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Interpolation theorem for Nikol'skii-Besov type spaceswith mixed metric

In this paper we study the interpolation properties of Nikol'skii-Besov spaces with a dominant mixed derivative and mixed metric with respect to anisotropic and complex interpolation methods. An interpolation theorem is proved for a weighted discrete space of vector-valued sequences $l_{\mathbf{q}}^{\alpha}(A)$. It is shown that the Nikol'skii-Besov space under study is a retract of the space $l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}})$. Based on the above results, interpolation theorems were obtained for Nikol'skii-Besov spaces with the dominant mixed derivative and mixed metric.

Keywords: Nikol'skii-Besov type spaces, method of anisotropic interpolation, complex interpolation.

Introduction

The embedding theorems for spaces of differentiable functions play an important role in the study of boundary value problems for equations of mathematical physics and approximation theory. At the same time, interpolation of smooth function spaces is of great interest.

The interpolation of the Sobolev and Besov spaces was first studied by J. Petre [1], J.-L. Lions and J. Petre [2]. Further results on interpolation of spaces of smooth functions with respect to the classical real and complex methods can be found in the monographs of J. Berg and J. Lofstrom [3], H. Tribel [4]. In V.L. Krepkogorskii's papers [5], [6], I. Asecritova's and others [7] interpolation properties of Besov and Lizorkin-Triebel spaces were studied with respect to Sparr's method. E.D. Nursultanov and K.A. Bekmaganbetov considered interpolation properties of Besov spaces with respect to a method of multiparametric interpolation (see [8]). They also considered interpolation properties of classical Besov and Lizorkin-Triebel spaces with respect to an anisotropic interpolation method (see [9], [10]). In works [11]– [13] E.D. Nursultanov, K.A. Bekmaganbetov and Ye. Toleugazy considered interpolation properties of Besov spaces with dominant mixed derivative with anisotropic and mixed metric. The use of interpolation theorems for receiving embedding theorems and their further applications in approximation theory is shown in works [14]– [16].

In this paper we study the interpolation properties of Nikol'skii-Besov spaces with a dominant mixed derivative and mixed metric with respect to anisotropic and complex interpolation methods.

An interpolation theorem is proved for a weighted discrete space of vector-valued sequences $l_{\mathbf{q}}^{\alpha}(A)$. It is shown that the Nikol'skii-Besov space under study is a retract of the space $l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}})$. Based on the above results, interpolation theorems were obtained for Nikol'skii-Besov spaces with the dominant mixed derivative and mixed metric.

Preliminaries and auxiliary results

Let $E = \{\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) : \varepsilon_i = 0$ or $\varepsilon_i = 1, i = 1, \dots, n\}$ be the set of vertices of an n-dimensional unit cube in $\mathbb{R}^n$, $\mathbf{A} = \{A_\varepsilon\}_{\varepsilon \in E}$ is a set of Banach spaces that are subspaces of some linear Hausdorff

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space, which is called a compatible set of Banach spaces [3]. For element a from space $\sum_{\varepsilon\in E} A_\varepsilon$ we define

$$K(\mathbf{t}, a; \mathbf{A}) = \inf_{a=\sum_{\varepsilon\in E} a_\varepsilon} \sum_{\varepsilon\in E} \mathbf{t}^\varepsilon \|a_\varepsilon\|_{A_\varepsilon},$$

where $\mathbf{t}^\varepsilon = t_1^{\varepsilon_1} \cdot \ldots \cdot t_n^{\varepsilon_n}$.

Let $\mathbf{0} < \theta = (\theta_1, \ldots, \theta_n) < \mathbf{1}$, $\mathbf{0} < \mathbf{r} = (r_1, \ldots, r_n) \leq \infty$. By $\mathbf{A}_{\theta\mathbf{r}} = (A_\varepsilon; \varepsilon \in E)_{\theta\mathbf{r}}$ we denote the linear subspace of $\sum_{\varepsilon\in E} A_\varepsilon$ such that for its elements the following condition holds:

$$\|a\|_{\mathbf{A}_{\theta\mathbf{r}}} = \left(\int_{\mathbb{R}^n_+} \left(\mathbf{t}^{-\theta} K(\mathbf{t}, a; \mathbf{A}) \right)^{\mathbf{r}} \frac{\mathbf{dt}}{\mathbf{t}} \right)^{\mathbf{1/r}} =$$

$$= \left(\int_0^\infty \left(t_n^{-\theta_n} \ldots \left(\int_0^\infty \left(t_1^{-\theta_1} K(\mathbf{t}, a; \mathbf{A}) \right)^{r_1} \frac{dt_1}{t_1} \right)^{r_2/r_1} \ldots \right)^{r_n/r_{n-1}} \frac{dt_n}{t_n} \right)^{1/r_n} < \infty.$$

Lemma 1 ([9]). Let $\mathbf{0} < \theta < \mathbf{1}$, $\mathbf{0} < \mathbf{r} \leq \infty$ and let $\mathbf{A} = \{A_\varepsilon\}_{\varepsilon\in E}$ and $\mathbf{B} = \{B_\varepsilon\}_{\varepsilon\in E}$ be two compatible sets of Banach spaces. Suppose that for a linear operator $T : \mathbf{A}_\varepsilon \to \mathbf{B}_\varepsilon$ there are two vectors $\mathbf{M_0} = (M_1^0, \ldots, M_n^0), \mathbf{M_1} = (M_1^1, \ldots, M_n^1)$ with positive components such that $\|T\|_{\mathbf{A}_\varepsilon\to\mathbf{B}_\varepsilon} \leq$ $\leq C_\varepsilon \prod_{i=1}^n M_i^{\varepsilon_i}$ for any $\varepsilon \in E$, then

$$T : \mathbf{A}_{\theta\mathbf{r}} \to \mathbf{B}_{\theta\mathbf{r}}$$

with estimate $\|T\|_{\mathbf{A}_{\theta\mathbf{r}}\to\mathbf{B}_{\theta\mathbf{r}}} \leq \max\limits_{\varepsilon\in E} C_\varepsilon \prod\limits_{i=1}^n \left(M_i^0\right)^{1-\theta_i} \left(M_i^1\right)^{\theta_i}$.

Lemma 2 ([4]). Let $\alpha_1 < \alpha < \alpha_0$ and $1 \leq q \leq \infty$. For a sequence of non-negative numbers $\{a_k\}_{k\in\mathbb{Z}}$ define transformations

$$I_0(a; j) = \sum_{k=-\infty}^{j} 2^{\alpha_0(k-j)} a_k,$$

$$I_1(a; j) = \sum_{k=j+1}^{\infty} 2^{\alpha_1(k-j)} a_k.$$

Then the following inequalities hold:

$$\left(\sum_{j=-\infty}^{\infty} \left(2^{\alpha j} I_0(a; j) \right)^q \right)^{1/q} \leq C_1 \left(\sum_{j=-\infty}^{\infty} \left(2^{\alpha j} a_j \right)^q \right)^{1/q}, \tag{1}$$

$$\left(\sum_{j=-\infty}^{\infty} \left(2^{\alpha j} I_1(a; j) \right)^q \right)^{1/q} \leq C_2 \left(\sum_{j=-\infty}^{\infty} \left(2^{\alpha j} a_j \right)^q \right)^{1/q}. \tag{2}$$

For multi-indices $\mathbf{b}_0 = (b_1^0, \ldots, b_n^0), \mathbf{b}_1 = (b_1^1, \ldots, b_n^1)$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in E$ we introduce the notation $\mathbf{b}_\varepsilon = (b_1^{\varepsilon_1}, \ldots, b_n^{\varepsilon_n})$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, $\mathbf{1} \leq \mathbf{q} = (q_1, \ldots, q_n) \leq \infty$ and A be a Banach space. By $l_{\mathbf{q}}^\alpha(A)$ we denote the set of multi-sequences $\{a_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^n}$ with values in A for which the norm

$$\|a\|_{l_{\mathbf{q}}^\alpha(A)} = \left(\sum_{\mathbf{k}\in\mathbb{Z}^n} \left(2^{(\alpha,\mathbf{k})} \|a_{\mathbf{k}}\|_A \right)^{\mathbf{q}} \right)^{\mathbf{1/q}} =$$

$$= \left(\sum_{k_n=-\infty}^{+\infty} \left(\dots \left(\sum_{k_1=-\infty}^{+\infty} \left(2^{\sum_{i=1}^{n} \alpha_i k_i} \|a_{k_1,\dots,k_n}\|_A \right)^{q_1} \right)^{q_2/q_1} \dots \right)^{q_n/q_{n-1}} \right)^{1/q_n}$$

is finite, here $(\alpha, \mathbf{k}) = \sum_{i=1}^{n} \alpha_i k_i$ is the inner product.

Lemma 3. Let $\alpha_0 = (\alpha_1^0, \ \dots, \ \alpha_n^0) \neq \alpha_1 = (\alpha_1^1, \ \dots, \ \alpha_n^1)$, $\mathbf{1} \leq \mathbf{q_0} = (q_1^0, \ \dots, \ q_n^0)$, $\mathbf{q_1} = (q_1^1, \ \dots, \ q_n^1) \leq \infty$. Then for $\mathbf{0} < \theta = (\theta_1, \dots, \theta_n) < \mathbf{1}$, $\mathbf{1} \leq \mathbf{q} = (q_1, \dots, q_n)\infty$ the equality

$$\left(l_{\mathbf{q}_\varepsilon}^{\alpha_\varepsilon}(A); \varepsilon \in E \right)_{\theta \mathbf{q}} = l_{\mathbf{q}}^{\alpha}(A)$$

holds, here $\alpha = (\mathbf{1} - \theta)\alpha_{\mathbf{0}} + \theta \alpha_{\mathbf{1}}$.

Proof. Without loss of generality, we can assume that $\alpha_0 = (\alpha_1^0, \dots, \alpha_n^0) > \alpha_1 = (\alpha_1^1, \dots, \alpha_n^1)$. Due to the embeddings

$$l_{\mathbf{1}}^{\alpha}(A) \hookrightarrow l_{\mathbf{q}}^{\alpha}(A) \hookrightarrow l_{\infty}^{\alpha}(A)$$

it is enough to prove the embeddings

$$(l_{\infty}^{\alpha_\varepsilon}(A); \varepsilon \in E)_{\theta \mathbf{q}} \hookrightarrow l_{\mathbf{q}}^{\alpha}(A) \tag{3}$$

and

$$l_{\mathbf{q}}^{\alpha}(A) \hookrightarrow (l_{\mathbf{1}}^{\alpha_\varepsilon}(A); \varepsilon \in E)_{\theta \mathbf{q}}, \tag{4}$$

where $\alpha = (\mathbf{1} - \theta)\alpha_{\mathbf{0}} + \theta \alpha_{\mathbf{1}}$.

First we prove the embedding (3). If $a = \{a_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}^n} \in l_{\infty}^{\alpha_{\mathbf{1}}}(A)$, then

$$K(\mathbf{t}, a; l_{\infty}^{\alpha_\varepsilon}(A); \varepsilon \in E) = \inf_{a = \sum_{\varepsilon \in E} a_\varepsilon} \sum_{\varepsilon \in E} \mathbf{t}^{\varepsilon} \|a_\varepsilon\|_{l_{\infty}^{\alpha_\varepsilon}(A)} =$$

$$= \inf_{a = \sum_{\varepsilon \in E} a_\varepsilon} \sum_{\varepsilon \in E} \mathbf{t}^{\varepsilon} \sup_{\mathbf{k} \in \mathbb{Z}^n} 2^{\langle \alpha_\varepsilon, \mathbf{k} \rangle} \left\| a_{\mathbf{k}}^{(\varepsilon)} \right\|_A = \inf_{a = \sum_{\varepsilon \in E} a_\varepsilon} \sum_{\varepsilon \in E} \sup_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{t}^{\varepsilon} 2^{\langle \alpha_\varepsilon, \mathbf{k} \rangle} \left\| a_{\mathbf{k}}^{(\varepsilon)} \right\|_A \geq$$

$$\geq \inf_{a = \sum_{\varepsilon \in E} a_\varepsilon} \sum_{\varepsilon \in E} \sup_{\mathbf{k} \in \mathbb{Z}^n} \min_{\varepsilon \in E} \left(\mathbf{t}^{\varepsilon} 2^{\langle \alpha_\varepsilon, \mathbf{k} \rangle} \right) \left\| a_{\mathbf{k}}^{(\varepsilon)} \right\|_A =$$

$$= \inf_{a = \sum_{\varepsilon \in E} a_\varepsilon} \sup_{\mathbf{k} \in \mathbb{Z}^n} \min_{\varepsilon \in E} \left(\mathbf{t}^{\varepsilon} 2^{\langle \alpha_\varepsilon, \mathbf{k} \rangle} \right) \sum_{\varepsilon \in E} \left\| a_{\mathbf{k}}^{(\varepsilon)} \right\|_A \geq$$

$$\geq \inf_{a = \sum_{\varepsilon \in E} a_\varepsilon} \sup_{\mathbf{k} \in \mathbb{Z}^n} \min_{\varepsilon \in E} \left(\mathbf{t}^{\varepsilon} 2^{\langle \alpha_\varepsilon, \mathbf{k} \rangle} \right) \left\| \sum_{\varepsilon \in E} a_{\mathbf{k}}^{(\varepsilon)} \right\|_A =$$

$$= \inf_{a = \sum_{\varepsilon \in E} a_\varepsilon} \sup_{\mathbf{k} \in \mathbb{Z}^n} \min_{\varepsilon \in E} \left(\mathbf{t}^{\varepsilon} 2^{\langle \alpha_\varepsilon, \mathbf{k} \rangle} \right) \|a_{\mathbf{k}}\|_A = \sup_{\mathbf{k} \in \mathbb{Z}^n} \min_{\varepsilon \in E} \left(\mathbf{t}^{\varepsilon} 2^{\langle \alpha_\varepsilon, \mathbf{k} \rangle} \right) \|a_{\mathbf{k}}\|_A .$$

Since $\alpha_0 > \alpha_1$, then $\mathbb{R}_+^n$ can be divided into parallelepipeds of the form $\left[2^{(\alpha_0 - \alpha_1)(\mathbf{j} - \mathbf{1})}; 2^{(\alpha_0 - \alpha_1)\mathbf{j}} \right)$, $\mathbf{j} \in \mathbb{Z}^n$. Then

$$\|a\|_{(l_{\infty}^{\alpha_\varepsilon}(A); \varepsilon \in E)_{\theta \mathbf{q}}} = \left(\int_{\mathbb{R}_+^n} \left(\mathbf{t}^{-\theta} K(\mathbf{t}, a; l_{\infty}^{\alpha_\varepsilon}(A); \varepsilon \in E) \right)^{\mathbf{q}} \frac{\mathbf{dt}}{\mathbf{t}} \right)^{\mathbf{1}/\mathbf{q}} \geq$$

$$\geq \left(\int_{\mathbb{R}_+^n} \left(\mathbf{t}^{-\theta} \sup_{\mathbf{k} \in \mathbb{Z}^n} \min_{\varepsilon \in E} \left(\mathbf{t}^{\varepsilon} 2^{\langle \alpha_\varepsilon, \mathbf{k} \rangle} \right) \|a_{\mathbf{k}}\|_A \right)^{\mathbf{q}} \frac{\mathbf{dt}}{\mathbf{t}} \right)^{\mathbf{1}/\mathbf{q}} =$$

$$= \left(\sum_{\mathbf{j} \in \mathbb{Z}^n} \int_{2^{(\alpha_0-\alpha_1)(\mathbf{j}-\mathbf{1})}}^{2^{(\alpha_0-\alpha_1)\mathbf{j}}} \left(\mathbf{t}^{-\theta} \sup_{\mathbf{k}\in\mathbb{Z}^n} \min_{\varepsilon\in E} \left(\mathbf{t}^{\varepsilon} 2^{\langle \alpha_\varepsilon, \mathbf{k}\rangle} \right) \|a_{\mathbf{k}}\|_A \right)^{\mathbf{q}} \frac{\mathbf{dt}}{\mathbf{t}} \right)^{\mathbf{1}/\mathbf{q}} \geq$$

$$\geq C_1 \left(\sum_{\mathbf{j} \in \mathbb{Z}^n} \left(2^{-\langle \theta(\alpha_0-\alpha_1), \mathbf{j}\rangle} \sup_{\mathbf{k}\in\mathbb{Z}^n} \min_{\varepsilon\in E} \left(2^{\langle \varepsilon(\alpha_0-\alpha_1), \mathbf{j}\rangle + \langle \alpha_\varepsilon, \mathbf{k}\rangle} \right) \|a_{\mathbf{k}}\|_A \right)^{\mathbf{q}} \right)^{\mathbf{1}/\mathbf{q}} =$$

$$= C_1 \left(\sum_{\mathbf{j} \in \mathbb{Z}^n} \left(\sup_{\mathbf{k}\in\mathbb{Z}^n} \min_{\varepsilon\in E} \left(2^{\langle (\varepsilon-\theta)(\alpha_0-\alpha_1), \mathbf{j}\rangle + \langle \alpha_\varepsilon, \mathbf{k}\rangle} \right) \|a_{\mathbf{k}}\|_A \right)^{\mathbf{q}} \right)^{\mathbf{1}/\mathbf{q}} =$$

$$= C_1 \left(\sum_{\mathbf{j} \in \mathbb{Z}^n} \left(\sup_{\mathbf{k}\in\mathbb{Z}^n} \min_{\varepsilon\in E} \left(2^{\langle (\varepsilon-\theta)(\alpha_0-\alpha_1)+\alpha_\varepsilon, \mathbf{j}\rangle + \langle \alpha_\varepsilon, (\mathbf{k}-\mathbf{j})\rangle} \right) \|a_{\mathbf{k}}\|_A \right)^{\mathbf{q}} \right)^{\mathbf{1}/\mathbf{q}}.$$

Since for any $\varepsilon \in E$ the equality

$$(\varepsilon-\theta)(\alpha_0-\alpha_1) + \alpha_\varepsilon = (\varepsilon-\theta)(\alpha_0-\alpha_1) + (\mathbf{1}-\varepsilon)\alpha_0 + \varepsilon\alpha_1 =$$

$$= (\mathbf{1}-\theta)\alpha_0 + \theta\alpha_1 = \alpha$$

holds, then we get

$$\|a\|_{(l_\infty^{\alpha_\varepsilon}(A); \varepsilon\in E)_{\theta\mathbf{q}}} \geq C_1 \left(\sum_{\mathbf{j} \in \mathbb{Z}^n} \left(\sup_{\mathbf{k}\in\mathbb{Z}^n} \min_{\varepsilon\in E} \left(2^{\langle \alpha, \mathbf{j}\rangle + \langle \alpha_\varepsilon, (\mathbf{k}-\mathbf{j})\rangle} \right) \|a_{\mathbf{k}}\|_A \right)^{\mathbf{q}} \right)^{\mathbf{1}/\mathbf{q}} =$$

$$= C_1 \left(\sum_{\mathbf{j} \in \mathbb{Z}^n} \left(2^{\langle \alpha, \mathbf{j}\rangle} \sup_{\mathbf{k}\in\mathbb{Z}^n} \min_{\varepsilon\in E} \left(2^{\langle \alpha_\varepsilon, (\mathbf{k}-\mathbf{j})\rangle} \right) \|a_{\mathbf{k}}\|_A \right)^{\mathbf{q}} \right)^{\mathbf{1}/\mathbf{q}} \geq$$

$$\geq C_1 \left(\sum_{\mathbf{j} \in \mathbb{Z}^n} \left(2^{\langle \alpha, \mathbf{j}\rangle} \|a_{\mathbf{j}}\|_A \right)^{\mathbf{q}} \right)^{\mathbf{1}/\mathbf{q}} = C_1 \|a\|_{l_{\mathbf{q}}^{\alpha}(A)}.$$

The last inequality means the embedding (3).

Next, we prove the embedding (4). Let $a = \{a_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^n} \in l_{\mathbf{q}}^{\alpha}(A)$. We have

$$K(\mathbf{t}, a; l_{\mathbf{1}}^{\alpha_\varepsilon}(A); \varepsilon \in E) = \inf_{a=\sum_{\varepsilon\in E} a_\varepsilon} \sum_{\varepsilon\in E} \mathbf{t}^{\varepsilon} \|a_\varepsilon\|_{l_{\mathbf{1}}^{\alpha_\varepsilon}(A)} =$$

$$= \inf_{a=\sum_{\varepsilon\in E} a_\varepsilon} \sum_{\varepsilon\in E} \mathbf{t}^{\varepsilon} \sum_{\mathbf{k}\in\mathbb{Z}^n} 2^{\langle \alpha_\varepsilon, \mathbf{k}\rangle} \left\| a_{\mathbf{k}}^{(\varepsilon)} \right\|_A = \inf_{a=\sum_{\varepsilon\in E} a_\varepsilon} \sum_{\varepsilon\in E} \sum_{\mathbf{k}\in\mathbb{Z}^n} \mathbf{t}^{\varepsilon} 2^{\langle \alpha_\varepsilon, \mathbf{k}\rangle} \left\| a_{\mathbf{k}}^{(\varepsilon)} \right\|_A \leq$$

$$\leq \sum_{\mathbf{k}\in\mathbb{Z}^n} \min_{\varepsilon\in E} \left(\mathbf{t}^{\varepsilon} 2^{\langle \alpha_\varepsilon, \mathbf{k}\rangle} \right) \|a_{\mathbf{k}}\|_A,$$

here we put $a_{\mathbf{k}}^{(\varepsilon)} = a_{\mathbf{k}}$ for ε that corresponds to $\min_{\varepsilon\in E} \left(\mathbf{t}^{\varepsilon} 2^{\langle \alpha_\varepsilon, \mathbf{k}\rangle} \right)$.

As in the proof of (3), we obtain

$$\|a\|_{(l_{\mathbf{1}}^{\alpha_\varepsilon}; \varepsilon\in E)_{\theta\mathbf{q}}} = \left(\int_{\mathbb{R}_+^n} \left(\mathbf{t}^{-\theta} K(\mathbf{t}, a; l_{\mathbf{1}}^{\alpha_\varepsilon}(A); \varepsilon \in E) \right)^{\mathbf{q}} \frac{\mathbf{dt}}{\mathbf{t}} \right)^{\mathbf{1}/\mathbf{q}} \leq$$

$$\left(\int_{\mathbb{R}^n_+}\left(\mathbf{t}^{-\theta}\sum_{\mathbf{k}\in\mathbb{Z}^n}\min_{\varepsilon\in E}\left(\mathbf{t}^{\varepsilon}2^{\langle\alpha_\varepsilon,\mathbf{k}\rangle}\right)\|a_{\mathbf{k}}\|_A\right)^{\mathbf{q}}\frac{\mathbf{dt}}{\mathbf{t}}\right)^{\mathbf{1/q}}=$$

$$\left(\sum_{\mathbf{j}\in\mathbb{Z}^n}\int_{2^{(\alpha_0-\alpha_1)(\mathbf{j-1})}}^{2^{(\alpha_0-\alpha_1)\mathbf{j}}}\left(\mathbf{t}^{-\theta}\sum_{\mathbf{k}\in\mathbb{Z}^n}\min_{\varepsilon\in E}\left(\mathbf{t}^{\varepsilon}2^{\langle\alpha_\varepsilon,\mathbf{k}\rangle}\right)\|a_{\mathbf{k}}\|_A\right)^{\mathbf{q}}\frac{\mathbf{dt}}{\mathbf{t}}\right)^{\mathbf{1/q}}\leq$$

$$C_2\left(\sum_{\mathbf{j}\in\mathbb{Z}^n}\left(2^{-\langle\theta(\alpha_0-\alpha_1),\mathbf{j}\rangle}\sum_{\mathbf{k}\in\mathbb{Z}^n}\min_{\varepsilon\in E}\left(2^{\langle\varepsilon(\alpha_0-\alpha_1),\mathbf{j}\rangle+\langle\alpha_\varepsilon,\mathbf{k}\rangle}\right)\|a_{\mathbf{k}}\|_A\right)^{\mathbf{q}}\right)^{\mathbf{1/q}}=$$

$$C_2\left(\sum_{\mathbf{j}\in\mathbb{Z}^n}\left(2^{\langle\alpha-\alpha_0,\mathbf{j}\rangle}\sum_{\mathbf{k}\in\mathbb{Z}^n}\min_{\varepsilon\in E}\left(2^{\langle\varepsilon(\alpha_0-\alpha_1),\mathbf{j}\rangle+\langle\alpha_\varepsilon,\mathbf{k}\rangle}\right)\|a_{\mathbf{k}}\|_A\right)^{\mathbf{q}}\right)^{\mathbf{1/q}}=$$

$$C_2\left(\sum_{\mathbf{j}\in\mathbb{Z}^n}\left(2^{\langle\alpha,\mathbf{j}\rangle}\sum_{\mathbf{k}\in\mathbb{Z}^n}\min_{\varepsilon\in E}\left(2^{\langle\alpha_\varepsilon,\mathbf{k-j}\rangle}\right)\|a_{\mathbf{k}}\|_A\right)^{\mathbf{q}}\right)^{\mathbf{1/q}}=$$

$$C_2\left(\sum_{\mathbf{j}\in\mathbb{Z}^n}\left(2^{\langle\alpha,\mathbf{j}\rangle}\sum_{\varepsilon\in E}I_\varepsilon(\|a\|_A;\mathbf{j})\right)^{\mathbf{q}}\right)^{\mathbf{1/q}}\leq C_2\sum_{\varepsilon\in E}\left(\sum_{\mathbf{j}\in\mathbb{Z}^n}\left(2^{\langle\alpha,\mathbf{j}\rangle}I_\varepsilon(\|a\|_A;\mathbf{j})\right)^{\mathbf{q}}\right)^{\mathbf{1/q}},$$

where $I_\varepsilon(\|a\|_A;\mathbf{j})=I_{\varepsilon_n}(\ldots I_{\varepsilon_1}(\|a\|_A;\mathbf{j}))$ is a composition of transformations from Lemma 2.

Further, using the Minkowski inequalities, (1) and (2) we obtain

$$\|a\|_{\left(l_{\mathbf{1}}^{\alpha_\varepsilon};\varepsilon\in E\right)_{\theta\mathbf{q}}}\leq C_2\sum_{\varepsilon\in E}\left(\sum_{\mathbf{j}\in\mathbb{Z}^n}\left(2^{\langle\alpha,\mathbf{j}\rangle}I_\varepsilon(\|a\|_A;\mathbf{j})\right)^{\mathbf{q}}\right)^{\mathbf{1/q}}=$$

$$=C_2\sum_{\varepsilon\in E}\left(\sum_{j_n=-\infty}^{\infty}\left(2^{\alpha_nj_n}\ldots\left(\sum_{j_1=-\infty}^{\infty}\left(2^{\alpha_1j_1}I_{\varepsilon_n}(..I_{\varepsilon_1}(\|a\|_A;\mathbf{j}))\right)^{q_1}\right)^{q_2/q_1}\ldots\right)^{q_n/q_{n-1}}\right)^{1/q_n}\leq$$

$$\leq C_2\sum_{\varepsilon\in E}\left(\sum_{j_n=-\infty}^{\infty}\left(2^{\alpha_nj_n}I_{\varepsilon_n}\left(\ldots\left(\sum_{j_1=-\infty}^{\infty}\left(2^{\alpha_1j_1}I_{\varepsilon_1}(\|a\|_A;\mathbf{j})\right)^{q_1}\right)^{q_2/q_1}\ldots\right)\right)^{q_n}\right)^{1/q_n}\leq$$

$$\leq C_3\sum_{\varepsilon\in E}\left(\sum_{j_n=-\infty}^{\infty}\left(2^{\alpha_nj_n}\ldots\left(\sum_{j_1=-\infty}^{\infty}\left(2^{\alpha_1j_1}\|a_{j_1,\ldots,j_n}\|_A\right)^{q_1}\right)^{q_2/q_1}\ldots\right)^{q_n/q_{n-1}}\right)^{1/q_n}=$$

$$=C_4\left(\sum_{\mathbf{j}\in\mathbb{Z}^n}\left(2^{\langle\alpha,\mathbf{j}\rangle}\|a_{\mathbf{j}}\|_A\right)^{\mathbf{q}}\right)^{\mathbf{1/q}}=C_4\|a\|_{l_{\mathbf{q}}^{\alpha}(A)}.$$

The last inequality means the embedding (4).

The lemma is completely proved. □

Nikol'skii-Besov type spaces and their interpolation

Let multi-index $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$, $\mathbb{T}^{\mathbf{d}} = \{\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) : \mathbf{x}_i = (x_1^i, \ldots, x_{d_i}^i) \in [0, 2\pi)^{d_i}$, $i = 1, \ldots, n\}$. Let $f(\mathbf{x}) = f(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ be measurable function on $\mathbb{T}^{\mathbf{d}}$.

Further, let $\mathbf{1} \leq \mathbf{p} = (p_1, \ldots, p_n) \leq \infty$. The Lebesgue space $L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})$ with a mixed metric is a set of functions for which the following expression is finite

$$\|f\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})}=\left(\int_{\mathbb{T}^{d_n}}\left(\ldots\left(\int_{\mathbb{T}^{d_1}}|f(\mathbf{x}_1,\ldots,\mathbf{x}_n)|^{p_1}\mathrm{d}\mathbf{x}_1\right)^{p_2/p_1}\ldots\right)^{p_n/p_{n-1}}\mathrm{d}\mathbf{x}_n\right)^{1/p_n}.$$

Here, the expression $\left(\int_{\mathbb{T}^{d_i}}|f(\mathbf{x})|^{p_i}\mathrm{d}\mathbf{x}_i\right)^{1/p_i}$ for $p_i{=}\infty$ we undestand as $\sup_{x\in\mathbb{T}^{d_i}}|f(\mathbf{x})|$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, $\mathbf{1} \leq \mathbf{q} = (q_1, \ldots, q_n) \leq \infty$, $\mathbf{1} \leq \mathbf{p} = (p_1, \ldots, p_n) < \infty$.

For trigonometric series $f \sim \sum_{\mathbf{k}\in\mathbb{Z}^{\mathbf{d}}} a_{\mathbf{k}} e^{i\langle\mathbf{k},\mathbf{x}\rangle}$ denote by

$$\Delta_{\mathbf{s}}(f, \mathbf{x}) = \sum_{\mathbf{k}\in\rho(\mathbf{s})} a_{\mathbf{k}} e^{i\langle\mathbf{k},\mathbf{x}\rangle},$$

where $\langle\mathbf{k},\mathbf{x}\rangle = \sum_{i=1}^{n}\sum_{j=1}^{d_i} k_j^i x_j^i$ is the inner product, $\rho(\mathbf{s}) = \{\mathbf{k} = (\mathbf{k}_1, \ldots, \mathbf{k}_n) \in \mathbb{Z}^{\mathbf{d}} : \left[2^{s_i-1}\right] \leq$ $\leq \max_{j=1,\ldots,d_i}|k_j^i| < 2^{s_i}, i = 1, \ldots, n\}$.

The Nikol'skii-Besov space $B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})$ with a mixed metric is the set of trigonometric series $f \sim \sum_{\mathbf{k}\in\mathbb{Z}^{\mathbf{d}}} a_{\mathbf{k}} e^{i\langle\mathbf{k},\mathbf{x}\rangle}$, for which the norm

$$\|f\|_{B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})} = \left\|\left\{\mathbf{2}^{\langle\alpha,\mathbf{s}\rangle}\|\Delta_{\mathbf{s}}(f)\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})}\right\}_{\mathbf{s}\in\mathbb{Z}_+^{\mathbf{n}}}\right\|_{l_{\mathbf{q}}}$$

is finite, where $\|\cdot\|_{l_{\mathbf{q}}}$ is the norm of a discrete Lebesgue space $l_{\mathbf{q}}$ with a mixed metric.

Definition 1. Let A and B be Banach spaces. An operator $R \in L(A, B)$ is called a retraction if there exists an operator $S \in L(B, A)$ such

$$RS = E \quad (\text{identity operator from } L(B, B)).$$

Moreover, the operator S is called the coretraction (corresponding to R).

Theorem 1. Let $-\infty < \alpha = (\alpha_1, \ldots, \alpha_n) < \infty$, $\mathbf{1} < \mathbf{p} = (p_1, \ldots, p_n) < \infty$ and $\mathbf{1} \leq \mathbf{q} =$ $= (q_1, \ldots, q_n) \leq \infty$. Then the space $B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})$ is a retraction of the space $l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}}))$.

Proof. We prove first the S-property. For the function $f \in B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})$ we define the operator S as follows

$$Sf = \{\Delta_{\mathbf{s}}(f, \mathbf{x})\}_{\mathbf{s}\in\mathbb{Z}_+^n} = \{(\Delta_{\mathbf{s}} * f)(\mathbf{x})\}_{\mathbf{s}\in\mathbb{Z}_+^n},$$

here $\Delta_{\mathbf{s}}(\mathbf{x})$ is the Dirichlet kernel corresponding to the block $\rho(\mathbf{s})$.

Then, by definition, we have

$$\|Sf\|_{l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}}))} = \|\{\Delta_{\mathbf{s}}(f, \cdot)\}\|_{l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}}))} = \|f\|_{B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})},$$

which means the fulfillment of the S-property.

Next, we check the fulfillment of the R-property. For the sequence $f = \{f_{\mathbf{s}}(\mathbf{x})\}_{\mathbf{s}\in\mathbb{Z}_+^n}$ we define the operator

$$Rf = \sum_{\mathbf{s}\in\mathbb{Z}_+^n} (\Delta_{\mathbf{s}} * f_{\mathbf{s}})(\mathbf{x}).$$

Then, according to the inequality of M. Riesz about the boundedness of parallelepipedal partial sums, we obtain

$$\|\Delta_{\mathbf{m}} * f\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})} \le C \|f\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})},$$

where C is the absolute constant, and further

$$\|Rf\|_{B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})} = \|\{(\Delta_{\mathbf{s}} * f_{\mathbf{s}}) * \Delta_{\mathbf{s}}\}\|_{l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}}))} =$$

$$= \|\{\Delta_{\mathbf{s}} * f_{\mathbf{s}}\}\|_{l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}}))} \le C \|\{f_{\mathbf{s}}\}\|_{l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}}))} = C \|f\|_{l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}}))}.$$

The last inequality means the fulfillment of the R-property.

It remains to show that $RS = E$. Indeed,

$$RSf(\mathbf{x}) = R(\{\Delta_{\mathbf{s}}(f,\mathbf{x})\}) = \sum_{\mathbf{s}\in\mathbb{Z}_+^n} (\Delta_{\mathbf{s}}(f,\mathbf{x}) * \Delta_{\mathbf{s}}(\mathbf{x})) =$$

$$= \sum_{\mathbf{s}\in\mathbb{Z}_+^n} ((S_{2^{\mathbf{s}}} - S_{2^{\mathbf{s}-\mathbf{1}}}) * f)(\mathbf{x}) = f(\mathbf{x}).$$

The theorem is completely proved. □

Theorem 2. Let $\mathbf{1} < \mathbf{p} = (p_1, \ldots, p_n) < \infty$, $\alpha_{\mathbf{0}} = (\alpha_1^0, \ldots, \alpha_n^0) \neq \alpha_{\mathbf{1}} = (\alpha_1^1, \ldots, \alpha_n^1)$, $\mathbf{1} \le \mathbf{q_0} = (q_1^0, \ldots, q_n^0), \mathbf{q_1} = (q_1^1, \ldots, q_n^1) \le \infty$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in E$. Then for $\mathbf{0} < \theta = (\theta_1, \ldots, \theta_n) < \mathbf{1}$ and $\mathbf{1} \le \mathbf{q} = (q_1, \ldots, q_n) \le \infty$ the equality

$$\left(B_{\mathbf{p}}^{\alpha_\varepsilon \mathbf{q}_\varepsilon}(\mathbb{T}^{\mathbf{d}}); \varepsilon \in E\right)_{\theta\mathbf{q}} = B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})$$

holds, where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$.

Proof. follows from Theorem 1 and the Lemma 3. □

Remark 1. In the case $\mathbf{d} = (1, \cdots, 1)$, the result of Theorem 2 was announced by E.D. Nursultanov in [9].

Theorem 3. Let $\mathbf{1} < \mathbf{p_0} = (p_1^0, \ldots, p_n^0), \mathbf{p_1} = (p_1^1, \ldots, p_n^1) < +\infty$, $\alpha_{\mathbf{0}} = (\alpha_1^0, \ldots, \alpha_n^0) \neq \alpha_{\mathbf{1}} = (\alpha_1^1, \ldots, \alpha_n^1)$, $\mathbf{1} \le \mathbf{q_0} = (q_1^0, \ldots, q_n^0), \mathbf{q_1} = (q_1^1, \ldots, q_n^1) \le \infty$. Then for $0 < \theta < 1$ the equality

$$\left(B_{\mathbf{p_0}}^{\alpha_{\mathbf{0}}\mathbf{q_0}}(\mathbb{T}^{\mathbf{d}}); B_{\mathbf{p_1}}^{\alpha_{\mathbf{1}}\mathbf{q_1}}(\mathbb{T}^{\mathbf{d}})\right)_{[\theta]} = B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})$$

holds, where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\mathbf{1}/\mathbf{p} = (\mathbf{1}-\theta)/\mathbf{p_0} + \theta/\mathbf{p_1}$ and $\mathbf{1}/\mathbf{q} = (\mathbf{1}-\theta)/\mathbf{q_0} + \theta/\mathbf{q_1}$. Here $(\cdot,\cdot)_{[\theta]}$ is a complex interpolation functor (see [3]).

Proof. follows from Theorem 1, Theorems 5.1.2 and 5.6.3 from [3]. □

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Аралас метрикалы Никольский-Бесов типтес кеңістіктер үшін интерполяциялық теорема

Мақалада аралас туындылы және аралас метрикалы Никольский-Бесов кеңістігінің интерполяциялық қасиеттері анизотропты және кешенді интерполяция әдістері бойынша зерттелді. $l_{\mathbf{q}}^{\alpha}(A)$ векторлы мәнді салмақтық дискретті кеңістік үшін интерполяциялық теорема дәлелденді. Зерттелген Никольский-Бесов кеңістіктері $l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}})$ кеңістігінің ретракты болатындығы көрсетілді. Жоғарыда келтірілген нәтижелерге сүйене отырып, үстемдік ететін аралас туындылы және аралас метрикалы Никольский-Бесов кеңістіктері үшін интерполяциялық теоремалар алынды.

Кілт сөздер: Никольский-Бесов типтес кеңістіктер, анизотропты интерполяция әдісі, кешенді интерполяция.

К.А. Бекмаганбетов, К.Е. Кервенев, Е. Толеугазы

Интерполяционная теорема для пространств типа Никольского-Бесова со смешанной метрикой

В статье изучены интерполяционные свойства пространств Никольского-Бесова с доминирующей смешанной производной и смешанной метрикой относительно анизотропного и комплексного методов интерполяции. Доказана интерполяционная теорема для весового дискретного пространства векторнозначных последовательностей $l_{\mathbf{q}}^{\alpha}(A)$. Показано, что изучаемые пространства Никольского-Бесова являются ретрактом пространства $l_{\mathbf{q}}^{\alpha}(L_{\mathbf{p}})$. На основании перечисленных выше результатов получены интерполяционные теоремы для пространств Никольского-Бесова с доминирующей смешанной производной и смешанной метрикой.

Ключевые слова: пространства типа Никольского-Бесова, метод анизотропной интерполяции, комплексная интерполяция.

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