leq$$

$$\leq \sqrt{\frac{2}{l}}\sqrt{\sum_{n=1}^{\infty}\frac{1}{\mu_n^4}}\sqrt{\sum_{n=1}^{\infty}|f_n(0)|^2} + p\sqrt{\frac{2}{l}}\sqrt{\sum_{n=1}^{\infty}\frac{1}{\mu_n^2}}\sqrt{\sum_{n=1}^{\infty}\max_{-p\leq t\leq 0}|f_n(t)|^2} \leq$$

$$\leq 2\sqrt{\frac{2}{l}}\, N_0 N_1 N_2 \max_{-p\leq t\leq 0}\| f(t,x)\|_{L_2[0;l]} < \infty,\ (t,x)\in\Omega_2, \tag{31}$$

where

$$N_0 = \max\{p;\ 1\};\ \ N_1 = \max\left\{\frac{l}{\pi};\ \left(\frac{l}{\pi}\right)^2\right\};\ \ N_2 = \sqrt{\sum_{n=1}^{\infty}\frac{1}{n^2}}.$$

By virtue of the estimates (30) and (31), we conclude that the series (28) and (29) absolutely and uniformly converge.

Similarly to the case of series (28) and (29), it is easy to check, that the series

$$t^{2-\alpha}U_{tt}(t,x) = \sum_{n=1}^{\infty} t^{2-\alpha}u_n''(t)\,\vartheta_n(x),\ (t,x)\in\Omega_1,$$

$$U_{tt}(t,x) = \sum_{n=1}^{\infty} u_n''(t)\,\vartheta_n(x),\ (t,x)\in\Omega_2.$$

are convergent.

Now we prove the convergence of the following series

$$U_{xx}(t,x) = -\sum_{n=1}^{\infty}\rho_n^2\vartheta_n(x)\left[\frac{f_n(0)}{\rho_n^2}E_{\frac{1}{\alpha}}(-\rho_n^2 t^{\alpha},\ 1) + \right.$$

$$\left. + \int_0^t (t-\tau)^{\alpha-1}E_{\frac{1}{\alpha}}\left(-\rho_n^2(t-\tau)^{\alpha},\ \alpha\right) f_n(\tau)\, d\tau\right],\quad (t,x)\in\Omega_1, \tag{32}$$

$$U_{xx}(t,x) = -\sum_{n=1}^{\infty}\rho_n^2\vartheta_n(x)\ \times$$

$$\times\left[\frac{f_n(0)}{\rho_n^2}\cos\rho_n t + \frac{1}{\rho_n}\int_t^0 f_n(\tau)\sin\rho_n(\tau - t)\, d\tau\right],\quad (t,x)\in\Omega_2. \tag{33}$$

Integrating twice in parts the integral $f_n(t) = \int\limits_0^l f(t,x)\,\vartheta_n(x)\,dx$ with respect to x, we obtain $f_n(t) = -\rho_n^{-2} f_n''(t)$, where $f_n''(t) = \int\limits_0^l f_{xx}(t,x)\,\vartheta_n(x)\,dx$.

Then for the series (32) and (33), respectively, we derive the following estimates

$$|U_{xx}(t,x)| \leq \sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\rho_n^2\left[\left|\frac{f_n(0)}{\rho_n^2}\right|\cdot\left|E_{\frac{1}{\alpha}}\left(-\rho_n^2 t^{\alpha},\ 1\right)\right| + \right.$$

$$+\int_0^t (t-\tau)^{\alpha-1}\left|E_{\frac{1}{\alpha}}\left(-\rho_n^2(t-\tau)^{\alpha},\alpha\right)\right|\cdot|f_n(\tau)|\,d\tau\Bigg]\leq$$

$$\leq\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\frac{|f_n''(0)|}{\rho_n^2}\cdot\frac{M_0}{\rho_n^2}+\sum_{n=1}^{\infty}M_0\frac{\max\limits_{0\leq t\leq q}|f_n''(t)|}{\rho_n^2}\left|\int_0^t (t-\tau)^{\alpha-1}\,d\tau\right|\leq$$

$$\leq 2\sqrt{\frac{2}{l}}M_0M_1M_2M_3\max_{0\leq t\leq q}\|f_{xx}(t,x)\|_{L_2[0;l]}<\infty,\ (t,x)\in\Omega_1, \quad (34)$$

$$|U_{xx}(t,x)|\leq\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\frac{|f_n''(0)|}{\rho_n^2}+p\sum_{n=1}^{\infty}M_0\frac{\max\limits_{-p\leq t\leq 0}|f_n''(t)|}{\rho_n}\leq$$

$$\leq 2\sqrt{\frac{2}{l}}N_0N_1N_2\max_{-p\leq t\leq 0}\|f_{xx}(t,x)\|_{L_2[0;l]}<\infty,\ (t,x)\in\Omega_2. \quad (35)$$

By virtue of the estimates (34) and (35), we conclude that the series (32) and (33) absolutely and uniformly converge. It follows that the series (28) and (29) satisfy all the properties of the problem 1. Theorem 2 is proved.

Using the same method, we can establish a unique solvability of the following problem.

Problem 2. Find a solution $U(t,x)$ of equation (1) that is regular in the domain Ω and satisfies all the conditions of problem 1 except (8), which is replaced by the following condition

$$U_x(t,0)=0,\quad U_x(t,0)=0,\quad -p\leq t\leq q.$$

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Т.К. Юлдашев, Б.И. Исломов, У.Ш. Убайдуллаев

Капуто операторы бар аралас типті бөлшек дифференциалдық теңдеуге арналған шеттік есептер туралы

Мақалада тікбұрышты аймақтағы бөлшек Капуто операторы бар параболалық гиперболалық типтес біртекті емес аралас дифференциалдық теңдеу үшін кеңістіктік айнымалыға қатысты бірінші және екінші типтегі шеттік есептер зерттелді. Мұндай шеттік есептерді зерттеу кезінде бірінші аргумент бойынша шекаралық шарт қоюдан бас тартылды және оның орнына қосымша үзіліссіздік шарты қолданылды. Бұл ретте біржақты шешілуін негіздеу үшін шекаралық облысқа арналған шарттар алынып тасталады. Бұл мақала авторлары қарастырған шеттік есептердің біржақты шешілу критерийін әлсіретуге мүмкіндік береді. Шешім біртекті спектрлік есептерге сәйкес келетін өз функциялары бар Фурье қатарлары түрінде жасалды. Фурье қатарларының конвергенциясы осы аралас теңдеудің тұрақты шешімі ретінде алынды.

Кілт сөздер: аралас дифференциалдық теңдеу, бөлшек реті, Капуто операторы, модельдік емес теңдеу, Фурье қатары, үзіліссіздік шарттары, біржақты шешілу.

Т.К. Юлдашев, Б.И. Исломов, У.Ш. Убайдуллаев

О краевых задачах для дробного дифференциального уравнения смешанного типа с оператором Капуто

В статье исследованы краевые задачи первого и второго родов относительно пространственного переменного для смешанного неоднородного дифференциального уравнения параболо-гиперболического типа с дробным оператором Капуто в прямоугольной области. При исследовании таких краевых задач авторы отказались от задания граничного условия по первому аргументу и использовали вместо этого дополнительное условие склеивания. При этом в обосновании однозначной разрешимости снимаются условия на граничную область. Это позволило авторам статьи ослабить критерий однозначной разрешимости рассматриваемых краевых задач. Решение построено в виде рядов Фурье с собственными функциями, соответствующими однородным спектральным задачам. Получены оценки сходимости рядов Фурье как регулярное решение этого смешанного уравнения.

Ключевые слова: смешанное дифференциальное уравнение, дробный порядок, оператор Капуто, немодельное уравнение, ряд Фурье, условия склеивания, однозначная разрешимость.

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