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Decomposition formulas for some quadruple hypergeometric series

Abstract: In the present work, the authors obtained operator identities and decomposition formulas for second order Gauss hypergeometric series of four variables into products containing simpler hypergeometric functions. A Choi–Hasanov method based on the inverse pairs of symbolic operators is used. The obtained expansion formulas for the hypergeometric functions of four variables will allow us to study the properties of these functions. These decompositions are used to study the solvability of boundary value problems for degenerate multidimensional partial differential equations.

Keywords: Appell hypergeometric function, Lauricella function, Saran function, Quadruple hypergeometric series, Decomposition formulas, Operator identities, Inverse symbolic operators.

Introduction

A variety of tasks related to almost all of the most significant sections of mathematical physics and answering urgent technical questions are associated with the special functions applying, such as the Bessel, Hermite, Gaussian hypergeometric functions, etc. Thus, for example, Bessel functions are actively used in solving hydrodynamics, radiophysics, acoustics problems of atomic and nuclear physics. There are applications of Bessel functions in problems of elasticity and thermal conductivity theories (determination of stress concentration near faults, plate oscillation) [1]. Many functions used in astronomy are arranged in series of hypergeometric functions [2]. Also, the hypergeometric functions of many complex variables are applicable to the research of analytic continuation problems of Mellin – Barnes type integrals [3], in the superstring theory [4], and in theoretical aspects of algebraic geometry [5].

Generalized hypergeometric functions are used in solving boundary value problems for shell theory equations whose applications are used in mechanical engineering. A.D. Kovalenko developed the application of the theory of generalized hypergeometric functions to determine the stress state in disks, circular plates of alternant thickness and conical shells of rotation according to the equilibrium linear theory [6]. Multiple hypergeometric series are used in research and development of aerospace systems [7]. At the same time, hypergeometric functions of many variables arise in quantum field theory as a solution of Knizhnik-Zamolodchikov equations [5]. In [8–10], the connection of special functions of the hypergeometric type with the actual problems of the theory of representations of Lie algebras and quantum groups is shown, as well as the application of hypergeometric functions and series to applied problems of various fields.

It should be noted that the Riemann functions and the fundamental solutions of degenerate partial differential equations are expressed in terms of multiple hypergeometric functions. Thus, hypergeometric functions are used in solving boundary value problems for degenerate differential equations [11]. In particular, hypergeometric functions are used in [12] to find the fundamental solutions of a four-dimensional degenerate equation of elliptic type, which can be used in solving known boundary value

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problems. Also in [13], Appel hypergeometric functions are used to construct a double layer potential theory.

Second order hypergeometric functions of four variables were introduced in [14, 15]. For one class of hypergeometric functions of four variables, various properties, such as decomposition formulas, integral representations were obtained in [16,17]. However, it should be noted that decompositions into products of simpler hypergeometric functions can be obtained not for all the introduced second order hypergeometric functions of four variables.

In this paper, we obtain decomposition formulas using operator identities for the following quadruple hypergeometric functions:

$$F_1^{(4)}\left(a_1, a_2, b; c_1, c_2, c_3, c_4; x, y, z, t\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n+p}(a_2)_q(b)_{m+n+p+q}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}, \tag{1}$$

$$F_3^{(4)}\left(a_1, a_2, a_3, b; c_1, c_2, c_3, c_4; x, y, z, t\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_p(a_3)_q(b)_{m+n+p+q}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}, \tag{2}$$

$$F_4^{(4)}\left(a, b, c; c_1, c_2, c_3, c_4; x, y, z, t\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}(b)_{m+n+q}(c)_p}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}, \tag{3}$$

$$F_5^{(4)}\left(a_1, a_2, b_1, b_2; c_1, c_2, c_3, c_4; x, y, z, t\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n}(b_2)_{p+q}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}, \tag{4}$$

$$F_6^{(4)}\left(a_1,a_2,b_1,b_2,b_3; c_1,c_2,c_3,c_4; x,y,z,t\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+q}(b_2)_n(b_3)_p}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}, \tag{5}$$

$$F_8^{(4)}\left(a_1, a_2, b_1, b_2; c_1, c_2, c_3, c_4; x, y, z, t\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{p+q}(b_1)_{m+p}(b_2)_n(b_3)_q}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}, \tag{6}$$

$$F_{11}^{(4)}\left(a_1, a_2, b; c_1, c_2, c_3; x, y, z, t\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_p(a_3)_q(b)_{m+n+p+q}}{(c_1)_{m+p}(c_2)_n(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}, \tag{7}$$

$$F_{13}^{(4)}\left(a_1, a_2, a_3, a_4, b; c_1, c_2, c_3; x, y, z, t\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_m(a_2)_n(a_3)_p(a_4)_q(b)_{m+n+p+q}}{(c_1)_{m+n}(c_2)_p(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}, \tag{8}$$

where $(a)_n = \Gamma\left(a+m\right)/\Gamma\left(a\right)$ is a Pochhammer symbol.

Operator identities

By means of Burchnall–Chaundy pair of mutually inverse symbolic operators $\nabla_{x,y}\left(h\right)$ and $\Delta_{x,y}\left(h\right)$ [18–20], decomposition formulas were obtained for the Appel's hypergeometric functions of two variables by the products of hypergeometric functions of one variable [21].

To decompose multiple hypergeometric functions, a multivariable analogue of the above pair of mutually inverse symbolic operators

$$\tilde{\nabla}_{x_1;x_2,...,x_r}(h) = \frac{\Gamma(h)\Gamma(h+\delta_1+\delta_2+...+\delta_r)}{\Gamma(h+\delta_1)\Gamma(h+\delta_2+...+\delta_r)} = \sum_{k_2,k_3,...,k_r=0}^{\infty} \frac{(-\delta_1)_{k_2+...+k_r}(-\delta_2)_{k_2}\cdots(-\delta_r)_{k_r}}{(h)_{k_2+...+k_r}k_2!k_3!\cdots k_r!}$$

and

$$\tilde{\Delta}_{x_1;x_2,...,x_r}(h) = \frac{\Gamma(h+\delta_1)\Gamma(h+\delta_2+...+\delta_r)}{\Gamma(h)\Gamma(h+\delta_1+\delta_2+...+\delta_r)} =$$

$$= \sum_{k_2,k_3,...,k_r=0}^{\infty} \frac{(-\delta_1)_{k_2+...+k_r}(-\delta_2)_{k_2}\cdots(-\delta_r)_{k_r}}{(1-h-\delta_1-...-\delta_r)_{k_2+...+k_r}k_2!k_3!\cdots k_r!}$$

$\left(\delta_{x_j} = x_j\frac{\partial}{\partial x_j};\ j=1,2,...,r\right)$ was introduced in [22].

To study the various properties of another class of generalized multidimensional hypergeometric functions, J. Choi and A. Hasanov [23] introduced the following reciprocal operators:

$$H_{x_1,...x_r}(\alpha,\beta) = \frac{\Gamma(\beta)\Gamma(\alpha+\delta_1+\cdots+\delta_r)}{\Gamma(\alpha)\Gamma(\beta+\delta_1+\cdots+\delta_r)} = \sum_{k_1,\cdots,k_r=0}^{\infty} \frac{(\beta-\alpha)_{k_1+\cdots+k_r}(-\delta_1)_{k_1}\cdots(-\delta_r)_{k_r}}{(\beta)_{k_1+\cdots+k_r}k_1!\cdots k_r!},$$

$$\overline{H}_{x_1,...x_r}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta+\delta_1+\cdots+\delta_r)}{\Gamma(\beta)\Gamma(\alpha+\delta_1+\cdots+\delta_r)} = \sum_{k_1,...k_r=0}^{\infty} \frac{(\beta-\alpha)_{k_1+\cdots+k_r}(-\delta_1)_{k_1}\cdots(-\delta_r)_{k_r}}{(1-\alpha-\delta_1-\cdots-\delta_r)_{i+j}i!j!}$$

$$\left(\delta_{x_j} = x_j\frac{\partial}{\partial x_j}\,(j=1,...,r, r\in\mathbb{N}=\{1,2,...\})\right).$$

Theorem 1. For the second order hypergeometric functions of four variables (1)–(8), the following operator identities are valid:

$$F_1^{(4)}(a_1,a_2,b;c_1,c_2,c_3,c_4;x,y,z,t) = H_t(a_2,c_4)(1-t)^{-b}F_C^{(3)}\left(a_1,b;c_1,c_2,c_3;\frac{x}{1-t},\frac{y}{1-t},\frac{z}{1-t}\right), \tag{9}$$

$$(1-t)^{-b}F_C^{(3)}\left(a_1,b;c_1,c_2,c_3;\frac{x}{1-t},\frac{y}{1-t},\frac{z}{1-t}\right) = \overline{H}_t(a_2,c_4)F_1^{(4)}(a_1,a_2,b;c_1,c_2,c_3,c_4;x,y,z,t), \tag{10}$$

$$F_3^{(4)}(a_1,a_2,a_3,b;c_1,c_2,c_3,c_4;x,y,z,t)$$
$$= H_z(a_2,c_3)H_t(a_3,c_4)(1-t-z)^{-b}F_4\left(a_1,b;c_1,c_2;\frac{x}{1-t-z},\frac{y}{1-t-z}\right), \tag{11}$$

$$(1-t-z)^{-b}F_4\left(a_1,b;c_1,c_2;\frac{x}{1-t-z},\frac{y}{1-t-z}\right)$$
$$= \overline{H}_z(a_2,c_3)\overline{H}_t(a_3,c_4)F_3^{(4)}(a_1,a_2,a_3,b;c_1,c_2,c_3,c_4;x,y,z,t), \tag{12}$$

$$F_4^{(4)}(a,b,c;c_1,c_2,c_3,c_4;x,y,z,t) =$$

$$= H_z\left(c, c_3\right)(1-z)^{-a} F_C^{(3)}\left(a, b; c_1, c_2, c_3; \frac{x}{1-z}, \frac{y}{1-z}, \frac{t}{1-z}\right), \tag{13}$$

$$(1-z)^{-a} F_C^{(3)}\left(a, b; c_1, c_2, c_3; \frac{x}{1-z}, \frac{y}{1-z}, \frac{t}{1-z}\right)$$
$$= \overline{H}_z\left(c, c_3\right) F_4^{(4)}\left(a, b, c; c_1, c_2, c_3, c_4; x, y, z, t\right), \tag{14}$$

$$F_5^{(4)}\left(a_1, a_2, b_1, b_2; c_1, c_2, c_3, c_4; x, y, z, t\right) = H_t\left(a_2, c_4\right)(1-t)^{-b_2} F_E\left(a_1; b_1, b_2; c_1, c_2, c_3; x, y, \frac{z}{1-t}\right), \tag{15}$$

$$(1-t)^{-b_2} F_E\left(a_1; b_1, b_2; c_1, c_2, c_3; x, y, \frac{z}{1-t}\right) = \overline{H}_t\left(a_2, c_4\right) F_5^{(4)}\left(a_1, a_2, b_1, b_2; c_1, c_2, c_3, c_4; x, y, z, t\right), \tag{16}$$

$$F_6^{(4)}\left(a_1, a_2, b_1, b_2, b_3; c_1, c_2, c_3, c_4; x, y, z, t\right)$$
$$= H_y\left(b_2, c_2\right) H_z\left(b_3, c_3\right)(1-y-z)^{-a_1} F_2\left(b_1; a_1, a_2; c_1, c_4; \frac{x}{1-y-z}, t\right), \tag{17}$$

$$(1-y-z)^{-a_1} F_2\left(b_1; a_1, a_2; c_1, c_4; \frac{x}{1-y-z}, t\right)$$
$$= \overline{H}_y\left(b_2, c_2\right) \overline{H}_z\left(b_3, c_3\right) F_6^{(4)}\left(a_1, a_2, b_1, b_2, b_3; c_1, c_2, c_3, c_4; x, y, z, t\right), \tag{18}$$

$$F_8^{(4)}\left(a_1, a_2, b_1, b_2; c_1, c_2, c_3, c_4; x, y, z, t\right)$$
$$= H_y\left(b_2, c_2\right) H_t\left(b_3, c_4\right)(1-y)^{-a_1}(1-t)^{-a_2} F_2\left(b_1; a_1, a_2; c_1, c_3; \frac{x}{1-y}, \frac{x}{1-y}\right), \tag{19}$$

$$(1-y)^{-a_1}(1-t)^{-a_2} F_2\left(b_1; a_1, a_2; c_1, c_3; \frac{x}{1-y}, \frac{x}{1-y}\right)$$
$$= \overline{H}_y\left(b_2, c_2\right) \overline{H}_t\left(b_3, c_4\right) F_8^{(4)}\left(a_1, a_2, b_1, b_2; c_1, c_2, c_3, c_4; x, y, z, t\right), \tag{20}$$

$$F_{11}^{(4)}\left(a_1, a_2, b; c_1, c_2, c_3; x, y, z, t\right) = H_t\left(a_3, c_3\right)(1-t)^{-b} F_F\left(b; a_1, a_2; c_2, c_1; \frac{y}{1-t}, \frac{z}{1-t}, \frac{x}{1-t}\right), \tag{21}$$

$$(1-t)^{-b} F_F\left(b; a_1, a_2; c_2, c_1; \frac{y}{1-t}, \frac{z}{1-t}, \frac{x}{1-t}\right) = \overline{H}_t\left(a_3, c_3\right) F_{11}^{(4)}\left(a_1, a_2, b; c_1, c_2, c_3; x, y, z, t\right), \tag{22}$$

$$F_{13}^{(4)}\left(a_1, a_2, a_3, a_4, b; c_1, c_2, c_3; x, y, z, t\right)$$
$$= H_z\left(a_3, c_2\right) H_t\left(a_4, c_3\right)(1-z-t)^{-b} F_1\left(b; a_1, a_2; c_1; \frac{x}{1-z-t}, \frac{y}{1-z-t}\right), \tag{23}$$

$$(1-z-t)^{-b}F_1\left(b;a_1,a_2;c_1;\frac{x}{1-z-t},\frac{y}{1-z-t}\right)$$
$$=\overline{H}_z\left(a_3,c_2\right)\overline{H}_t\left(a_4,c_3\right)F_{13}^{(4)}\left(a_1,a_2,a_3,a_4,b;c_1,c_2,c_3;x,y,z,t\right), \quad (24)$$

where F_1, F_2, F_4 are Appel hypergeometric functions [21], $F_C^{(3)}$ is Lauricella function [24], and F_E, F_F are Saran functions [25]:

$$F_E\left(\alpha;\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{p+m+n}(\beta_1)_{m+n}(\beta_2)_p}{(\gamma_1)_m(\gamma_2)_n(\gamma_3)_p m!n!p!}x^m y^n z^p,$$

$$F_F\left(\alpha;\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{m+n+p}(\beta_1)_{m+p}(\beta_2)_n}{(\gamma_1)_m(\gamma_2)_{n+p} m!n!p!}x^m y^n z^p.$$

Proof. Theorem 1 is proved by dint of Mellin's transformations [26].

Decomposition formulas

Theorem 2. For second order hypergeometric functions (1)–(8) the following decomposition formulas are valid:

$$F_1^{(4)}\left(a_1,a_2,b;c_1,c_2,c_3,c_4;x,y,z,t\right)$$
$$=(1-t)^{-b}\sum_{i=0}^{\infty}\frac{(-1)^i(b)_i(c_4-a_2)_i}{(c_4)_i i!}\left(\frac{t}{1-t}\right)^i F_C^{(3)}\left(a_1,b+i;c_1,c_2,c_3;\frac{x}{1-t},\frac{y}{1-t},\frac{z}{1-t}\right), \quad (25)$$

$$(1-t)^{-b}F_C^{(3)}\left(a_1,b;c_1,c_2,c_3;\frac{x}{1-t},\frac{y}{1-t},\frac{z}{1-t}\right)$$
$$=\sum_{i=0}^{\infty}\frac{(c_4-a_2)_i(b)_i}{(c_4)_i i!}t^i F_1^{(4)}\left(a_1,a_2,b+i;c_1,c_2,c_3,c_4+i;x,y,z,t\right), \quad (26)$$

$$F_3^{(4)}\left(a_1,a_2,a_3,b;c_1,c_2,c_3,c_4;x,y,z,t\right)$$
$$=(1-t-z)^{-b}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(b)_{i+j}(c_3-a_2)_i(c_4-a_3)_j}{(c_3)_i(c_4)_j i!j!}\left(\frac{z}{1-t-z}\right)^i$$
$$\times\left(\frac{t}{1-t-z}\right)^j F_4\left(a_1,b+i+j;c_1,c_2;\frac{x}{1-t-z},\frac{y}{1-t-z}\right), \quad (27)$$

$$(1-t-z)^{-b}F_4\left(a_1,b;c_1,c_2;\frac{x}{1-t-z},\frac{y}{1-t-z}\right)$$
$$=\sum_{i,j=0}^{\infty}\frac{(c_3-a_2)_i(c_4-a_3)_j(b)_{i+j}}{(c_3)_i(c_4)_j i!j!}z^i t^j F_3^{(4)}\left(a_1,a_2,a_3,b+i+j;c_1,c_2,c_3+i,c_4+j;x,y,z,t\right), \quad (28)$$

$$F_4^{(4)}\left(a,b,c;c_1,c_2,c_3,c_4;x,y,z,t\right)$$
$$=(1-z)^{-a}\sum_{i=0}^{\infty}\frac{(-1)^i(a)_i(c_3-c)_i}{(c_3)_i i!}\left(\frac{z}{1-z}\right)^i F_C^{(3)}\left(a+i,b;c_1,c_2,c_4;\frac{x}{1-z},\frac{y}{1-z},\frac{t}{1-z}\right), \quad (29)$$

$$(1-z)^{-a}F_C^{(3)}\left(a,b;c_1,c_2,c_3;\frac{x}{1-z},\frac{y}{1-z},\frac{t}{1-z}\right) = \sum_{i=0}^{\infty}\frac{(a)_i(c_3-c)_i}{(c_3)_i i!}z^i F_4^{(4)}\left(a+i,b,c;c_1,c_2,c_3+i,c_4;x,y,z,t\right), \quad (30)$$

$$F_5^{(4)}\left(a_1,a_2,b_1,b_2;c_1,c_2,c_3,c_4;x,y,z,t\right) = (1-t)^{-b_2}\sum_{i=0}^{\infty}\frac{(-1)^i(b_2)_i(c_4-a_2)_i}{(c_4)_i i!}\left(\frac{t}{1-t}\right)^i F_E\left(a_1;b_1,b_2+i;\gamma_1,\gamma_2,\gamma_3;x,y,\frac{z}{1-t}\right), \quad (31)$$

$$(1-t)^{-b_2}F_E\left(a_1;b_1,b_2;c_1,c_2,c_3;x,y,\frac{z}{1-t}\right) = \sum_{i=0}^{\infty}\frac{(b_2)_i(c_4-a_2)_i}{(c_4)_i i!}t^i F_5^{(4)}\left(a_1,a_2,b_1,b_2+i;c_1,c_2,c_3,c_4+i;x,y,z,t\right), \quad (32)$$

$$F_6^{(4)}\left(a_1,a_2,b_1,b_2,b_3;c_1,c_2,c_3,c_4;x,y,z,t\right) = (1-y-z)^{-a_1}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(a_1)_{i+j}(c_2-b_2)_i(c_3-b_3)_j}{(c_2)_i(c_3)_j i!j!}\left(\frac{y}{1-y-z}\right)^i \times\left(\frac{z}{1-y-z}\right)^j F_2\left(b_1;a_1+i+j,a_2;c_1,c_4;\frac{x}{1-y-z},t\right), \quad (33)$$

$$(1-y-z)^{-a_1}F_2\left(b_1;a_1,a_2;c_1,c_4;\frac{x}{1-y-z},t\right) = \sum_{i,j=0}^{\infty}\frac{(a_1)_{i+j}(c_2-b_2)_i(c_3-b_3)_j}{(c_2)_i(c_3)_j i!j!}y^i z^j \times F_6^{(4)}\left(a_1+i+j,a_2,b_1,b_2,b_3;c_1,c_2+i,c_3+j,c_4;x,y,z,t\right), \quad (34)$$

$$F_8^{(4)}\left(a_1,a_2,b_1,b_2;c_1,c_2,c_3,c_4;x,y,z,t\right) = (1-y)^{-a_1}(1-t)^{-a_2} \times\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(a_1)_i(a_2)_j(c_2-b_2)_i(c_4-b_3)_j}{(c_2)_i(c_4)_j i!j!}\left(\frac{y}{1-y}\right)^i\left(\frac{t}{1-t}\right)^j \times F_2\left(b_1;a_1+i,a_2+j;c_1,c_3;\frac{x}{1-y},\frac{z}{1-y}\right), \quad (35)$$

$$(1-y)^{-a_1}(1-t)^{-a_2}F_2\left(b_1;a_1,a_2;c_1,c_3;\frac{x}{1-y},\frac{x}{1-y}\right) = \sum_{i,j=0}^{\infty}\frac{(a_1)_i(a_2)_j(c_2-b_2)_i(c_4-b_3)_j}{(c_2)_i(c_4)_j i!j!}y^i t^j \times F_8^{(4)}\left(a_1+i,a_2+j,b_1,b_2;c_1,c_2+i,c_3,c_4+j;x,y,z,t\right), \quad (36)$$

$$F_{11}^{(4)}\left(a_1,a_2,b;c_1,c_2,c_3;x,y,z,t\right) = (1-t)^{-b}\sum_{i=0}^{\infty}\frac{(-1)^i(b)_i(c_3-a_3)_i}{(c_3)_i i!}\left(\frac{t}{1-t}\right)^i F_F\left(b+i;a_1,a_2;c_2,c_1;\frac{y}{1-t},\frac{z}{1-t},\frac{x}{1-t}\right), \quad (37)$$

$$(1-t)^{-b}F_F\left(b;a_1,a_2;c_2,c_1;\frac{y}{1-t},\frac{z}{1-t},\frac{x}{1-t}\right)$$
$$=\sum_{i=0}^{\infty}\frac{(b)_i(c_3-a_3)_i}{(c_3)_i i!}t^i F_{11}^{(4)}\left(a_1,a_2,b+i;c_1,c_2,c_3+i;x,y,z,t\right), \quad (38)$$

$$F_{13}^{(4)}\left(a_1,a_2,a_3,a_4,b;c_1,c_2,c_3;x,y,z,t\right)$$
$$=(1-z-t)^{-b}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(b)_{i+j}(c_2-a_3)_i(c_3-a_4)_j}{(c_2)_i(c_3)_j i!j!}\left(\frac{z}{1-z-t}\right)^i\left(\frac{t}{1-z-t}\right)^j$$
$$\times F_1\left(b+i+j;a_1,a_2;c_1;\frac{x}{1-z-t},\frac{y}{1-z-t}\right), \quad (39)$$

$$(1-z-t)^{-b}F_1\left(b;a_1,a_2;c_1;\frac{x}{1-z-t},\frac{y}{1-z-t}\right)$$
$$=\sum_{i=0}^{\infty}\frac{(b)_{i+j}(c_2-a_3)_i(c_3-a_4)_j}{(c_2)_i(c_3)_j i!j!}z^it^jF_{13}^{(4)}\left(a_1,a_2,a_3,a_4,b+i+j;c_1,c_2+i,c_3+j;x,y,z,t\right). \quad (40)$$

Proof. The proof of Theorem 2 is realized utilizing operator identities (9)–(24), some properties of hypergeometric functions of many variables and the following operator identities [27, p. 93]:

$$(\delta+\alpha)_n\left\{f(\xi)\right\}=\xi^{1-\alpha}\frac{d^n}{d\xi^n}\left\{\xi^{\alpha+n-1}f(\xi)\right\},$$
$$(-\delta)_n\left\{f(\xi)\right\}=(-\xi)^n\frac{d^n}{d\xi^n}\left\{f(\xi)\right\}, \quad (41)$$

$\delta=\xi\frac{d}{d\xi}$; $\alpha\in\mathbb{C}$; $n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$; $\mathbb{N}=\{1,2,3,...\}$, where $f(\xi)$ is analytical function.

As an example, we give a brief proof of the decomposition (25).

The following equality holds:

$$(1-t)^{-b}F_C^{(3)}\left(a_1,b;c_1,c_2,c_3;\frac{x}{1-t},\frac{y}{1-t},\frac{z}{1-t}\right)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(b)_{m+n+p+q}}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}. \quad (42)$$

Considering operator definition $H_t(a_2,c_4)$ and identity (42), from (9) we have

$$F_1^{(4)}\left(a_1,a_2,b;c_1,c_2,c_3,c_4;x,y,z,t\right)$$
$$=\sum_{j=0}^{\infty}\frac{(c_4-a_2)_j(-\delta_t)_j}{(c_4)_j j!}\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(b)_{m+n+p+q}}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}.$$

Using formula (41), we obtain

$$F_1^{(4)}\left(a_1,a_2,b;c_1,c_2,c_3,c_4;x,y,z,t\right)$$
$$=\sum_{i=0}^{\infty}\frac{(-1)^i(b)_i(c_4-a_2)_i}{(c_4)_i i!}t^i\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(b+i)_{m+n+p+q}}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}.$$

By virtue of the validity of the identity $(\lambda)_{m+n} = (\lambda)_m(\lambda+m)_n$ we get

$$F_1^{(4)}(a_1,a_2,b;c_1,c_2,c_3,c_4;x,y,z,t) = (1-t)^{-b}\sum_{i=0}^{\infty}\frac{(-1)^i(b)_i(c_4-a_2)_i}{(c_4)_i i!}\left(\frac{t}{1-t}\right)^i$$
$$\times\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n+p}(b+i)_{m+n+p}}{(c_1)_m(c_2)_n(c_3)_p}\frac{\left(\frac{x}{1-t}\right)^m}{m!}\frac{\left(\frac{y}{1-t}\right)^n}{n!}\frac{\frac{z}{1-t}^p}{p!}. \quad (43)$$

In view of the $F_C^{(3)}$ Lauricella hypergeometric function definition, from expression (43) we obtain decomposition (25).

Thus, the decomposition formula (25) is proved.

Similarly, we can prove each of the decomposition formulas (26)–(40).

Remark 1. The decomposition formulas $(25)-(40)$ can also be proved by comparing the coefficients before the factor $x^m y^n z^p t^q$ in both sides of the equality.

Conclusion

In conclusion, we proved the operator identities written via the mutually inverse operators H and $\overline{H}$ for the hypergeometric functions of four variables $F_1^{(4)}$, $F_3^{(4)}-F_6^{(4)}$, $F_8^{(4)}$, $F_{11}^{(4)}$, $F_{13}^{(4)}$, the validity of the former is proved using the Mellin transforms. By applying the obtained operator identities, differentiation formulas for hypergeometric functions, and properties of hypergeometric functions, we have proved decompositions for the functions $F_1^{(4)}$, $F_3^{(4)}-F_6^{(4)}$, $F_8^{(4)}$, $F_{11}^{(4)}$, $F_{13}^{(4)}$ by products of such known hypergeometric functions as the Appell's functions F_1, F_2, F_4; Lauricella's function $F_C^{(3)}$; the Saran functions F_E, F_F. Similarly, the decomposition formulas for hypergeometric functions of four variables $F_{17}^{(4)}$, $F_{18}^{(4)}$, $F_{19}^{(4)}$, $F_{20}^{(4)}$, $F_{21}^{(4)}$, etc. can be obtained.

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А.С. Бердышев, А. Хасанов, А.Р. Рысқан

Кейбір төрт айнымалы гипергеометриялық қатарлар үшін жіктеу формулалары

Мақалада төрт айнымалы гипергеометриялық Гаусс қатарлары үшін операторлық тепе-теңдік пен қарапайым функцияларға жіктеу формулалары алынды. Символдық операторлардың кері жұптарына негізделген Чои-Хасанов әдісі қолданылды. Алынған төрт айнымалы гипергеометриялық қатарлары үшін жіктеу формулалары осы функциялардың қасиеттерін зерттеп білуге мүмкіндік береді. Алынған жіктеулер көпөлшемді азғындалған дербес туындылы дифференциалдық теңдеулер үшін шеттік есептердің шешілімділік мәселелерін зерттеуде пайданылады.

Кілт сөздер: Аппель гипергеометриялық функциясы, Лауричелл функциясы, Саран функциясы, төрт айнымалы гипергеометриялық қатар, жіктеу формулалары, операторлық тепе-теңдіктер, кері символдық операторлар.

А.С. Бердышев, А. Хасанов, А.Р. Рыскан

Формулы разложения для некоторых гипергеометрических рядов четырех переменных

В статье получены операторные тождества и формулы разложения для гипергеометрических рядов Гаусса второго порядка четырех переменных по произведениям, содержащим более простые гипергеометрические функции. Авторами использован метод Чои-Хасанова, основанный на обратных парах символических операторов. Полученные формулы разложения для гипергеометрических функций четырех переменных позволят изучить свойства этих функций. Данные разложения применяются при исследовании вопросов разрешимости краевых задач для вырождающихся многомерных дифференциальных уравнений в частных производных.

Ключевые слова: гипергеометрическая функция Аппеля, функция Лауричелла, функция Сарана, гипергеометрический ряд четырех переменных, формулы разложения, операторные тождества, обратные символические операторы.

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