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D.M. Dovletov[1,2]

[1]*Department of Mathematics, Near East University, Nicosia, TRNC, Mersin 10, Turkey*
[2]*Institute of Mathematics, Ashgabat, Turkmenistan*
(E-mail: dovlet.dovletov@gmail.com)

Nonlocal boundary value problem with Poisson's operator on a rectangle and its difference interpretation

In the present paper, differential and difference variants of nonlocal boundary value problem (NLBVP) for Poisson's equation in open rectangular domain are studied. The existence, uniqueness and a priori estimate of classical solution are established. The second order of accuracy difference scheme is presented. The applications with weighted integral condition are provided in differential and difference variants.

Keywords: Poisson's operator, nonlocal boundary value problem, rectangle, difference scheme.

Introduction

Firstly, NLBVP for Laplace's equation in a rectangular domain was considered by A.V. Bitsadze and A.A. Samarskii [1]. Later, the n-dimensional problem was studied by A. L. Skubachevskii [2].

V. A. Il'in and E. I. Moiseev [3] studied 2-d NLBVP with Poisson's operator on rectangle Π

$$\begin{cases} \Delta u = f(x,y),\ (x,y) \in \Pi = (0,1)\times(0,\pi), \\ u(x,0) = u(x,\pi) = u(0,y) = 0,\ u(1,y) = \sum\limits_{k=1}^{m} \alpha_k u(\xi_k, y),\ x\in[0,1],\ y\in[0,\pi],\ \xi_k\in(0,1) \end{cases}$$

and proved the existence and uniqness of classical solution when $\sum\limits_{k=1}^{m} \frac{1}{2}(\alpha_k + |\alpha_k|) \le 1$, established a priori estimate $||u||_{W_2^2(\Pi)} \le C||f||_{L_2(\Pi)}$ when $-\infty < \sum\limits_{k=1}^{m} \alpha_k \le 1$ and if all $\alpha_k,\ k=\overline{1,m}$ have the same sign and given this condition offered the second order of accuracy difference scheme on a uniform grid.

In [4], E. A. Volkov demonstrated a simple proof of the existence and uniqueness of classical solution for Laplace's equation with the original Bitsadze-Samarskii nonlocal boundary value condition (NLBVC), proposed a finite-difference method on a square mesh that produces a uniform approximation by the second order of accuracy in the difference metric C, applied the method to Poisson's equation $\Delta u = g$ when $g \in C^{2,\lambda}$ for $0<\lambda<1$. In [5], he studied a solvability of the multilevel NLBVP for Poisson's operator on rectangular domain by applying the contraction mapping principle.

In [6], A. Ashyralyev established well-posedness of NLBVP in the open square $\Omega = (0,1)\times(0,1)$ by proving the coercive inequalities for solution of the differential problem

$$u_{tt}(t,x) + a(x)u_{xx}(t,x) - \delta u(t,x) = f(t,x) \text{ in } \Omega,\ u(0,x) = u(t,0) = u(t,1) = 0,\ u(1,x) = u(\lambda,x) \text{ in } \overline{\Omega},$$

when smooth functions $a(x)$ and $f(t,x)$ satisfy the conditions

$$a(x) \ge 0,\ f(0,x) = 0,\ f(1,x) = f(\lambda,x),\ 0 \le x \le 1,\ 0 \le \lambda < 1,$$

where $\delta > 0$ is sufficiently large number. In Ω, under the condition $\int\limits_0^1 |\rho(t)|dt < 1$, E. Ozturk [7] studied well-posedness of NLBVP for elliptic equation with integral type of NLBVC (in $\overline{\Omega}$) by reaching the coercive inequalities for solution of the problem

$$u_{tt}(t,x) + (a(x)u_x(t,x))_x = f(t,x),\ u(t,0) = u(t,1) = 0,\ u(0,x) = \varphi(x),$$

$$u(1,x)=\int\limits_0^1 \rho(t)u(t,x)dt+\psi(x)$$

and offered the first order of accuracy difference scheme against the term $\sum\limits_{j=1}^{N}|\rho(t_j)\tau|<1,\ \tau=1/N.$

By returning to Laplace's operator on rectangular domain we note, that various numerical methods on multilevel and integral type of NLBVPs were researched in [8–11] and other papers.

In the present paper, we generalize and prove the statements of the preliminary abstract [19] and, additionally, apply our results to NLBVP with integral conditions. We study the problem

$$\begin{cases} \Delta u(x,y)=f(x,y),\ (x,y)\in\Pi, \\ u(x,0)=u(x,\pi)=u(0,y)=0,\ u(1,y)=\sum\limits_{r=1}^{n}\alpha_r u(\zeta_r,y)-\sum\limits_{s=1}^{m}\beta_s u(\eta_s,y)=0,\ x\in[0,1],\ y\in[0,\pi], \end{cases}$$

where $f\in C(\overline{\Pi})$, $\alpha_r>0$, $\beta_s>0$, $0<\zeta_1<...<\zeta_n<1$ and $0<\eta_1<...<\eta_m<1$, $-\infty<\sum\limits_{r=1}^{n}\alpha_r-\sum\limits_{s=1}^{m}\beta_s\le 1$ when $\zeta_n<\eta_1$; $\sum\limits_{r=1}^{n}\alpha_r\le 1$ when $\zeta_n\ge\eta_1$. We prove the existence, uniqueness and a priori estimate $||u||_{W_2^2(\Pi)}\le C||f||_{L_2(\Pi)}$ of the classical solution. Particularly, we consider the problem when $n=m$ and $\zeta_r<\eta_r$, $r=\overline{1,n}$ and for this special subcase we prove the existence, uniqueness and a priori estimate when $\sum\limits_{r=1}^{n}\frac{(\alpha_r-\beta_r)+|\alpha_r-\beta_r|}{2}\le 1$. We offer the finite difference variants on a uniform grid and prove the second order of accuracy in terms of $h=\sqrt{h_1^2+h_2^2}$ for $h_1\le c_0h_2$, $h_2\to 0$ in respect of each difference metrics C and W_2^2.

As an application, we study NLBVP for Poisson's equation with weighted integral condtion (WIC)

$$\begin{cases} \Delta u(x,y)=f(x,y),\ (x,y)\in\Pi, \\ u(x,0)=u(x,\pi)=u(0,y)=0,\ u(1,y)=\int\limits_0^1\rho(x)u(x,y)dx=0,\ 0\le x\le 1,\ 0\le y\le\pi \end{cases}$$

respectively the behavior of $\rho(x)$, $\rho(x)\in C^0[\tau_0,\tau_1]$, i.e., $[\tau_0,\tau_1]\subset(0,1)$, $\rho(x)\equiv 0$ in $[0,1]\setminus[\tau_0,\tau_1]$. We prove the existence, uniqueness and a priori estimate under the conditions on $\rho(x)$ subject to whether or no the weight function changes the sign, whether or no the sign changing acts from plus to minus or vice verca, whether or no the number of sign changes is an even or odd. Particularly, when $\rho(x)$ does not change the sign and $-\infty<\int\limits_{\tau_0}^{\tau_1}\rho(x)dx\le 1$, we prove the existence, uniqueness, a priori estimate and offer the second order of accuracy difference sheme.

Differential problem

We consider NLBVP in the rectangle $\Pi=(0<x<1)\times(0<y<\pi)$

$$\begin{cases} \Delta u(x,y)=f(x,y),\ (x,y)\in\Pi, \\ u(x,0)=u(x,\pi)=0,\ 0\le x<1,\ \ u(0,y)=0,\ \ell[u](y)=0,\ 0\le y\le\pi, \end{cases} \tag{1}$$

where

$$\ell[u](y)\equiv u(1,y)-\sum_{r=1}^{n}\alpha_r u(\zeta_r,y)+\sum_{s=1}^{m}\beta_s u(\eta_s,y)\ , \tag{2}$$

$0<\zeta_1<...<\zeta_n<1,\ 0<\eta_1<...<\eta_m<1,\ \zeta_r\ne\eta_s,\ \alpha_r>0,\ \beta_s>0,\ r=\overline{1,n},\ s=\overline{1,m}$. We study the classical solution $u(x,y)\in C^2(\Pi)\cap C(\overline{\Pi})$ that satisfies the equation and all conditions of (1).

Further, on default, the symbol ***A1*** denotes the term: $-\infty<\sum\limits_{r=1}^{n}\alpha_r-\sum\limits_{s=1}^{m}\beta_s\le 1$ holds when $\zeta_n<\eta_1$.

The symbol ***A2*** denotes: $\sum\limits_{r=1}^{n}\alpha_r\le 1$ holds when $\zeta_n\ge\eta_1$. The ***A*** denotes that ***A1*** holds or ***A2*** holds.

Theorem 1. Let $f(x,y) \in C(\overline{\Pi})$. If $\boldsymbol{A}$ holds, then classical solution of (1) exists and it is an unique.

Proof. Assume that classical solution of (1) exists. To prove the uniqueness it is sufficiently to show that $u(x,y) \equiv 0$ if $f(x,y) \equiv 0$. Put $f(x,y) \equiv 0$ in $\overline{\Pi}$. Then $u(x,y)$ is the solution of Laplace's equation, therefore, for each natural number $k \in N$ the function

$$X_k(x) = \sqrt{2/\pi} \int_0^{\pi} u(x,y) \sin(ky) dy \tag{3}$$

satisfies the equation $X_k''(x) - k^2 X_k(x) = 0, \ 0 < x < 1$. Moreover, since $u(0,y) = \ell[u](y) = 0$, then

$$X_k(0) = 0, \ \ X_k(1) = \sum_{r=1}^{n} \alpha_r X_k(\zeta_r) - \sum_{s=1}^{m} \beta_s X_k(\eta_s).$$

Hence, $X_k(x)$ is the solution of the multipoint problem

$$X_k''(x) - k^2 X_k(x) = 0, \ \ 0 < x < 1, \quad X_k(0) = 0, \ \ \ell[X_k] = 0, \tag{4}$$

where $\ell[X_k] = X_k(1) - \sum\limits_{r=1}^{n} \alpha_r X_k(\zeta_r) + \sum\limits_{s=1}^{m} \beta_s X_k(\eta_s)$. By virtue of mean value (MV) property [12, p. 1198-1199] (see also [13, 18, 20]) we get that solution of (4) satisfies the problem[1] [17, p. 92-93]

$$X_k''(x) - k^2 X_k(x) = 0, \ 0 < x < 1, \quad X_k(0) = 0, \ X_k(1) = \alpha X_k(\zeta_{[k]}) - \beta X_k(\eta_{[k]}) \ , \tag{5}$$

where[2] $\alpha = \sum\limits_{r=1}^{n} \alpha_r, \ \beta = \sum\limits_{s=1}^{m} \beta_s, \ \zeta_{[k]} \in [\zeta_1, \zeta_n], \ \eta_{[k]} \in [\eta_1, \eta_m]$ and $\zeta_{[k]} < \eta_{[k]}$ when $\zeta_n < \eta_1$. By virtue of [16, p. 1298-1299] we conclude that (5) has only trivial solution since $\boldsymbol{A}$ holds, i.e., $X_k(x) \equiv 0$ in the interval $[0,1]$. Hence, from (3), using the completeness of orthonormal system $\{\sqrt{2/\pi}\sin(ky), \ k \in N\}$ on the interval $0 \le y \le \pi$, we result $u(x,y) \equiv 0$ in $\overline{\Pi}$. Since the uniqueness is proved, then the existence follows from Fredholm's property [2] inherent (1). Theorem 1 is proved.

Theorem 2. Let $f \in C(\overline{\Pi})$. If $\boldsymbol{A}$ holds, then for classical solution of (1) a priori estimate holds

$$||u||_{W_2^2(\Pi)} \le C||f||_{L_2(\Pi)}. \tag{6}$$

Proof. To prove (6) it is sufficiently to establish the estimates

$$||X_k||_{L_2[0,1]} \le \frac{C_1}{k^2}||f_k||_{L_2[0,1]}, \ \ ||X_k'||_{L_2[0,1]} \le \frac{C_2}{k}||f_k||_{L_2[0,1]}, \ \ ||X_k''||_{L_2[0,1]} \le C_3||f_k||_{L_2[0,1]} \tag{7}$$

for $k \in N$, where

$$f_k(x) = \sqrt{2/\pi} \int_0^{\pi} f(x,y) \sin(ky) dy \ , \tag{8}$$

so that (7) [3, p. 142-143] results in

$$||u||_{W_2^2(\Pi)} \le C_1||f||_{L_2(\Pi)}, \ \ ||u_{xx}||_{W_2^2(\Pi)} \le C_2||f||_{L_2(\Pi)}, \ \ ||u_{xy}||_{W_2^2(\Pi)} \le C_3||f||_{L_2(\Pi)}, \tag{9}$$

[1] Further in similar obstacles we will say, for example: the problem (4) is reducible to the problem (5), or the nonlocal condition (4) is reducible to the nonlocal condition (5), or we reduce (4) to (5).

[2] Further in this section the symbols α and β denote the sums $\alpha = \sum\limits_{r=1}^{n} \alpha_r$ and $\beta = \sum\limits_{s=1}^{m} \beta_s$.

and, after all, (9) results in (6). Hence, our target is to prove (7). Thereto, using (3) and (8) for equation $\Delta u(x,y)=f(x,y)$ and conditions $u(0,y)=0$, $u(1,y)=\sum\limits_{r=1}^{n}\alpha_r u(\zeta_r,y)-\sum\limits_{s=1}^{m}\beta_s u(\eta_s,y)$, we conclude that $X_k(x)$ satisfies the nonhomogeneous multipoint problem (this problem was studied in [16, 17])

$$X_k''(x)-k^2X_k(x)=f_k(x),\ 0<x<1,\ \ X_k(0)=0,\ X_k(1)=\sum_{r=1}^{n}\alpha_r X_k(\zeta_r)-\sum_{s=1}^{m}\beta_s X_k(\eta_s)\,. \tag{10}$$

Actually, the estimate

$$|\ X_k(1)\ |\ \le C\frac{\sqrt{2}}{k^{3/2}}\ ||f_k(x)||_{L_2[0,1]} \tag{11}$$

results in the estimates (7). Indeed, put $X_k(x)=\overline{X}_k(x)+\overline{\overline{X}}_k(x)$, where $\overline{X}_k(x)$ is the solution of

$$\overline{X}_k''(x)-k^2\overline{X}_k(x)=f_k(x),\ 0<x<1,\ \ \overline{X}_k(0)=\overline{X}_k(1)=0\ , \tag{12}$$

and $\overline{\overline{X}}_k(x)$ is the soluion of

$$\overline{\overline{X}}_k''(x)-k^2\overline{\overline{X}}_k(x)=0,\ 0<x<1,\ \ \overline{\overline{X}}_k(0)=0,\ \overline{\overline{X}}_k(1)=X_k(1)\ . \tag{13}$$

Thereby, it is sufficiently to show that the analog of (7) holds for each of the functions $\overline{X}_k(x)$ and $\overline{\overline{X}}_k(x)$. Thereto, we use the explicit solution of (13) to get

$$||\overline{\overline{X}}_k||_{L_2[0,1]}\le |\ X_k(1)\ |\ \Big(\frac{\int_0^1\sinh^2(kx)dx}{\sinh^2 k}\Big)^{1/2}, \tag{14}$$

$$||\overline{\overline{X}}_k'||_{L_2[0,1]}\le k\ |\ X_k(1)\ |\ \Big(\frac{\int_0^1\cosh^2(kx)dx}{\sinh^2 k}\Big)^{1/2}, \tag{15}$$

$$||\overline{\overline{X}}_k''||_{L_2[0,1]}\le k^2\ |\ X_k(1)\ |\ \Big(\frac{\int_0^1\sinh^2(kx)dx}{\sinh^2 k}\Big)^{1/2}, \tag{16}$$

and then, in view of $\frac{\int_0^1\sinh^2(kx)dx}{\sinh^2 k}\le\frac{1}{k}$ and $\frac{\int_0^1\cosh^2(kx)dx}{\sinh^2 k}\le\frac{5}{2k}$, from (14)-(16), we get

$$||\overline{\overline{X}}_k||_{L_2[0,1]}\le\frac{C\sqrt{2}}{k^2}||f_k||_{L_2[0,1]},\ \ ||\overline{\overline{X}}_k'||_{L_2[0,1]}\le\frac{C\sqrt{5}}{k}||f_k||_{L_2[0,1]},\ \ ||\overline{\overline{X}}_k''||_{L_2[0,1]}\le C\sqrt{2}||f_k||_{L_2[0,1]}. \tag{17}$$

It means that if (11) holds, then (7) holds for the function $\overline{\overline{X}}_k(x)$. Moreover, if (11) holds, then (7) holds for $\overline{X}_k(x)$ [3, p. 143-144]. Therefore, to establish (7) for $X_k(x)$ it is sufficiently to prove (11).

Let we prove (11). In view of [17, 92-93] the multipoint problem (10) is reducible to 3-point problem

$$X_k''(x)-k^2X_k(x)=f_k(x),\ 0<x<1,\ \ \ X_k(0)=0,\ \ X_k(1)=\alpha X_k(\zeta_{[k]})-\beta X_k(\eta_{[k]})\ , \tag{18}$$

where the points $\zeta_{[k]}\in[\zeta_1,\zeta_n]$, $\eta_{[k]}\in[\eta_1,\eta_m]$, so that $\zeta_{[k]}<\eta_{[k]}$ when $\zeta_n<\eta_1$. Therefore, it is sufficiently to obtain the estimate (11) for the solution of (18) when the term ***A*** holds.

Let ***A1*** holds, i.e., $-\infty<\alpha-\beta\le 1$ and $\zeta_n<\eta_1$. Put $sign\big(X_k(1)\ X_k(\eta_{[k]})\ X_k(\zeta_{[k]})\big)\ne 0$. We consider the alternate subcases: $sign\big(X_k(1)X_k(\eta_{[k]})\big)=-1$ and $sign\big(X_k(1)X_k(\eta_{[k]})\big)=1$. Note in advance, if $sign\big(X_k(1)\ X_k(\eta_{[k]})\ X_k(\zeta_{[k]})\big)=0$, then (11) results from the current proof.

Subcase 1.1 :

If $sign\big(X_k(1)X_k(\eta_{[k]})\big)=-1$, then in view of Bolzano theorem $X_k(\tau_k)=0$ for $\tau_k\in(\eta_{[k]},1)$. Then by virtue of [3, 143-144]

$$||\ X_k\ ||_{L_2[0,\tau_k]}\ \le\frac{1}{k^2}\ ||f_k||_{L_2[0,\tau_k]}\ ,\ \ ||\ X_k'\ ||_{L_2[0,\tau_k]}\ \le\frac{1}{k}\ ||f_k||_{L_2[0,\tau_k]}\ . \tag{19}$$

Since $X_k(0)=0$, then by virtue of Cauchy-Bunyakovskii inequality

$$X_k^2(\zeta_{[k]}) = |\int_0^{\zeta_{[k]}} [X_k^2(x)]'dx| = 2\ |\int_0^{\zeta_{[k]}} X_k(x)X_k'(x)dx| \le 2\ ||X_k||_{L_2[0,\zeta_{[k]}]}||X_k'||_{L_2[0,\zeta_{[k]}]}, \tag{20}$$

$$X_k^2(\eta_{[k]}) = |\int_0^{\eta_{[k]}} [X_k^2(x)]'dx| = 2\ |\int_0^{\eta_{[k]}} X_k(x)X_k'(x)dx| \le 2\ ||X_k||_{L_2[0,\eta_{[k]}]}||X_k'||_{L_2[0,\eta_{[k]}]}. \tag{21}$$

Using (19) in (20) and (21) we get

$$|\ X_k(\zeta_{[k]})\ |\ \le \frac{\sqrt{2}}{k^{3/2}}\ ||f_k||_{L_2[0,1]}\ ,\ \ |\ X_k(\eta_{[k]})\ |\ \le \frac{\sqrt{2}}{k^{3/2}}\ ||f_k||_{L_2[0,1]}\ . \tag{22}$$

Put $c_1=\alpha+\beta$. From the 3-point condition (18), in view of (22), we obtain the desired estimate

$$|\ X_k(1)\ |\ \le c_1\frac{\sqrt{2}}{k^{3/2}}\ ||f_k||_{L_2[0,1]}\ . \tag{23}$$

Subcase 1.2 :
Let $sign\big(X_k(1)X_k(\eta_{[k]})\big)=1$. Then $sign\big(X_k(1)X_k(\zeta_{[k]})\big)=1$ in view of (18). By virtue of MV property [12, p. 1198-1199] we reduce the 3-point condition (18) to

$$X_k(0)=0,\ X_k(\xi_k)=\nu X_k(\zeta_{[k]}) \tag{24}$$

for $\xi_k\in[\eta_{[k]},1]$ and $\nu=\frac{\alpha}{1+\beta}$. Note, $0<\nu\le 1$ since $\alpha-\beta\le 1$, $\zeta_{[k]}<\xi_k$ since $\zeta_{[k]}<\eta_k$. By virtue of [12, p. 1199-1200] we specify an appropriate point $\tau_k\in[\zeta_{[k]},\xi_k]$, so that the solution of (18) satisfies the classical boundary value condition

$$X_k(0)=0,\ \ X_k'(\tau_k)+h_kX_k(\tau_k)=0 \tag{25}$$

for $h_k\ge 0$. Therefore, (19) holds [3, 143-144]. Since $\zeta_{[k]}\le\tau_k$, then (20) holds, and then the first estimate (22) holds. Since $X_k(1),\ X_k(\eta_{[k]}),\ X_k(\zeta_{[k]})$ have the same sign, then in view of (18)

$$(1+\beta)\min\{|X_k(1)|,|X_k(\eta_{[k]})|\}\le\alpha|\ X_k(\zeta_{[k]})\ |\ \le\alpha\frac{\sqrt{2}}{k^{3/2}}\ ||f_k||_{L_2[0,1]}\ ,$$

$$\min\{|X_k(1)|,|X_k(\eta_{[k]})|\}\le\frac{\alpha}{1+\beta}\frac{\sqrt{2}}{k^{3/2}}\ ||f_k||_{L_2[0,1]}\ . \tag{26}$$

Hence, the estimate (11) follows from (26) or, in view of (22), results from (18), i.e.:

$$|\ X_k(1)\ |\ \le c_2\frac{\sqrt{2}}{k^{3/2}}\ ||f_k||_{L_2[0,1]}\ , \tag{27}$$

$$c_2=\begin{cases}\frac{\alpha}{1+\beta}, & \text{if}\ \ |X_k(1)|\le|X_k(\eta_{[k]})|\ ,\\ \frac{\alpha\beta}{1+\beta}+\alpha, & \text{if}\ \ |X_k(1)|>|X_k(\eta_{[k]})|\ .\end{cases}$$

Let ***A2*** holds, i.e., $\alpha\le 1$ and $\zeta_n\not<\eta_1$. Put $\zeta_{[k]}\ne\eta_{[k]}$, because if this two points coincide, then NLBVC (18) transfoms to

$$X_k(0)=0,\ X_k(1)=(\alpha-\beta)X_k(\xi_k)\ \ \text{for}\ \ \xi_k=\zeta_{[k]}=\eta_{[k]}\ \ \text{while}\ \ -\infty<\alpha-\beta<1\ ,$$

so that the estimate (11) holds in view of [3]. Moreover, we consider the layout $\zeta_{[k]}>\eta_{[k]}$ only, since for the alternate order when $\zeta_{[k]}<\eta_{[k]}$ (note that $-\infty<\alpha-\beta<1$ since $\alpha\le 1$)

the estimate (16) is proved already in the above case under the term ***A1***. Additionally, we put $sign\big(X_k(1)\ X_k(\eta_{[k]})\ X_k(\zeta_{[k]})\big) \neq 0$. Note in advance, if $sign\big(X_k(1)\ X_k(\eta_{[k]})\ X_k(\zeta_{[k]})\big) = 0$, then the estimate (11) results from the current proof. In summary, we have to consider the alternate subcases when $sign\big(X_k(1)X_k(\zeta_{[k]})\big) = -1$ and $sign\big(X_k(1)X_k(\zeta_{[k]})\big) = 1$ for $\eta_{[k]} < \zeta_{[k]}$.

Subcase 2.1 :

If $sign\big(X_k(1)X_k(\zeta_{[k]})\big) = -1$ and $\eta_{[k]} < \zeta_{[k]}$, then by analogy with the subcase 1.1 we obtain all estimates (19)-(23).

Subcase 2.2 :

Put $sign\big(X_k(1)X_k(\zeta_{[k]})\big) = 1$ and $\eta_{[k]} < \zeta_{[k]}$. Then we have the alternate inequalities: $|X_k(\zeta_{[k]})| \geq$ $\geq |X_k(1)|$ and $|X_k(\zeta_{[k]})| < |X_k(1)|$.

If $X_k(\zeta_{[k]}) = X_k(1)$, then by virtue of Rolle's theorem $X_k'(\tau_{k1}) = 0$ for $\tau_{k1} \in [\zeta_{[k]}, 1]$.

If $|X_k(\zeta_{[k]})| > |X_k(1)|$, then $X_k(1) = \nu_k X_k(\zeta_{[k]})$ for an appropriate value $\nu_k,\ 0 < \nu_k < 1$. Hence, by virtue of [12, p. 1199-1200] we specify an appropriate point $\tau_{k2} \in [\zeta_{[k]}, 1]$, so that the classical boundary value condition holds for $h_k > 0$: $X_k(0) = 0,\ \ X_k'(\tau_{k2}) + h_k X_k(\tau_{k2}) = 0$. Thereby, if $|X_k(\zeta_{[k]})| \geq |X_k(1)|$, then for some $\tau_k \in [\zeta_{[k]}, 1]$ and $h_k \geq 0$

$$X_k(0) = 0,\ \ X_k'(\tau_k) + h_k X_k(\tau_k) = 0\ .$$

Since $\eta_{[k]} < \zeta_{[k]}$, then using the method of section 1.1 we succesively obtain the estimates (19)-(23).

If $|X_k(\zeta_{[k]})| < |X_k(1)|$, then $sign\big(X_k(\eta_{[k]})X_k(1)\big) = sign\big(X_k(\eta_{[k]})X_k(\zeta_{[k]})\big) = -1$ since $\alpha \leq 1$ and because $sign\big(X_k(1)X_k(\zeta_{[k]})\big) = 1$. By virtue of Bolzano theorem $X_k(\tau_k) = 0$ for $\tau_k \in [\eta_{[k]}, \zeta_{[k]}]$. Then, by analogy with subcase 1.1 we get (19), (21) and the second estimate in (22). Hence, if $\alpha < 1$, then in view of (18)

$$(1-\alpha)|X_k(\zeta_{[k]})| < \beta\frac{\sqrt{2}}{k^{3/2}}||f_k||_{L_2[0,1]}\ . \tag{28}$$

Put $c_3 = \frac{\alpha\beta}{1-\alpha} + \beta$. Using (18), in view of (22) and (28), we obtain the desired estimate

$$|\ X_k(1)\ |\ \leq c_3\frac{\sqrt{2}}{k^{3/2}}\ ||f_k||_{L_2[0,1]}\ . \tag{29}$$

At least, if $|X_k(\zeta_{[k]})| < |X_k(1)|$ but $\alpha = 1$, then to estimate $X_k(1)$ we reduce NLBVP (18) to

$$L[X_k(x)] = f_k(x),\ 0 < x < 1, \quad X_k(0) = 0,\ X_k(1) = X_k(\zeta_{[k]}) - \gamma_k\ , \tag{30}$$

where $L[X_k(x)] = X_k''(x) - k^2 X_k(x)$ and $\gamma_k = \beta X_k(\eta_{[k]})$. In view of the second estimate in (22)

$$|\gamma_k| < \beta\frac{\sqrt{2}}{k^{3/2}}||f_k||_{L_2[0,1]}\ . \tag{31}$$

Put $X_k(x)$ is the sum $X_k(x) = V_k(x) + W_k(x)$, so that $V_k(x)$ is the solution of

$$L[V_k(x)] = f_k(x),\ 0 < x < 1, \quad V_k(0) = 0,\ V_k(1) - V_k(\zeta_{[k]}) = 0\ , \tag{32}$$

and $W_k(x)$ is the solution of

$$L[W_k(x)] = 0,\ 0 < x < 1, \quad W_k(0) = 0,\ W_k(1) - W_k(\zeta_{[k]}) = -\gamma_k\ . \tag{33}$$

The classical solution of (32) exists and is a unique [12, p. 1198-1200]. By virtue of Rolle theorem $V_k'(\tau_k) = 0$ for $\tau_k \in (\zeta_{[k]}, 1)$. Then similar subcase 1.1 the analogs of (19)-(20) and the first estimate (22) hold for $V_k(x)$. Hence, since $V_k(1) = V_k(\zeta_{[k]})$

$$|\ V_k(1)\ |\ \leq \frac{\sqrt{2}}{k^{3/2}}\ ||f_k||_{L_2[0,1]}\ . \tag{34}$$

On the other hand, for $\mathcal{C}_k = -\gamma_k\Big(1-\frac{\sinh k\zeta_{[k]}}{\sinh k}\Big)^{-1}$ the function $W_k(x) = \mathcal{C}_k\frac{\sinh kx}{\sinh k}$ is the solution of (33) since $1-\frac{\sinh k\zeta_{[k]}}{\sinh k} > 0$ for $\zeta_{[k]} < 1$. Then, in view of 2-point condition (33),

$$|W_k(1)| \le \Big(1-\frac{\sinh k\zeta_{[k]}}{\sinh k}\Big)^{-1}\beta\frac{\sqrt{2}}{k^{3/2}}||f_k||_{L_2[0,1]}\ . \tag{35}$$

Hence, for $M = \frac{\sinh\zeta_n}{\sinh 1}$

$$|W_k(1)| \le \frac{1}{1-M}\beta\frac{\sqrt{2}}{k^{3/2}}||f_k||_{L_2[0,1]}\ . \tag{36}$$

Then, in view of (34) and (36), $|X_k(1)| \le c_4\frac{\sqrt{2}}{k^{3/2}}||f_k(x)||_{L_2[0,1]}$ for $c_4 = 1+\beta\frac{1}{1-M}$.

Finally we resume, that for the classical solution of (10) the estimate (11) is proved for the constant $C = \max\{c_1, c_2, c_3, c_4\}$. Theorem 2 is proved.

Theorem 3. Let $f \in C(\overline{\Pi})$, $m = n$ and $\zeta_r < \eta_r$, $r = \overline{1,n}$. If $\sum\limits_{r=1}^{n}\frac{(\alpha_r-\beta_r)+|\alpha_r-\beta_r|}{2} \le 1$, then classical solution of NLBVP (1) exists, it is an unique and a priori estimate (6) holds.

Proof. Suppose that classical solution exists. In view of Theorem 2, we rewrite (10) as

$$L[X_k(x)] = f_k(x),\ 0<x<1, \quad X_k(0)=0,\ \ \ell[X_k]=0, \tag{37}$$

where $L[X_k(x)] = X_k''(x) - k^2X_k(x)$ and $\ell[X_k] \equiv X_k(1) - \sum\limits_{r=1}^{n}[\alpha_rX_k(\zeta_r) - \beta_rX_k(\eta_r)]$. To obtain the estimate (11) we put $X_k(x) = V_k(x) + W_k(x)$, so that $V_k(x)$ is the solution of problem

$$L[V_k(x)] = f_k(x),\ 0<x<1, \quad V_k(0)=0,\ V_k(1)=0\ , \tag{38}$$

and $W_k(x)$ is the solution of problem

$$L[W_k(x)] = 0,\ 0<x<1, \quad W_k(0)=0,\ \ \ell[W_k] = -\ell[V_k]\ . \tag{39}$$

For solution of (38) the analog of (7) holds (see Theorem 2). Hence, since $V_k(0) = 0$ and $\zeta_r \in (0,1)$, $\eta_r \in (0,1)$, $r = \overline{1,n}$, then

$$|V_k(\zeta_r)| \le \frac{\sqrt{2}}{k^{3/2}}||f_k||_{L_2[0,1]}, \quad |V_k(\eta_r)| \le \frac{\sqrt{2}}{k^{3/2}}||f_k||_{L_2[0,1]}\ .$$

Therefore,

$$|\ell[V_k]| \le \Big(\sum_{r=1}^{n}(\alpha_r+\beta_r)\Big)\frac{\sqrt{2}}{k^{3/2}}||f_k||_{L_2[0,1]}\ . \tag{40}$$

The problem (39) has the solution $W_k(x) = \mathcal{W}_k\frac{\sinh kx}{\sinh k}$, $\mathcal{W}_k = \frac{-\ell[V_k]}{1-(\sinh k)^{-1}\sum\limits_{r=1}^{n}[\alpha_r\sinh k\zeta_r - \beta_r\sinh k\eta_r]}$,

where the denominator of $\mathcal{W}_k$ is nozero when $\frac{1}{2}\sum\limits_{r=1}^{n}\Big[(\alpha_r-\beta_r)+|\alpha_r-\beta_r|\Big] < 1$. In view of (40),

$$|W_k(1)| \le \frac{\sqrt{2}\sum\limits_{r=1}^{n}(\alpha_r+\beta_r)}{k^{3/2}\big[1-\frac{1}{2}\sum\limits_{r=1}^{n}(\alpha_r-\beta_r+|\alpha_r-\beta_r|)\big]}\,||f_k||_{L_2[0,1]}\ .$$

Hence, (11) holds since $V_k(1) = 0$, i.e., $|\ X_k(1)\ | \le C\frac{\sqrt{2}}{k^{3/2}}\ ||f_k||_{L_2[0,1]}$.

At least, put $\frac{1}{2}\sum\limits_{r=1}^{n}\Big[(\alpha_r-\beta_r)+|\alpha_r-\beta_r|\Big]=1$, then similar (35), but in view of (40), we get

$$|W_k(1)|\le\Big(1-\frac{\sinh\zeta_p}{\sinh 1}\Big)^{-1}\Big[\sum_{r=1}^{n}(\alpha_r+\beta_r)\Big]\frac{\sqrt{2}}{k^{3/2}}\,||f_k||_{L_2[0,1]}\ ,$$

where p , $1\le p\le n$ is a natural number, so that

$$\frac{(\alpha_p-\beta_p)+|\alpha_p-\beta_p|}{2}>0\ ,\quad\text{but}\quad\frac{(\alpha_{p+i}-\beta_{p+i})+|\alpha_{p+i}-\beta_{p+i}|}{2}=0\quad\text{for all}\quad i,\ \ p<i\le n\ ,$$

and $p=n$ if i does not exists. Hence, (11) holds for $\frac{1}{2}\sum\limits_{r=1}^{n}\big[(\alpha_r-\beta_r)+|\alpha_r-\beta_r|\big]=1$ since $V_k(1)=0$. In summary, for the solution of (37) the estimate (11) holds when $\frac{1}{2}\sum\limits_{r=1}^{n}\big[(\alpha_r-\beta_r)+|\alpha_r-\beta_r|\big]\le 1$. Hence, in view of Theorem 2, a priori estimate (6) holds for NLIVP (1), thereto the solution of (1) is a unique and, therefore, in view of Theorem 1 the solution exists. Theorem 3 is proved.

Difference variant

We consider the difference variant of NLBVP (1)

$$\begin{cases}\Lambda Y=Y_{\bar{x}x}+Y_{\bar{y}y}=f(x_i,y_j),\ \ (x_i,y_j)\in\Pi,\\ Y|_{y=0}=Y|_{y=\pi}=0,\ x_i\in[0,1)\,,\quad Y|_{x=0}=0,\ \ y_j\in[0,\pi]\,,\\ \mathcal{L}Y=\sum\limits_{r=1}^{n}\alpha_r\Big\{Y_{i_{\zeta_r},j}\frac{[(i_{\zeta_r}+1)h_1-\zeta_r]}{h_1}+Y_{i_{\zeta_r}+1,j}\frac{[\zeta_r-i_{\zeta_r}h_1]}{h_1}\Big\}-\\ -\sum\limits_{s=1}^{m}\beta_s\Big\{Y_{i_{\eta_s},j}\frac{[(i_{\eta_s}+1)h_1-\eta_s]}{h_1}+Y_{i_{\eta_s}+1,j}\frac{[\eta_s-i_{\eta_s}h_1]}{h_1}\Big\}-Y_{N_1,j}=0,\ j=\overline{1,N_2-1},\end{cases}\tag{41}$$

where $i_{\zeta_r}h_1\le\zeta_r<(i_{\zeta_r}+1)h_1$, $r=\overline{1,n}$, $i_{\eta_s}h_1\le\eta_s<(i_{\eta_s}+1)h_1$, $s=\overline{1,m}$ for $h_1=1/N_1$, $h_1<\frac{1}{2}\min\{\zeta_{r+1}-\zeta_r,\ r=\overline{0,n},\ \eta_{s+1}-\eta_s,\ s=\overline{0,m},\ |\zeta_r-\eta_s|,\ r=\overline{1,n},\ s=\overline{1,m}\}$, $\zeta_0=\eta_0=0$, $\zeta_{n+1}=\eta_{m+1}=1$, $h_1\le c_0h_2$, $h_2=\pi/N_2$.

Theorem 4. Let the term $\boldsymbol{A}$ holds and $u\in C^{(4)}(\overline{\Pi})$ is the solution of NLBVP (1). Then solution of the difference problem (41) approximates $u(x,y)$ by the second order of accuracy in terms of $h=\sqrt{h_1^2+h_2^2}$ when $h_2\to 0$ in respect of difference metrics C, W_2^2 .

Proof. We denote $z=Y-u$ and obtain the difference problem

$$\Lambda z=f-\Lambda u=F,\ \ (ih_1,jh_2)\in\Pi,\quad z|_{x=0}=z|_{y=0}=z|_{y=\pi}=0,\ \ \mathcal{L}z=-\mathcal{L}u.\tag{42}$$

For this problem $F=O(h^2)$, $\mathcal{L}u=O(h^2)$ [14, p. 81, 229]. Put $z=\tilde{z}+\hat{z}$, where $\tilde{z}$ is the solution of

$$\Lambda\tilde{z}=0,\ \ (ih_1,jh_2)\in\Pi,\quad \tilde{z}|_{x=0}=\tilde{z}|_{y=0}=\tilde{z}|_{y=\pi}=0,\ \ \mathcal{L}\tilde{z}=-\mathcal{L}u\ ,\tag{43}$$

and $\hat{z}$ is the solution of

$$\Lambda\hat{z}=F,\ (ih_1,jh_2)\in\Pi,\quad \hat{z}|_{x=0}=\hat{z}|_{y=0}=\hat{z}|_{y=\pi}=0,\ \ \mathcal{L}\hat{z}=0\ .\tag{44}$$

To estimate $\tilde{z}$ we use [14, p. 113] the orthogonal system of mesh functions $\{\sin(ky)\}_{k=1}^{k=N_2-1}$, so that

$$\tilde{z}=\sum_{k=1}^{N_2-1}\tilde{z}_k\sin(ky),\quad y=jh_2,\ \ \overline{j=0,N_2}$$

thereto $\tilde{z}_k$, $k=\overline{1,N_2-1}$ is the solution of difference problem

$$\Lambda_1\tilde{z}_k-\lambda_k\tilde{z}_k=0,\quad \tilde{z}_k|_{x=0}=0,\ \ \mathcal{L}\tilde{z}_k=-Q_k\ , \tag{45}$$

where $\Lambda_1\tilde{z}=\tilde{z}_{\bar{x}x}$, $\lambda_k=4h_2^{-2}\sin^2(kh_2)$, $Q_k=(\mathcal{L}u)_k$ so that, in view of [3, p. 142-143],

$$\tilde{z}_{k_i}=A_k\sinh(i\ln q_k),\ \ A_k=-Q_k/\mathcal{L}[\sinh(i\ln q_k)],\ \ i=\overline{0,N_1},\ \ q_k=1+\lambda_k h_1^2/2+\sqrt{\lambda_k h_1^2+\lambda_k^2h_1^4/4}\ .$$

By acting $\mathcal{L}$ in the denominator of the fraction A_k, we get

$$-\mathcal{L}[\sinh(i\ln q_k)]\geq\sinh(N_1\ln q_k)-\sum_{r=1}^{n}\alpha_r\sinh((i_{\zeta_n}+1)\ln q_k)+\sum_{s=1}^{m}\beta_s\sinh(i_{\eta_1}\ln q_k)\ . \tag{46}$$

Hence,

$$-\mathcal{L}[\sinh(i\ln q_k)]\geq\sinh(N_1\ln q_k)-S\sinh((i_{\zeta_n}+1)\ln q_k) \tag{47}$$

for

$$S=\begin{cases}\sum\limits_{r=1}^{n}\alpha_r-\sum\limits_{s=1}^{m}\beta_s\ ,\ \text{if}\ \ \zeta_n<\eta_1,\\ \sum\limits_{r=1}^{n}\alpha_r\ ,\quad \text{if}\quad \zeta_n>\eta_1.\end{cases}$$

Then

$$-\mathcal{L}[\sinh(i\ln q_k)]\geq C\sinh(N_1\ln q_k) \tag{48}$$

for $C>0$,

$$C=\begin{cases}1\ ,\quad \text{if}\quad -\infty<\sum\limits_{r=1}^{n}\alpha_r-\sum\limits_{s=1}^{m}\beta_s\leq 0,\ \ \zeta_n<\eta_1;\\ 1-\big(\sum\limits_{r=1}^{n}\alpha_r-\sum\limits_{s=1}^{m}\beta_s\big)\ ,\quad \text{if}\quad 0<\sum\limits_{r=1}^{n}\alpha_r-\sum\limits_{s=1}^{m}\beta_s<1,\ \ \zeta_n<\eta_1;\\ 1-\sum\limits_{r=1}^{n}\alpha_r\ ,\quad \text{if}\quad \alpha_r<1,\ \ \zeta_n>\eta_1.\end{cases}$$

Let we show that when $S=1$ in (47), then the inequality (48) holds for $C=1-\frac{1}{(1+4/\pi)^\delta}$ subject to an appropriate δ, $0<\delta\leq 1$. Indeed, in view of (47)

$$-\mathcal{L}[\sinh(i\ln q_k)]\geq\sinh(N_1\ln q_k)\Big[1-\frac{\sinh((i_{\zeta_n}+1)\ln q_k)}{\sinh(N_1\ln q_k)}\Big]\geq 0\ .$$

Hence,

$$-\mathcal{L}[\sinh(i\ln q_k)]\geq\sinh(N_1\ln q_k)\Big[1-\frac{q_k^{i_{\zeta_n}+1}-q_k^{-(i_{\zeta_n}+1)}}{q_k^{N_1}-q_k^{-N_1}}\Big]\ . \tag{49}$$

Since $q_k\geq 1$, then

$$\frac{q_k^{i_{\zeta_n}+1}-q_k^{-(i_{\zeta_n}+1)}}{q_k^{N_1}-q_k^{-N_1}}\leq\frac{q_k^{i_{\zeta_n}+1}[1-q_k^{-2(i_{\zeta_n}+1)}]}{q_k^{N_1}[1-q_k^{-2N_1}]}\leq\frac{q_k^{i_{\zeta_n}+1}}{q_k^{N_1}}\ . \tag{50}$$

Since $h_1<\theta$ for $\theta=\frac{1}{2}\min\{\zeta_{r+1}-\zeta_r,\ r=\overline{0,n},\ \eta_{s+1}-\eta_s,\ s=\overline{0,m},\ |\zeta_r-\eta_s|,\ r=\overline{1,n},\ s=\overline{1,m}\}$, then for specified $\delta=1-\zeta_n-\theta$ the inequality $\zeta_n+h_1\leq 1-\delta$ holds. Hence, $i_{\zeta_n}+1\leq h_1^{-1}(1-\delta)$. Then from (60) it follows that

$$\frac{q_k^{i_{\zeta_n}+1}-q_k^{-(i_{\zeta_n}+1)}}{q_k^{N_1}-q_k^{-N_1}}\leq\frac{q_k^{N_1(1-\delta)}}{q_k^{N_1}}\leq\frac{1}{q_k^{N_1\delta}}\ .$$

Hence, in view of (49),

$$-\mathcal{L}[\sinh(i\ln q_k)] \geq \Big(1-\frac{1}{q_k^{N_1\delta}}\Big)\sinh(N_1\ln q_k)\ . \tag{51}$$

Since $q_k^{N_1} \geq (1+\sqrt{\lambda_k}h_1)^{N_1} \geq (1+\sqrt{\lambda_1}h_1)^{N_1} \geq (1+\sqrt{\lambda_1}) \geq 1+\frac{4}{\pi}$, then from (51) we obtain

$$-\mathcal{L}[\sinh(i\ln q_k)] \geq \Big[1-\frac{1}{(1+4/\pi)^{\delta}}\Big]\sinh(N_1\ln q_k)\ . \tag{52}$$

In summary, if the term $\boldsymbol{A}$ holds, then

$$-\mathcal{L}[\sinh(i\ln q_k)] \geq C\sinh(N_1\ln q_k) > 0\ . \tag{53}$$

Finally, in view of (53), by virtue of [3, 150-151], we obtain the estimates

$$\max_{i,j}|\tilde{z}_{ij}| = O(h^2),\ \ ||\tilde{z}||_{W_2^2} = O(h^2),\ \ \max_{i,j}|\hat{z}_{ij}| = O(h^2),\ \ ||\hat{z}||_{W_2^2} = O(h^2).$$

Therefore, $\max_{i,j}|z_{ij}| = O(h^2)$, $||z||_{W_2^2} = O(h^2)$. Theorem 4 is proved.

Corollary 1. Let $n=m$, $\zeta_r < \eta_r$, $r=\overline{1,n}$. Let $u \in C^{(4)}(\overline{\Pi})$ is the solution of NLBVP (1). If $\sum\limits_{r=1}^{n}\frac{(\alpha_r-\beta_r)+|\alpha_r-\beta_r|}{2} \leq 1$, then difference solution of (41) approximates $u(x,y)$ by the second order of accuracy in terms of $h=\sqrt{h_1^2+h_2^2}$ when $h_2 \to 0$ in respect of difference metrics C, W_2^2 .

Proof. By virtue of (42)-(46) we get the inequality for the denominator of the fraction A_k:

$$-\mathcal{L}[\sinh(i\ln q_k)] \geq \sinh(N_1\ln q_k) - \sum_{r=1}^{n}\alpha_r\sinh((i_{\zeta_r}+1)\ln q_k) + \sum_{r=1}^{n}\beta_r\sinh(i_{\eta_r}\ln q_k)\ .$$

Since $i_{\zeta_r}+1 < i_{\eta_r}$, $r=\overline{1,n}$, then

$$-\mathcal{L}[\sinh(i\ln q_k)] \geq \sinh(N_1\ln q_k) - \sum_{r=1}^{n}(\alpha_r-\beta_r)\ \sinh((i_{\zeta_r}+1)\ln q_k)\ .$$

Hence,

$$-\mathcal{L}[\sinh(i\ln q_k)] \geq\ \Big[1-\sum_{r=1}^{n}(\alpha_r-\beta_r)\ \Big(\frac{q_k^{i_{\zeta_r}+1}-q_k^{-(i_{\zeta_r}+1)}}{q_k^{N_1}-q_k^{-N_1}}\Big)\Big]\ \sinh(N_1\ln q_k).$$

Then

$$-\mathcal{L}[\sinh(i\ln q_k)] \geq \Big[1-\sum_{r=1}^{n}\Big(\frac{(\alpha_r-\beta_r)+|\alpha_r-\beta_r|}{2}\Big)\Big(\frac{q_k^{i_{\zeta_r}+1}-q_k^{-(i_{\zeta_r}+1)}}{q_k^{N_1}-q_k^{-N_1}}\Big)\Big]\sinh(N_1\ln q_k). \tag{54}$$

Put p is a natural number, $1 \leq p \leq n$, so that

$$\frac{(\alpha_p-\beta_p)+|\alpha_p-\beta_p|}{2} > 0\ ,\ \ \text{but}\ \ \frac{(\alpha_{p+i}-\beta_{p+i})+|\alpha_{p+i}-\beta_{p+i}|}{2} = 0\ \ \text{for all}\ \ i,\ p<i\leq n$$

(if such p does not exists, or if such i does not exists, then put $p=n$). Hence, in view of (54),

$$-\mathcal{L}[\sinh(i\ln q_k)] \geq \Big[1-S\ \frac{q_k^{i_{\zeta_p}+1}-q_k^{-(i_{\zeta_p}+1)}}{q_k^{N_1}-q_k^{-N_1}}\Big]\ \sinh(N_1\ln q_k) \tag{55}$$

for $S=\sum\limits_{r=1}^{n}\frac{(\alpha_r-\beta_r)+|\alpha_r-\beta_r|}{2}$. By analogy with (50), for $q_k\geq 1$ and for $\delta=1-\zeta_p-\theta$ we get

$$\frac{q_k^{i_{\zeta_p}+1}-q_k^{-(i_{\zeta_p}+1)}}{q_k^{N_1}-q_k^{-N_1}}\leq\frac{1}{q_k^{N_1\delta}} \tag{56}$$

since the inequalities $\zeta_p+h_1\leq 1-\delta$ and $i_{\zeta_p}+1\leq h_1^{-1}(1-\delta)$ hold. Hence, the analog of (47) holds, then (51)-(53) hold, too. Thereby, in view of Theorem 4, the proof is finished. Corollary 1 is proved

NLBVP with integral condition

Here we apply the results of the previous sections to NLBVP with weighted integral condition (WIC). We consider the differential problem in the rectangular Π

$$\begin{cases}\Delta u(x,y)=f(x,y),\ (x,y)\in\Pi,\\ u(x,0)=u(x,\pi)=0,\ 0\leq x<1,\ \ u(0,y)=0,\ \mathcal{I}[u](y)=0,\ 0\leq y\leq\pi\ ,\end{cases} \tag{57}$$

$$\mathcal{I}[u](y)\equiv u(1,y)-\int\limits_{\tau_0}^{\tau_1}\rho(x)u(x,y)dx\ , \tag{58}$$

where $\rho(x)\in C[\tau_0,\tau_1]$, $[\tau_0,\tau_1]\subset(0,1)$, $\tau_0<\tau_1$ and $\rho(x)\not\equiv 0$ in $[\tau_0,\tau_1]$.

Theorem 5. Let the function $\rho(x)$ changes the sign[3] no more than once in the interval (τ_0,τ_1). Let :

$$-\infty<\int\limits_{\tau_0}^{\tau_1}\rho(x)dx\leq 1\ ,\ \text{ if } \rho(x) \text{ does not change the sign, or changes it from plus to minus ;}$$

$$\int\limits_{\tau_0}^{\tau_1}\frac{\rho(x)+|\rho(x)|}{2}dx\leq 1\ ,\ \text{ if } \rho(x) \text{ changes the sign from minus to plus .}$$

Then classical solution of (57) exists, it is an unique and a priori estimate (6) holds.

Proof. Assume that classical solution exits. Since

$$\int\limits_0^{\pi}u(1,y)\sin(ky)dy=\int\limits_0^{\pi}\int\limits_{\tau_0}^{\tau_1}\rho(x)u(x,y)dx\ \sin(ky)dy=\int\limits_{\tau_0}^{\tau_1}\rho(x)\left(\int\limits_0^{\pi}u(x,y)\sin(ky)dy\right)dx\ ,$$

then from (57)-(58), in view of (3) and by virtue of Theorem 1, we conclude that the function $X_k(x)$ satisfies the problem

$$X_k''(x)-k^2X_k(x)=0,\ 0<x<1,\quad X_k(0)=0,\ \ \mathcal{I}[X_k]=0\ , \tag{59}$$

where $\mathcal{I}[X_k]=X_k(1)-\int\limits_{\tau_0}^{\tau_1}\rho(x)X_k(x)dx$. By virtue of the integral type of mean value theorem, we reduce WIC problem (59) to the 3-point problem

$$X_k''(x)-k^2X_k(x)=0,\ 0<x<1,\quad X_k(0)=0,\ \ \ell[X_k]=0\ , \tag{60}$$

where

$$\ell[X_k]=X_k(1)-\left(\int\limits_{\tau_0}^{\tau_1}\frac{\rho(x)+|\rho(x)|}{2}dx\right)X_k(\zeta_k)+\left(\int\limits_{\tau_0}^{\tau_1}\frac{|\rho(x)|-\rho(x)}{2}dx\right)X_k(\eta_k) \tag{61}$$

[3]The sign changing number and order are regarded as argument x shifs towards τ_1.

for some $\zeta_k \in (\tau_0, \tau_1)$ and $\eta_k \in (\tau_0, \tau_1)$. Denote

$$\alpha = \int\limits_{\tau_0}^{\tau_1} \frac{\rho(x) + |\rho(x)|}{2} dx \ , \quad \beta = \int\limits_{\tau_0}^{\tau_1} \frac{|\rho(x)| - \rho(x)}{2} dx \ . \tag{62}$$

If $\rho(x)$ does not change the sign, then:

$$\ell[X_k] = X_k(1) - \alpha X_k(\zeta_k) \text{ and } 0 \le \alpha \le 1 \ , \text{ if } \rho(x) \text{ is a nonnegative function} \ ,$$

$$\ell[X_k] = X_k(1) + \beta X_k(\eta_k) \text{ and } -\infty < -\beta \le 0 \ , \text{ if } \rho(x) \text{ is a nonpositive function} \ .$$

If $\rho(x)$ changes the sign, then $\ell[X_k] = X(1) - \alpha X_k(\zeta_k) + \beta X_k(\eta_k)$, so that

$$-\infty < \alpha - \beta \le 1, \ \zeta_k < \eta_k \text{ if } \rho(x) \text{ changes the sign from plus to minus} \ ,$$

$$\alpha \le 1, \ \eta_k < \zeta_k \text{ if } \rho(x) \text{ changes the sign from minus to plus} \ .$$

Hence, in view of (61)-(62), for the 3-point NLBVP (60) the term $\boldsymbol{A}$ holds in extended form [16, p. 917], i.e., includes the option when $\alpha = 0$ or $\beta = 0$. Then, in view of Theorem 1, the problem (60) (and in turn the problem (59) of course) has only trivial solution $X_k(x) \equiv 0$, and, therefore, $u(x, y) \equiv 0$ in the rectangle Π. Since the uniqueness for the problem (57) is proved, then the existence follows from the Fredholm's property inherent such NLBVP with WIC [15, p. 68-70].

To prove a priori estimate (6) we follow Theorem 2 and, in view of (8), get WIC problem

$$X_k''(x) - k^2 X_k(x) = f_k(x), \ 0 < x < 1, \quad X_k(0) = 0, \ \mathcal{I}[X_k] = 0 \tag{63}$$

(this problem was studied in [17]) and, in view of (60), reduce it to the multipoint problem

$$X_k''(x) - k^2 X_k(x) = f_k(x) \ , \ 0 < x < 1, \quad X_k(0) = 0, \ \ell[X_k] = 0 \ . \tag{64}$$

In view of (61)-(62) and by virtue of Theorem 2, we ascertain that (11) holds for solution of (64) and, thereby, it holds for solution of (63). Further proof is similarly of Theorem 2. Theorem 5 is proved.

Corollary 2. Let the function $\rho(x)$ has an arbitrary order and a finite number of sign changings. If $\int\limits_{\tau_0}^{\tau_1} \frac{\rho(x)+|\rho(x)|}{2} dx \le 1$, then classical solution of (57) exists, it is an unique and a priori estimate (6) holds.

Proof. The proof results from Theorem 1 and Theorem 2 by using Theorem 5. Corollary 2 is proved.

Corollary 3. Let starting from plus to minus the function $\rho(x)$ changes the sign $2n - 1$ times in the interval (τ_0, τ_1) for specified natural number n and $\xi_1, ..., \xi_{2n-1}$ are the sign changing points. Put $\xi_0 = \tau_0$ and $\xi_{2n} = \tau_1$. If

$$\sum_{k=1}^{n} \frac{1}{2} \Big(\int\limits_{\xi_{2(k-1)}}^{\xi_{2k}} \rho(x) dx \ + \Big| \int\limits_{\xi_{2(k-1)}}^{\xi_{2k}} \rho(x) dx \Big| \Big) \le 1 \ ,$$

then classical solution of (57) exists, it is an unique and a priori estimate (6) holds

Proof. It results from Theorem 1-2 and by using of Theorem 3, Theorem 5. Corollary 3 is proved.

Difference application for WIC

We consider the difference problem

$$\begin{cases} \Lambda Y = Y_{\bar{x}x} + Y_{\bar{y}y} = f(x_i, y_j), \ \ (x_i, y_j) \in \Pi, \\ Y|_{y=0} = Y|_{y=\pi} = 0, \ x_i \in [0,1) \,, \ \ Y|_{x=0} = 0, \ \ y_j \in [0,\pi] \,, \\ \mathcal{T}Y = \sum\limits_{i=1}^{N_1} 2^{-1}(\rho_i Y_{i,j} + \rho_{i-1} Y_{i-1,j}) h_1 - Y_{N_1,j} = 0, \ j = \overline{1, N_2 - 1}, \end{cases} \tag{65}$$

where $\rho(x)$ does not change the sign, $\rho(x) \in C[0,1]$ and $\rho(x) \equiv 0$ in $[0,\tau_0] \cup [\tau_1, 1]$, $\rho_i = \rho(x_i)$, $h_1 < \frac{1}{2}\min\{\tau_0, \ 1-\tau_1\}$, $h_1 \le c_0 h_2$, $h_1 = 1/N_1$, $h_2 = \pi/N_2$.

Corollary 4. Let $u \in C^{(4)}(\overline{\Pi})$ is solution of WIC NLBVP (57). If $-\infty < \int\limits_{\tau_0}^{\tau_1} \rho(x)dx < 1$, then the solution of (65) approximates $u(x,y)$ by the second order of accuracy in terms of $h = \sqrt{h_1^2 + h_2^2}$ when $h_2 \to 0$ in respect of difference metrics C, W_2^2.

Proof. Following Theorem 4, for $z = Y - u$ we obtain the difference problem

$$\Lambda z = f - \Lambda u = F \ , \ (ih_1, jh_2) \in \Pi, \ \ z|_{x=0} = z|_{y=0} = z|_{y=\pi} = 0 \ , \ \ \mathcal{T}z = -\mathcal{T}u \ , \tag{66}$$

thereto $F = O(h^2)$ and $\mathcal{T}u = O(h^2)$ as a neglect of the trapezoid method. Put $z = \tilde{z} + \hat{z}$, where $\tilde{z}$ is the solution of

$$\Lambda \tilde{z} = 0 \ , \ (ih_1, jh_2) \in \Pi \ , \ \ \tilde{z}|_{x=0} = \tilde{z}|_{y=0} = \tilde{z}|_{y=\pi} = 0, \ \ \mathcal{T}\tilde{z} = -\mathcal{T}u \ , \tag{67}$$

and $\hat{z}$ is the solution of

$$\Lambda \hat{z} = F \ , \ (ih_1, jh_2) \in \Pi \ , \ \ \hat{z}|_{x=0} = \hat{z}|_{y=0} = \hat{z}|_{y=\pi} = 0 \ , \ \ \mathcal{T}\hat{z} = 0 \tag{68}$$

By virtue of the orthogonal system [14, p. 113] of the mesh functions $\{\sin(ky)\}_{k=1}^{k=N_2-1}$

$$\tilde{z} = \sum_{k=1}^{N_2-1} \tilde{z}_k \sin(ky), \quad y = jh_2, \ \ \overline{j = 0, N_2} \ ,$$

thereto $\tilde{z}_k$, $k = \overline{1, N_2 - 1}$ is solution of the problem

$$\Lambda_1 \tilde{z}_k - \lambda_k \tilde{z}_k = 0 \ , \ \ \tilde{z}_k|_{x=0} = 0, \ \ \mathcal{T}\tilde{z}_k = -Q_k \tag{69}$$

for $\Lambda_1 \tilde{z} = \tilde{z}_{\bar{x}x}$, $\lambda_k = 4h_2^{-2}\sin^2(kh_2)$, $Q_k = (\mathcal{T}u)_k$ and, in view of [3, p. 142-143],

$$\tilde{z}_{k_i} = A_k \sinh(i \ln q_k), \ A_k = -Q_k/\mathcal{T}[\sinh(i \ln q_k)], \ \ i = \overline{0, N_1}, \ q_k = 1 + \lambda_k h_1^2/2 + \sqrt{\lambda_k h_1^2 + \lambda_k^2 h_1^4/4}.$$

Acting by $\mathcal{T}$ we get the inequality for the denominator of the fraction A_k:

$$-\mathcal{T}[\sinh(i \ln q_k)] \ge \sinh(N_1 \ln q_k) - \sum_{i=1}^{N_1} 2^{-1}\Big(\rho_i \sinh(i \ln q_k) + \rho_{i-1}\sinh([i-1]\ln q_k)\Big) h_1 \ . \tag{70}$$

If $\rho(x) \le 0$, then $-\mathcal{T}[\sinh(i \ln q_k)] \ge \sinh(N_1 \ln q_k)$. If $\rho(x) \ge 0$, then for $i_{\tau_0} h_1 \le \tau_0 < (i_{\tau_0} + 1)h_1$ and $i_{\tau_1} h_1 \le \tau_1 < (i_{\tau_1} + 1)h_1$

$$-\mathcal{T}[\sinh(i \ln q_k)] \ge \sinh(N_1 \ln q_k) - \sinh\left((i_{\tau_1} + 1)\ln q_k\right) \sum_{i=i_{\tau_0}+1}^{i_{\tau_1}+1} 2^{-1}(\rho_i + \rho_{i-1}) h_1 \ .$$

Denote $S_{h_1} = \sum\limits_{i=i_{\tau_0}+1}^{i_{\tau_1}+1} 2^{-1}(\rho_i + \rho_{i-1})h_1$, then

$$-\mathcal{T}[\sinh(i \ln q_k)] \geq (1 - S_{h_1}) \sinh(N_1 \ln q_k) \ .$$

Since $\int\limits_0^1 \rho(x)dx < \lambda$ for specified $\lambda,\ 0 < \lambda < 1$, then $S_{h_1} < \lambda$ for sufficiently small h_1. Hence,

$$-\mathcal{T}[\sinh(i \ln q_k)] \geq (1 - \lambda) \sinh(N_1 \ln q_k) > 0 \ .$$

In summary,

$$-\mathcal{T}[\sinh(i \ln q_k)] \geq C \sinh(N_1 \ln q_k) \qquad (71)$$

for

$$C = \begin{cases} 1 > 0, & \text{if} \quad \rho(x) \leq 0 \ , \\ 1 - \lambda > 0, & \text{if} \quad \rho(x) \geq 0 \ . \end{cases}$$

In view of (71) and by virtue of Theorem 4, the proof is finished. Corollary 4 is proved.

Conclusion

We considered NLBVP for the Poisson's operator on a rectangular domain and obtained new accurate conditions of the existence, uniqueness and a priori estimate of classical solution. We applied our results and researched NLBVPs with weighted integral condition. We offered the difference variants and proved the second order of accuracy on a uniform grid.

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Д.М. Довлетов

Тікбұрышта Пуассон операторымен берілген бейлокальді шеттік есебі және оның айырымдық интерпретациясы

Жұмыста ашық тікбұрышты облыста Пуассон теңдеуі үшін бейлокальді шеттік есебінің дифференциалдық және айырымдық нұсқалары қарастырылған. Классикалық шешімінің бар болуы, жалғыздығы және априорлық бағамы анықталған. Екінші ретті дәлдікпен айырымдық схемасы көрсетілген. Салмақты интегралдық шарттары бар қосымшалар дифференциалдық және айырымдық нұсқада ұсынылған.

Кілт сөздер: пуассон операторы, бейлокальді шеттік есебі, тікбұрыш, айырымдық схемасы.

Д.М. Довлетов

Нелокальная краевая задача с оператором Пуассона на прямоугольнике и ее разностная интерпретация

В статье изучены дифференциальные и разностные варианты нелокальной краевой задачи для уравнения Пуассона в открытой прямоугольной области. Установлены существование, единственность и априорная оценка классического решения. Представлена разностная схема второго порядка точности. Приложения с весовым интегральным условием даны в дифференциальном и разностном вариантах.

Ключевые слова: оператор Пуассона, нелокальная краевая задача, прямоугольник, разностная схема.

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