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Smoothness and approximative properties of solutions of the singular nonlinear Sturm-Liouville equation

It is known that the eigenvalues $\lambda_n (n = 1, 2, ...)$ numbered in decreasing order and taking the multiplicity of the self-adjoint Sturm-Liouville operator with a completely continuous inverse operator L^{-1} have the following property

(*) $\lambda_n \to 0$, when $n \to \infty$, moreover, than the faster convergence to zero so the operator L^{-1} is best approximated by finite rank operators.

The following question:

- Is it possible for a given nonlinear operator to indicate a decreasing numerical sequence characterized by the property (*)?

naturally arises for nonlinear operators.

In this paper, we study the above question for the nonlinear Sturm-Liouville operator. To solve the above problem the theorem on the maximum regularity of the solutions of the nonlinear Sturm-Liouville equation with greatly growing and rapidly oscillating potential in the space $L_2(R)\,(R = (-\infty, \infty))$ is proved. Two-sided estimates of the Kolmogorov widths of the sets associated with solutions of the nonlinear Sturm-Liouville equation are also obtained. As is known, the obtained estimates of Kolmogorov widths give the opportunity to choose approximation apparatus that guarantees the minimum possible error.

Keywords: maximum regularity; singular nonlinear equation; Sturm-Liouville equation; smoothness of solutions; approximative properties; approximate numbers; Kolmogorov widths; rapidly oscillating potential; greatly growing potential; two-sided estimates.

Introduction

In this paper we study the nonlinear Sturm-Liouville equation

$$Ly = -y'' + q(x, y)y = f(x) \in L_2(R),\ \ R = (-\infty, \infty).$$

The existence and the smoothness of nonlinear elliptic equations solutions in a bounded domain have been studied quite well. A very comprehensive bibliography is contained, for example, in [1-6] and the works cited there.

However, nonlinear equations in an unbounded domain with greatly increasing and rapidly oscillating coefficients arise in applications. For example, the nonlinear Sturm-Liouville equation, which is especially interesting for quantum mechanics.

Here we are interested in the question:

A) to find out the conditions on the potential function $q\,(x, y)$ which provide $y'' \in L_2\,(R)$, when $y\,(x)$ is a solution of the nonlinear equation $Ly = f \in L_2\,(R)$.

We note that the linear case is well studied and reviews are available in [7–12].

It is known that eigenvalues $\lambda_n\,(n = 1, 2, ...)$ of the self-adjoint positive completely continuous operator A in the Hilbert space H are numbered according to their decreasing magnitude and observing their multiplicities have the following approximative properties

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a) $\lambda_n = \min\limits_{k \in l_n} \|A - K\|$, where l_n is the set of all finite-dimensional operators with dimension no greater than n;
b) $\lambda_n \to 0$, when $n \to \infty$, wherein the faster convergence to zero, the operator A better approximated by finite rank operators.

It will be natural to explore a similar issues for a nonlinear Sturm-Liouville operator, i.e. to study the question

B) Is it possible for a given non-linear operator to specify a numerical sequence that characterizes properties a)–b)?

This paper is devoted to the study of the issues A) and B) for the nonlinear Sturm-Liouville equation.

Formulation of the main results. Example

We will make some notation and definitions for the statement of results.

The set of integrable functions with respect to the square of the module in each strictly internal subdomain $\Omega \subset R$ is denoted by $L_{2,loc}(R)$.

The set of functions from $L_{2,loc}(R)$ having generalized first-order derivatives (from $L_{2,loc}(R)$) will be denoted by $W^1_{2,loc}(R)$. We denote the subset of $W^1_{2,loc}(R)$ by $W^1_2(R)$, which elements together with the first generalized derivatives belong to $L_2(R)$. By $W^2_{2,loc}(R)$ we denote the set of all functions $u \in L_{2,loc}(R)$ which with their generalized derivatives up to and including the second order belong to $L_{2,loc}(R)$.

$\|\cdot\|_2$ is the norm of an element in $L_2(R)$, $\|\cdot\|_{2,1}$ is the norm of an element in $W^1_2(R)$, $\|\cdot\|_{2,loc}$ is the norm of an element in $L_{2,loc}(R)$.

Consider the nonlinear Sturm-Liouville equation

$$Ly = -y'' + q(x,y)\,y = f(x) \in L_2(R), \quad R = (-\infty, \infty). \tag{1}$$

Suppose that $q(x,y)$ satisfies the conditions:

i) $q(x,y)$ is a continuous mapping $R \times C$ in $[\delta, \infty)$, $\delta > 0$, C is a set of complex numbers;

ii) $\sup\limits_{|x-\eta|\le 1} \sup\limits_{|c_1-c_2|\le A} \frac{q(x,c_1)}{q(\eta,c_2)} \le \mu(A) < \infty$, where A is a finite value, $\mu(A)$ is a continuous function from A.

Definition 1. The function $y \in L_2(R)$ is called a solution of the equation (1) if there exist a sequence $\{y_n\}_{n=1}^{\infty} \subset W^1_2(R)$ such that $\{y_n\}_{n=1}^{\infty} \subset W^2_{2,loc}(R)$, $\|y_n - y\|_{L_{2,loc}} \to 0$ $\|Ly_n - f\|_{L_{2,loc}} \to 0$ as $n \to \infty$.

Definition 2. Following [13-15], we say that the solution $y(x) \in L_2(R)$ of equation (1) is called the maximal regular in $L_2(R)$ if $q(x,y)\,y \in L_2(R)$, $y'' \in L_2(R)$.

Theorem 1.1. Let the conditions $i) - ii)$ be fulfilled. Then there is the most regular solution to equation (1).

The condition $ii)$, imposed in Theorem 1.1 and in [16], limits the potential oscillations. This condition is removed in the following theorem. In order to formulate the theorem, we introduce the following condition:

$i_0)$ $\sup\limits_{x \in R} \sup\limits_{|c_1-c_2|} \frac{q(x,c_1)}{Q^2(x,c_2)} < \infty$, $Q(x,c_2)$ is a special averaging of the function $q(x,c_1)$ [11], i.e.

$$Q(x,c_2) = \inf_{d>0}\left(d^{-1} + \int_{x-\frac{d}{2}}^{x+\frac{d}{2}} q(t,c_2)\,dt\right),$$

where A is a finite value.

Theorem 1.2. Let the conditions $i) - i_0)$ be fulfilled. Then there exist the maximal regular solution to equation (1).

Example 1. Let $q(x,y)=e^{|x|}\cdot\sin^2 e^{|x|}+e^{|y|}$. Then it is not difficult to verify that all conditions of Theorem 1.2 are satisfied for the equation

$$Ly=-y''+\left(e^{|x|}\cdot\sin^2 e^{|x|}+e^{|y|}\right)y=f(x).$$

Therefore, there exists a solution $y(x)$ for the equation such that $y''(x)\in L_2(R)$.

This shows that Theorem 1.2 holds for a very wide class of nonlinear equations, including equations with potentials that are rapidly oscillating at infinity.

Now, we consider consider the question B), i.e. finding such sequences of numbers that have approximative properties of the type a)-b). To do this, we study the behavior of the Kolmogorov k-widths of the set

$$M=\left\{u\in W_2^1(R):\left\|-y''+q(x,y)y\right\|_2^2\le T\right\}.$$

By definition [17], the Kolmogorov k-width of the set M is called the quantity

$$d_k(M,L_2)=d_k=\inf_{\{\ell_k\}}\sup_{u\in M}\inf_{v\in\ell_k}\|u-v\|_2,$$

where ℓ_k is a subspace of dimension k.

Note that the Kolmogorov widths of a compact set have the following properties: 1) $d_0\ge d_1\ge d_2\ge\ldots\ge d_k\ge\ldots$, 2) $d_k\to 0$ as $k\to\infty$.

By $L_2^2(R,q(x,0))$ we denote the space obtained by completing $C_0^\infty(R)$ with respect to the norm

$$\left\|y\cdot L_2^2(R,q(x,0))\right\|_2=\left(\int_{-\infty}^{\infty}\left(\left|y''\right|^2+q(x,0)|y|^2\right)dx\right)^{1/2}$$

Theorem 1.3. Let the conditions $i)$-$ii)$ be fulfilled. Then any bounded set is compact in $L_2^2(R,q(x,0))$ if and only if

$$\lim_{|x|\to\infty}q(x,0)=\infty. \qquad (*)$$

We introduce the following counting function $N(\lambda)=\sum_{d_k>\lambda}1$ of those widths d_k greater than $\lambda>0$.

Theorem 1.4. Let the conditions $i)$-$ii)$ be fulfilled. Then the estimate

$$c^{-1}\lambda^{-1/2}mes\left(x\in R:q(x,0)\le c^{-1}\lambda^{-1}\right)\le$$
$$\le N(\lambda)\le c\lambda^{-1/2}mes\left(x\in R:q(x,0)\le c\lambda^{-1}\right),$$

holds, $c>0$ is a constant depending, generally speaking, on T.

Example 2. Let us take $q(x)=|x|+|y|+1$. Then, by virtue of Theorem 1.4, the estimate $c^{-1}\lambda^{-3/2}\le N(\lambda)\le c\lambda^{-3/2}$ holds for the distribution function of the widths of the set $M=\left\{y\in W_2^1(R):\ \|-y''+q(x,y)y\|_2^2\le 1\right\}$, where $c>0$ is a constant. Since $N(\lambda)$ is a monotone function then we have $d_k\sim\frac{c}{k^{2/3}},\ k=1,2,3,\ldots$.

On the existence of solutions of the nonlinear Sturm-Liouville equation

In this section we prove a lemma on the existence of solutions.

Lemma 2.1. Let the condition $i)$ be fulfilled. Then there exists a solution of the equation (1) in the space $W_2^1(R)$ for any $f\in L_2(R)$.

To prove this lemma we need some auxiliary assertions.

Consider the following problem

$$L_{n,v}y=-y''(x)+q(x,v)y=f\cdot\chi_n, \qquad (2)$$

$$y(-n) = y(n) = 0, \tag{3}$$

where χ_n is the characteristic function of the segment $[-n, n]$, $n = 1, 2, ..., v(x) \in C[-n, n]$, $C[-n, n]$ is a space of continuous functions.

Lemma 2.2. Let the condition *i*) be fulfilled and let $v \in C[-n, n]$. Then there exists a unique solution of problem (2)-(3) for any $f \cdot \chi_n \in L_2(-n, n)$ such that

$$\|y\|_{W_2^1[-n,n]} \leq c_0 \|f\|_2 , \tag{4}$$

$$\|y\|_{C[-n,n]} \leq c \|f\|_2 . \tag{5}$$

where $c_0 > 0$ and $c > 0$ are constant numbers.

Proof. Assume $q(x) = q(x, v)$. Then the problem (2)-(3) takes the form

$$L_{n,v}y = -y''(x) + q(x)y = f \cdot \chi_n, \tag{6}$$

$$y(-n) = y(n) = 0. \tag{7}$$

From the general theory of boundary value problems [7] it follows that the problem (6)-(7) has a unique solution $W_2^2(-n, n)$ such that

$$\|y\|_{W_2^1[-n,n]} \leq c_0(\delta) \|f\chi_n\|_2 \leq c_0(\delta) \|f\|_2 . \tag{8}$$

It is known that $W_2^1(-n, n)$ is completely continuously embedded in the space $C[-n, n]$. Therefore, the following estimate

$$\|y\|_{C[-n,n]} \leq c_1 \|y\|_{W_2^1(-n,n)} , \tag{9}$$

holds, where $c_1 > 0$ is the constant of the embedding theorem.

So problem (6)-(7) has a unique solution

$$y_{n,v} = L_{n,v}^{-1}(f\chi_n) , \tag{10}$$

where $L_{n,v}^{-1}$ is the inverse operator of the operator $L_{n,v}$ corresponding to the problem (6)-(7). And

$$\left\|L_{n,v}^{-1}\right\|_{C[-n,n]} \leq c, \tag{11}$$

where $c = c_1 \cdot c_0(\delta)$.

Lemma 2.3. Let the condition *i*) be fulfilled. Then the operator $L_{n,v}^{-1}$ maps the ball $\bar{s}$ into itself, where $\bar{s} = \left\{v \in C[-n, n] : \|v\|_{C[-n,n]} \leq A\right\}$ is a ball in the space $C[-n, n]$ and A is an arbitrary positive number.

Proof. If the radius A of the ball $\bar{s}$ is equal to the right side of the inequality (5), i.e. $A = c\|f\|_2$, then Lemma 2.2 implies that the operator $L_{n,v}^{-1}$ maps the set $\bar{s}$ into itself. Lemma 2.3 is proved.

Let $K = \left\{y_{n,v} \in C[-n, n] : y_{n,v} = L_{n,v}^{-1}(f\chi_n), v \in \bar{s}, f \in L_2(R)\right\}$ is the image of the ball $\bar{s}$ under the mapping $L_{n,v}^{-1}$.

Lemma 2.4. Let the condition *i*) be fulfilled. Then the set K is compact in the space $C[-n, n]$.

Proof. Lemma 2.2 implies that the inequality

$$\|y_{n,v}\|_{W_2^1(-n,n)} \leq c_0 \|f\| ,$$

holds for any function $y_{n,v}(x)$ from K, where $c_0 > 0$ is a constant.

This and the embedding theorem imply that the set K is compact in $C[-n, n]$. Lemma 2.4 is proved.

Lemma 2.5. Let the condition *i*) be fulfilled. Then the operator $L_{n,v}^{-1}$ is continuous.

Proof. Let $f(x) \in L_2(R)$ and let the sequence $\{v_k\}_{k=1}^{\infty}$ converge to the element $v(x)$ of the ball $\bar{s}$ in the norm of the space $C[-n,n]$ and

$$L_{n,v_k} y_{n,v_k} = f(x) \cdot \chi_n, \tag{12}$$

$$L_{n,v} y_{n,v} = f(x) \cdot \chi_n, \tag{13}$$

From the equality (12)-(13) we find that

$$-\left(y_{n,v_k} - y_{n,v}\right)'' + q(x, v_k)\left(y_{n,v_k} - y_{n,v}\right) + \left(q(x,v_k) - q(x,v)\right) y_{n,v} = 0.$$

Hence

$$L_{n,v_k}\left(y_{n,v_k} - y_{n,v}\right) = \left(q(x,v) - q(x,v_k)\right) y_{n,v}. \tag{14}$$

It is easy to verify that the coefficients of the operator L_{n,v_k} satisfy the conditions of Lemma 2.2, therefore there exist an inverse operator L_{n,v_k}^{-1} and the equality

$$y_{n,v_k} - y_{n,v} = L_{n,v_k}^{-1}\left(q(x,v) - q(x,v_k)\right) y_{n,v}$$

holds.

From this and the inequalities (4)-(5) and (9)-(11) we obtain that

$$\begin{aligned} \left\| y_{n,v_k} - y_{n,v} \right\|_{C[-n,n]} &= \left\| L_{n,v_k}^{-1}\left(q(x,v) - q(x,v_k)\right) y_{n,v} \right\|_{C[-n,n]} \leq \\ &\leq \left\| L_{n,v_k}^{-1} \right\|_{C[-n,n]} \cdot \left\| \left(q(x,v) - q(x,v_k)\right) y_{n,v} \right\|_{C[-n,n]} \leq \\ &\leq c \cdot \sup_{x \in [-n,n]} \left| q(x,v) - q(x,v_k) \right| \cdot \left\| y_{n,v} \right\|_{L_2(-n,n)}. \end{aligned}$$

From this and from the inequality (4) we have

$$\begin{aligned} \left\| y_{n,v_k} - y_{n,v} \right\|_{C[-n,n]} &\leq c \cdot \sup_{x \in [-n,n]} \left| q(x,v) - q(x,v_k) \right| \cdot A_0 \cdot \|f\|_2 = \\ &= c_1 \cdot \sup_{x \in [-n,n]} \left| q(x,v) - q(x,v_k) \right| \cdot \|f\|_2, \end{aligned} \tag{15}$$

where $c_1 = c \cdot c_0$.

Since $\|v_k - v\|_{C[-n,n]} \to 0$ for $k \to \infty$ then we obtain from (15) that

$$\lim_{k\to\infty} \left\| y_{n,v_k} - y_{n,v} \right\|_{C[-n,n]} \leq c_0 \cdot \lim_{k\to\infty} \sup_{x \in [-n,n]} \left| q(x,v) - q(x,v_k) \right| \cdot \|f\|_2 \to 0$$

The last relation shows that the operator L_{n,v_k}^{-1} is continuous. Lemma 2.5 is proved.

Now, consider the following nonlinear problem

$$L_n y_n \equiv -y_n'' + q(x, y_n) y_n = f \cdot \chi_n \tag{16}$$

$$y_n(-n) = y_n(n) = 0. \tag{17}$$

Lemma 2.6. Let the condition *i*) be fulfilled. Then there exist a solution of the problem (16)-(17) for any $f \in L_2(R)$ such that

$$\|y_n\|_{C[-n,n]} + \|y_n\|_{W_2^1(-n,n)} \leq c \cdot \|f\|_2, \tag{18}$$

where $c > 0$ is a constant.

Proof. The function $y_{n,v} = L_{n,v}^{-1}(f\chi_n)$ belongs to the domain $D(L_n)$ of the operator L_n for each function $v \in C[-n,n]$ corresponding to the problem (16)-(17). Therefore, the existence of a solution to problem (16)-(17) is equivalent to the existence of a fixed point of the operator $L_{n,v}^{-1}$ in the space $C[-n,n]$, i.e., to the existence of a function $y_n \in C[-n,n]$ such that $y_n = L_{n,y_n}^{-1} f \cdot x_n$. Thus $y_n \in D\left(L_n\right)$, since $L_{n,v}^{-1}\left(f\chi_n\right) \in D\left(L_n\right)$ for any $v\left(x\right)$ from $C[-n,n]$.

To find a fixed point, it remains to show that the operator $L_{n,v}^{-1}$ maps the convex set into itself and it is completely continuous. The proof of this assertion follows from Lemmas 2.2-2.5. Lemma 2.6 is proved.

Proof of Lemma 2.1. Each y_n is continued by zero outside $[-n,n]$ and the continuation is denoted by $\tilde{y}_n$. As you know, we get the elements $W_2^1\left(R\right)$ with such a continuation and (18) implies that their norm is bounded

$$\|\tilde{y}_n\|_{W_2^1(R)} \leq c \cdot \|f\|_{L_2(R)}\,. \tag{19}$$

Therefore, from the sequence $\{\tilde{y}_n\}$ one can select a subsequence $\tilde{y}_{n_k}$ such that

$$\tilde{y}_{n_k} \to y \text{ weakly in } W_2^1\left(R\right), \tag{20}$$

$$\tilde{y}_{n_k} \to y \text{ strongly in } L_{2,loc}, \tag{21}$$

and the estimate

$$\|y\|_{W_2^1(R)} \leq c \cdot \|f\|_2 \tag{22}$$

holds. The last estimate follows from (19) and (20).

Let $[-\alpha,\alpha]$ is an arbitrary fixed segment in R, where $\alpha > 0$ is any number. Then there exists a number N for any $\varepsilon > 0$ such that

$$\|L\tilde{y}_{n_k} - f\|_{L_2(-\alpha,\alpha)} \to 0 \text{ for } n_k \to \infty \tag{23}$$

for $n \geq N$ $[-\alpha,\alpha] \subset \operatorname{supp} \tilde{y}_{n_k}$ and by virtue of (16).

(21) and (23) imply that $y\left(x\right)$ is a solution to the equation (1). Lemma 2.1 is proved. □

On smoothness of solutions

Proof of Theorem 1.1. Let $|x-\eta| \leq 1$, then by Lemma 2.1 and from the inequality (22) we have

$$|y\left(x\right) - y\left(\eta\right)| = \left|\int_\eta^x y'\left(t\right) dt\right| \leq \sqrt{x-\eta} \cdot c\,\|f\|_2 \leq c\,\|f\|_2\,.$$

Now supposing $y\left(x\right) = c,\;\; y\left(\eta\right) = c_2\;\; A = c\,\|f\|_2$ we obtain that

$$\sup_{|x-\eta|\leq 1} \frac{q\left(x, y\left(x\right)\right)}{q\left(\eta, y\left(\eta\right)\right)} \leq \sup_{|x-\eta|\leq 1} \sup_{|c_1-c_2|\leq A} \frac{q\left(x, c_1\right)}{q\left(\eta, c_2\right)} \leq \mu\left(A\right) < \infty.$$

Hence, according to Theorem 3 in [11] $y'',\;\; q\left(x,y\right) y$ belongs to $L_2\left(R\right)$. Theorem 1.1 is proved. □

Proof of Theorem 1.2. By Lemma 2.1, there exist a solution $y\left(x\right)$ for the equation (1) such that $y\left(x\right) \in W_2^1\left(R\right)$. Consequently, by the Sobolev embedding theorem $y\left(x\right) \in C\left(R\right)$. In the space $C\left(R\right)$ the norm is defined by the formula

$$\|y\|_{C(R)} = \sup_{x\in R} |y\left(x\right)|\,.$$

Then, according to the condition *i)* $q\left(x, y\left(x\right)\right) \in C_{loc}\left(R\right)$. Further, the inequality

$$|y\left(x\right) - y\left(\eta\right)| \leq |c_1 - c_2| \leq A$$

holds, where $y\left(x\right) = c_1,\;\; y\left(\eta\right) = c_2$.

Hence, we have:

$$\sup_{x\in R}\frac{q\left(x,y\left(x\right)\right)}{Q^{2}\left(x,y\left(x\right)\right)}\leq\sup_{x\in R}\sup_{\left|c_{1}\right|\leq A}\frac{q\left(x,c_{1}\right)}{Q_{A}^{2}\left(x,c_{1}\right)}\leq\sup_{x\in R}\sup_{\left|c_{1}-c_{2}\right|\leq A}\frac{q\left(x,c_{1}\right)}{Q^{2}\left(x,c_{2}\right)}$$

From the last inequality according to the condition $i)$ we find that

$$\sup_{x\in R}\frac{q\left(x,y\left(x\right)\right)}{Q^{2}\left(x,y\left(x\right)\right)}<\sup_{x\in R}\sup_{\left|c_{1}-c_{2}\right|\leq A}\frac{q\left(x,c_{1}\right)}{Q^{2}\left(x,c_{2}\right)}<\infty.$$

It follows that all the conditions of Theorem 4 of [11] are fulfilled. Consequently, $q\left(x,y\right)y\left(x\right)$, $y''\in L_{2}\left(R\right)$. Theorem 1.2 is proved. □

Two-sided estimates of the approximate numbers of solutions of the nonlinear Sturm-Liouville equation

As is known for a compact set, especially, when it contains solutions of a differential equation, the problem of the asymptotics of their widths is meaningful. The Kolmogorov widths estimation of the set M can be used to determine for the equation $Ly=f$ the convergence rate of approximate solutions to the exact one.

In order to prove Theorem 1.3, first we prove several lemmas.

Lemma 3.1 Let the conditions $i)$ – $ii)$ be fulfilled. Then there exist a number $K\left(T\right)$ such that

$$\tilde{M}\subseteq M\subseteq\tilde{\tilde{M}},$$

where

$$\tilde{\tilde{M}}=\left\{y\in L_{2}\left(R\right):\left\|-y''\right\|_{2}^{2}+\left\|q\left(x,y\right)y\right\|_{2}^{2}\leq K\left(T\right)\right\},$$

$$\tilde{M}=\left\{y\in L_{2}\left(R\right):\left\|-y''\right\|_{2}^{2}+\left\|q\left(x,y\right)y\right\|_{2}^{2}\leq\frac{T}{2}\right\}.$$

Proof. Let $y\in\tilde{M}$. Then, using the triangle inequality, we get

$$\left\|-y''+q\left(x,y\right)y\right\|_{2}^{2}\leq2\left(\left\|-y''\right\|_{2}^{2}+\left\|q\left(x,y\right)y\right\|_{2}^{2}\right)\leq2\cdot\frac{T}{2}\leq T.$$

It follows that $y\in M$, i.e. $\tilde{M}\subseteq M$.

Let $y\in M$. Then, by virtue of Lemma 2.1 and the estimate (22) and the embedding theorem $W_{2}^{1}\left(R\right)$ in the space $C\left(R\right)$ we have

$$\|y\|_{C(R)}\leq c\left\|-y''+q\left(x,y\right)y\right\|_{2},$$

where c is independent of $y\left(x\right)$ и $q\left(x,y\right)$.

It follows that

$$\sup_{y\in M}\|y\left(x\right)\|_{C(R)}\leq c\cdot T^{1/2}\tag{24}$$

On the other hand, using the estimate (22), we have

$$\left|y\left(x\right)-y\left(\eta\right)\right|\leq c\left\|-y''+q\left(x,y\right)y\right\|\leq c\cdot T^{1/2}\tag{25}$$

for any $y\in M$, where $c>0$ is a constant independent of $y\left(x\right)$.

Now, supposing $y\left(x\right)=c_{1},\ y\left(\eta\right)=c_{2},\ A=c\cdot T^{1/2}$ from (25) we obtain that $\left|c_{1}-c_{2}\right|\leq A$.

Let $y_0(x) \in M$ and suppose $q_0(x) = q(x, y_0(x))$. Denote by L the closure in the norm of $L_2(R)$ of the operator defined on $C_0^\infty(R)$ by the equality

$$L_0 y = -y''(x) + q_0(x)\, y.$$

It is easy to verify that the operator L is self-adjoint, positive definite and $y_0(x) \in D(L)$, wherein the estimate

$$\left\|-y_0''\right\|_2 \le \mu(A) \left\|-y_0 + q(x, y_0)\, y\right\|_2 \tag{26}$$

holds. The estimate (26) follows from Theorem 1.1.

This shows that the inequality

$$\left\|-y''\right\|_2 \le \mu(A)\, T^{1/2}. \tag{27}$$

holds for all $y \in M$.

From the inequality (27) we have

$$\left\|q(x,y)\, y\right\|_2 = \left\|-y'' + q(x,y)\, y + y''\right\|_2 \le \left\|y''\right\|_{,2} + \left\|-y'' + q(x,y)\, y\right\|_2 \le$$

$$\le \mu(A) \cdot T^{1/2} + T^{1/2} \le 2\mu(A) \cdot T^{1/2} \tag{28}$$

for any $y \in M$. Here we take into account that the condition $ii)$ implies that $\mu(A) \ge 1$.

From the inequalities (27) and (28) we find

$$\left\|-y''\right\|_2^2 + \left\|q(x,y)\, y\right\|_2^2 \le \mu^2(A) \cdot T + 4\mu^2(A) \cdot T \le K(T) \tag{29}$$

for any $y \in M$, where $K(T) = 5\mu^2(A) \cdot T$. The estimate (29) proves Lemma 3.1.

Lemma 3.2. Let the conditions $i) - ii)$ be fulfilled. Then $\tilde{\tilde{M}} \subseteq \tilde{\tilde{B}}$, where

$$\tilde{\tilde{B}} = \left\{u \in L_2(R) : \left\|-y''\right\|_2^2 + \left\|q(x,0)\, y\right\|_2^2 \le K_1(T)\right\}.$$

Proof. By the embedding theorems, we have

$$\|y\|_{C(R)} \le c \left(\left\|-y''\right\|_2^2 + \left\|q(x,y)\, y\right\|_2^2\right)^{1/2} \le c \cdot K(T) \tag{30}$$

for any $y(x) \in \tilde{\tilde{M}}$, where $c > 0$ is the constant of the embedding theorem.

Hence, using the computations and arguments used in the proof of (29), we obtain that

$$y(x) = c_1,\ \ y(\eta) = c_2,\ \ |c_1 - c_2| \le A,\ \ A = 2c \cdot K^{1/2}(T). \tag{31}$$

Hence, using the conditions of $ii)$ for all $y(x) \in \tilde{\tilde{M}}$, we have

$$\mu^{-1}(A)\, q(x,0) \le q(x, y(x)) \le \mu(A)\, q(x,0), \tag{32}$$

where $A = 2c \cdot K^{1/2}(T)$, $\mu(A) = \mu\left(2cK^{1/2}(T)\right)$.
From (32) we have

$$\left\|-y''\right\|_2^2 + \left\|q(x,0)\, y\right\|_2^2 \le \left\|-y''\right\|_2^2 + \mu^2(A) \left\|q(x,y)\, y\right\|_2^2 \le \mu^2(A) \left(\left\|-y''\right\|_{,2}^2 +\right.$$

$$\left. + \left\|q(x,y)\, y\right\|_2^2\right) \le \mu^2(A) \cdot K(T) \le K_1(T),\ K_1(T) = \mu^2\left(2cK^{1/2}\right) \cdot K(T)$$

for any $y(x) \in \tilde{\tilde{M}}$. This implies $\tilde{\tilde{M}} \subseteq \tilde{\tilde{B}}$.

Lemma 3.3. Let the conditions $i) - ii)$ be fulfilled. Then $\tilde{B} \subseteq \tilde{M}$, where

$$\tilde{B} = \left\{ u \in L_2(R) : \left\| -y'' \right\|_2^2 + \| q(x,0)\, y \|_2^2 \leq K_2(T) \right\},$$

$K_2(T)$ is a positive number depending on T, such that $K_2(T) \leq \frac{T}{2}$.

Proof. Let $u \in \tilde{B}$. Then, using the embedding theorem, we have

$$\|y\|_{C(R)} \leq c \cdot K_2(T) \leq c \cdot \frac{T}{2}, \tag{33}$$

$c > 0$ is the constant of the embedding theorem from $W_2^2(R)$ to $C(R)$.

Now, using the condition $ii)$, we obtain from (33) that for all $u \in \tilde{B}$

$$\mu^{-1}\left(c \cdot \frac{T}{2}\right) q(x,0) \leq q(x, y(x)) \leq \mu \left(c \cdot \frac{T}{2}\right) q(x,0). \tag{34}$$

Hence, we find

$$\left\| -y'' \right\|_2^2 + \| q(x,y)\, y \|_2^2 \leq \left\| -y'' \right\|_2^2 + \mu^2\left(c \cdot \frac{T}{2}\right) \cdot \| q(x,0)\, y \|_2^2 \leq$$

$$\leq \mu^2\left(c \cdot \frac{T}{2}\right)\left(\| -y \|_{,2}^2 + \| q(x,0)\, y \|_2^2\right) \leq \mu^2\left(c \cdot \frac{T}{2}\right) K_2(T)$$

for any $y \in \tilde{B}$.

If we assume $K_2(T) = \frac{\frac{T}{2}}{\mu^2\left(c \cdot \frac{T}{2}\right)}$ then the inequality $\| -y'' \|_2^2 + \| q(x,y)\, y \|_2^2 \leq \frac{T}{2}$ holds for all $y \in \tilde{B}$. Therefore $\tilde{B} \subseteq \tilde{M}$. Lemma 3.3 is proved.

Lemma 3.4. Let the conditions $i) - ii)$ be fulfilled. Then the estimates

$$c^{-1} d_k \leq \tilde{d}_k \leq c d_k, \;\; k = 1, 2, ...$$

hold, where $c > 0$ depends only on T, $\tilde{d}_k, d_k$ k are the Kolmogorov widths of the sets M and B, respectively, where $B = \left\{ y \in L_2(R) : \| -y'' \|_2^2 + \| q(x,0)\, y \|_2^2 \leq 1 \right\}$.

This lemma is proved in the same way as Lemma 4.3 in [18].

Lemma 3.5. Let the conditions $i) - ii)$ be fulfilled. Then the estimates

$$N(c\lambda) \leq \tilde{N}(\lambda) \leq N\left(c^{-1}\lambda\right)$$

hold, where $N(\lambda) = \sum_{d_k > \lambda} 1$ is the counting function of those d_k greater than $\lambda > 0$, $\tilde{N}(\lambda) = \sum_{\tilde{d}_k > \lambda} 1$ is the counting function of those $\tilde{d}_k$ greater than $\lambda > 0$, $c > 0$ is a constant.

The proof of this lemma follows from Lemma 3.4.

Proof of Theorems 1.3-1.4. Repeating the computations and arguments used in the proof of Theorems 1.2–1.3 from [18] we obtain the proof of Theorem 1.3. □

Proof of Theorem 1.4. Using Lemmas 3.4–3.5, the proofs of Theorems 1.1—1.4 from [18] and the results from [19], we obtain the proof of Theorem 1.4. □

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Список литературы

1 Ладыженская О.А. Линейные и квазилинейные уравнения эллиптического типа / О.А. Ладыженская, Н.Н. Уральцева. — М.: Наука, 1973. — 576 с.

2 Лионс Ж.Л. Некоторые методы решения нелинейных краевых задач / Ж.Л. Лионс. — М.: Мир, 1972. — 586 с.

3 Похожаев С.Н. О квазилинейных эллиптических уравнениях высокого порядка / С.Н. Похожаев // Дифференциальные уравнения. — 1981. — 17. — № 1. — С. 115-128.

4 Похожаев С.Н. О некоторых весовых тождествах для решений обобщенных уравнений Кортевега-де Фриза / С.Н. Похожаев // Математические заметки. — 2011. — 89. — № 3. — С. 382-396.

5 Куфнер А. Нелинейные дифференциальные уравнения / А. Куфнер, С. Фучик. — М.: Наука, 1988. — 304 с.

6 Zidkov P.E. Korteweg-de Vries and nonlinear Schrödinger equations: Qualitative Theory. — Berlin: Springer, 2001. — 147 p.

7 Kato T. Perturbation theory for linear operators. — Berlin, Heidelberg, NewYork: Springer-Verlag Inc., 1966. — 623 p.

8 Костюченко А.Г. Распределение собственных значений / А.Г. Костюченко, И.С. Саргсян. — М.: Наука, 1979. — 399 с.

9 Рид М. Методы современной математической физики. Т. 2. Гармонический анализ. Самосопряженность / М. Рид, Б. Саймон. — М.: Мир, 1978. — 395 с.

10 Everitt W. On some properties of the powers of a formally self adjoins differential expression / W. Everitt, M. Giertz // Proc. London Math. Soc. — 1972. — 24. — No. 3. — P. 149-170.

11 Отелбаев М. О суммируемости с весом решения уравнения Штурма-Лиувилля / М. Отелбаев // Математические заметки. — 1974. — 16. — № 6. — С. 969-980.

12 Отелбаев М. Оценки спектра оператора Штурма-Лиувилля / М. Отелбаев. — Алма-Ата: Ғылым, 1990. — 190 с.

13 Haller R. Muckenhoupt weights and maximal L^p-regularity / R. Haller, H. Heck, M. Hieber // Arch. Math. — 2003. — 81. — P. 422-430.

14 Kunstmann P.C. Maximal L^p-regularity for Parabolic Equations, Fourier Multiplier Theorems and H^∞-functional Calculus / P.C. Kunstmann, L. Weis // Da Prato et al: LNM. — 2004. — 1855. — P.65-311.

15 Weis L. Operator-valued Fourier Multiplier Theorems and maximal L^p-regularity // Math.Ann. — 2001. — 319. — P. 735-758.

16 Муратбеков М.Б. Разделимость и оценки поперечников множеств, связанных с областью определения нелинейного оператора типа Шредингера / М.Б. Муратбеков // Дифференциальные уравнения. — 1991. — 27. — № 6. — С. 1034-1044.

17 Гохберг И.Ц. Введение в теорию линейных несамосопряженных операторов в гильбертовом пространстве / И.Ц. Гохберг, М.Г. Крейн. — М.: Наука, 1965. — 448 с.

18 Muratbekov M.B. Sturm-Liouville operator with a parameter and it is usage to spectrum research of some differential operators / M.B. Muratbekov, M.M. Muratbekov // Complex Variables and Elliptic Equations. — 2019. — 64. — No. 9. — P. 1457-1476.

19 Отелбаев М. Теоремы вложения пространств с весом и их применения к изучению спектра оператора Шредингера / М. Отелбаев // Тр. МИ имени В.А. Стеклова АН СССР. — 1979. — 150. — С. 265-305.

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Сингулярлы сызықты емес Штурм-Лиувилль теңдеуінің шешімінің тегістігі мен аппроксимативті қасиеттері туралы

Кему тәртібімен еселігі бойынша реттелген өз-өзіне түйіндес Штурм-Лиувилль операторымен L^{-1} жете үзіліссіз кері операторының меншікті сандарының келесі қасиеті бар екендігі белгілі: $^{(*)}$ $\lambda_n \to 0$, егер $n \to \infty$, сонымен қатар нөлге ұмтылу жылдам болған сайын, L^{-1} операторы ақырлы рангілі операторлар арқылы жақсырақ жуықталады. Сонымен қатар сызықты емес операторларға келесі сұрақ туындайды: «Берілген сызықты емес операторға (*) қасиетімен сипатталатын кемімелі сандық тізбегін көрсетуге бола ма?» Мақалада сызықты емес Штурм-Лиувиль операторына арналған жоғарыда келтірілген сұрақ зерттелді. Аталған мәселені шешу үшін $L_2(R)\,(R = (-\infty, \infty))$ кеңістігінде жылдам өспелі және жылдам тербелмелі потенциалы бар сызықты емес Штурм-Лиувилль теңдеуінің шешімдерінің максималды регулярлығы туралы теорема дәлелденді және сызықты емес Штурм-Лиувилль теңдеуінің шешімдерімен байланысты жиындардың Колмогоров енінің екіжақты бағалаулары алынды. Белгілі болғандай, Колмогоров енінің алынған бағалаулар ең аз қателікке кепілдік беретін жуықтау аппаратын таңдауға мүмкіндік береді.

Кілт сөздер: максималды регулярлығы, сингулярлы сызықты емес теңдеу, Штурм-Лиувилль теңдеуі, шешімдердің тегістігі, аппроксимативті қасиеттер, аппроксимативті сандар, Колмогоров ені, жылдам тербелмелі потенциал, қатты өспелі потенциал, екіжақты бағалау.

М.Б. Муратбеков, М.М. Муратбеков

О гладкости и аппроксимативных свойствах решений сингулярного нелинейного уравнения Штурма-Лиувилля

Известно, что собственные числа $\lambda_n (n = 1, 2, ...)$, пронумерованные в порядке убывания и с учетом кратности самосопряженного оператора Штурма-Лиувилля с вполне непрерывным обратным оператором L^{-1}, обладают следующим свойством: $^{(*)}$ $\lambda_n \to 0$, когда $n \to \infty$, причем, чем быстрее стремление к нулю, тем оператор L^{-1} лучше аппроксимируется с операторами конечного ранга. Естественным образом возникает следующий вопрос для нелинейных операторов: «Можно ли для заданного нелинейного оператора указать убывающую числовую последовательность, которая характеризуется свойством (*)?». В статье изучен указанный выше вопрос для нелинейного оператора Штурма-Лиувилля. Для решения задачи доказана теорема о максимальной регулярности решений нелинейного уравнения Штурма-Лиувилля с сильно растущим и быстро колеблющимся потенциалом в пространстве $L_2(R)\,(R = (-\infty, \infty))$, а также получены двусторонние оценки поперечников по Колмогорову множества, связанные с решениями нелинейного уравнения Штурма-Лиувилля. Как известно, полученные оценки поперечников по Колмогорову дают возможность выбирать аппарат приближения, который гарантирует минимально возможную погрешность.

Ключевые слова: максимальная регулярность, сингулярное нелинейное уравнение, уравнение Штурма-Лиувилля, гладкость решений, аппроксимативные свойства, аппроксимативные числа, поперечники по Колмогорову, быстро колеблющийся потенциал, сильно растущий потенциал, двусторонние оценки.

References

1 Ladyzhenskaya, O.A. & Ural'ceva, N.N. (1973). *Lineinye i kvazilineinye uravneniia ellipticheskoho tipa [Linear and quasilinear equations of elliptic type]*. Moscow: Nauka [in Russian].

2 Lyons, J.L. (1972). *Nekotorye metody resheniia nelineinykh kraevykh zadach [Some methods for solving nonlinear boundary value problems]*. Moscow: Mir [in Russian].

3 Pokhozhaev, S.I. (1981). O kvazilineinykh ellipticheskikh uravneniiakh vysokoho poriadka [Higher-order quasilinear elliptic equations]. Differentsialnye uravneniia – Differential Equation, 17, 6, 78-88.

4 Pokhozhaev, S.I. (2011). O nekotorykh vesovykh tozhdestvakh dlia reshenii obobshchennykh uravnenii Korteveha-de Friza [Weighted identities for the solutions of generalized Korteweg–de Vries equations]. *Matematicheskie Zametki – Math. Notes, 89, 3,* 382-396 [in Russian].

5 Kufner, A. & Fučik, S. (1988). *Nelineinye differentsialnye uravneniia [Nonlinear differential equations].* Moscow: Nauka [in Russian].

6 Zidkov, P.E. (2001). *Korteweg-de Vries and nonlinear Schrödinger equations: Qualitative Theory.* Berlin: Springer.

7 Kato, T. (1966). *Perturbation theory for linear operators.* Berlin, Heidelberg, NewYork: Springer-Verlag Inc.

8 Kostyuchenko, A.G. & Sargsyan, I.S. (1979). *Raspredelenie sobstvennykh znachenii [Distribution of eigenvalues].* Moscow: Nauka [in Russian].

9 Reed, M. & Simon, B. (1978). *Metody sovremennoi matematicheskoi fiziki. T. 2. Harmonicheskii analiz. Samosopriazhennost [Methods of modern mathematical physics. V.2. Harmonic analysis. Self-conjugation].* Moscow: Mir [in Russian].

10 Everitt, W. & Giertz, M. (1972). On some properties of the powers of a formally self adjoins differential expression. *Proc. London Math. Soc., 24, 3,* 149-170.

11 Otelbaev, M. (1974). O summiruemosti s vesom resheniia uravneniia Shturma-Liuvillia [On summability with a weight of a solution of the Sturm-Liouville equation]. *Matematicheskie Zametki - Math. Notes, 16, 6,* 969-980 [in Russian].

12 Otelbaev, M. (1990). *Otsenki spektra operatora Shturma-Liuvillia [Estimates of the spectrum of the Sturm–Liouville operator].* Almaty: Hylym [in Russian].

13 Haller, R. & Heck, H. & Hieber, M. (2003). Muckenhoupt weights and maximal L^p-regularity. *Arch. Math., 81,* 422-430.

14 Kunstmann, P.C. & Weis, L. (2004). Maximal L^p-regularity for Parabolic Equations, Fourier Multiplier Theorems and H^∞- functional Calculus. Da Prato et al: LNM. 1855, 65-311.

15 Weis, L. (2001). Operator-valued Fourier Multiplier Theorems and maximal L^p-regularity. *Math. Ann., 319,* 735-758.

16 Muratbekov, M.B. (1991). Razdelimost i otsenki poperechnikov mnozhestv, sviazannykh s oblastiu opredeleniia nelineinoho operatora tipa Shredinhera [Separability and estimates of diameters of sets associated with the domain of definition of a nonlinear operator of Schrödinger type]. *Differentsialnye uravneniia — Differential Equation, 27, 6,* 734-741.

17 Gokhberg, I.T. & Krein, M.G. (1965). *Vvedenie v teoriiu lineinykh nesamosopriazhennykh operatorov v hilbertovom prostranstve [Introduction to the theory of linear non-self-adjoint operators in a Hilbert space].* Moscow: Nauka [in Russian].

18 Muratbekov, M.B. & Muratbekov, M.M. (2019). Sturm-Liouville operator with a parameter and it is usage to spectrum research of some differential operators. *Complex Variables and Elliptic Equations, 64, 9,* 1457-1476.

19 Otelbaev, M. (1979). Teoremy vlozheniia prostranstv s vesom i ikh primeneniia k izucheniiu spektra operatora Shredinhera [Imbedding theorems for spaces with a weight and their application to the study of the spectrum of a Schrödinger operator]. *Trudy Matematicheskoho Instituta imeni V.A. Steklova AN SSSR - Proceedings of the Steklov Mathematical Institute, 150,* 265-305 [in Russian].