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## Numerical solution to elliptic inverse problem with Neumann-type integral condition and overdetermination

In modeling various real processes, an important role is played by methods of solution source identification problem for partial differential equation. The current paper is devoted to approximate of elliptic over determined problem with integral condition for derivatives. In the beginning, inverse problem is reduced to some auxiliary nonlocal boundary value problem with integral boundary condition for derivatives. The parameter of equation is defined after solving that auxiliary nonlocal problem. The second order of accuracy difference scheme for approximately solving abstract elliptic overdetermined problem is proposed. By using operator approach existence of solution difference problem is proved. For solution of constructed difference scheme stability and coercive stability estimates are established. Later, obtained abstract results are applied to get stability estimates for solution Neumann-type overdetermined elliptic multidimensional difference problems with integral conditions. Finally, by using MATLAB program, we present numerical results for two dimensional and three dimensional test examples with short explanation on realization on computer.

*Keywords:* difference scheme, inverse elliptic problem, overdetermination, source identification problem, stability, coercive stability, estimate.

*Introduction*

Methods of solutions and theory nonlocal boundary value problems (BVPs) for differential equations have been studied by numerous authors (see [1–5, 7–12, 14–16, 18, 19] and references herein).

Let us $I$ is identity operator and $A$ is a selfadjoint and positive definite operator (SAPDO) in an arbitrary Hilbert space $H$. It is known that $A > \delta I$ for some positive number $\delta$, and the operator $C = \frac{\tau}{2}A + \sqrt{A + \frac{\tau^2 A^2}{4}})$ is also SAPDO.

Assume that given function $f \in C^1([0,T], H)$, elements $\phi, \eta, \zeta \in H$, number $\lambda_0 \in [0,1]$. Denote by $[0,1]_\tau = \{t_i = i\tau,\ i = 1, \cdots, N,\ \tau N = T\}$ the uniform grid space with step size $\tau > 0$, where $N$ is a fixed integer number. Let $\beta$ be known scalar continuous function satisfying condition

$$\sum_{j=1}^{N} \left|\beta\left(t_{j-\frac{1}{2}}\right)\right| \tau < 1. \tag{1}$$

In the study [10] established well-possedness of elliptic inverse problem with Neumann-type overdetermination and integral condition for obtaining a function $u \in C^2([0,T], H) \cap C([0,T], D(A))$ and an element $p \in H$ such that

$$\begin{cases} -u''(t) + Au(t) = f(t) + p, \quad t \in (0,T), \\ u'(0) = \phi, \ u'(T) = \int\limits_0^T \beta(\lambda) u'(\lambda) d\lambda + \eta, \ u(\lambda_0) = \zeta. \end{cases} \tag{2}$$

Moreover, in [10], the stability inequalities for solution of inverse problem (2) were applied to investigate the following source identificating problem (SIP) for multi dimensional elliptic partial differential equation

$$\begin{cases} -u_{tt}(t,x) - \sum\limits_{r=1}^{n} (a_r(x) u_{x_r}(t,x))_{x_r} + \sigma u(t,x) = f(t,x) + p(x), \ (t,x) \in (0,T) \times \Omega, \\ u_t(0,x) = \phi(x), \ u_t(T,x) = \int\limits_0^T \beta(\gamma) u_\gamma(\gamma,x) d\gamma + \eta(x), \ u(\lambda_0, x) = \zeta(x), x \in \overline{\Omega}, \\ u(t,x) = 0, \quad (t,x) \in [0,T] \times S. \end{cases} \tag{3}$$

Here $\Omega = (0,T)^n$ is open cube in $\mathbb{R}^n$ with boundary $S$, $\overline{\Omega} = \Omega \cup S$; $a_r, \zeta, \phi, \eta$ , $f$ are given sufficiently smooth functions; $\forall x \in \Omega$, $a_r(x) \geq a_0 > 0$; $\sigma > 0, 0 < \lambda_0 < T$ are known numbers.

We denote by $R, P$, and $D$, the corresponding operators $R = (I + \tau C)^{-1}$, $P = (I - R^{2N})^{-1}$, $D = (I + \tau C)(2I + \tau C)^{-1} C^{-1}$.

Now, let us to give some lemmas that will be used in further.

*Lemma 1.* [8] The following estimates hold:

$$\| R^k \|_{H \to H} \leq M(\delta) (1 + \delta^{\frac{1}{2}} \tau)^{-k}, \ \| CR^k \|_{H \to H} \leq \frac{1}{k\tau} M(\delta), k \geq 1, \| P \|_{H \to H} \leq M(\delta), \ \delta > 0. \tag{4}$$

*Lemma 2.*

Suppose that inequality (1) is satisfied, then the operator

$$\begin{aligned} G_2 = & \left[-3(I - R^{2N}) + 4\left(R - R^{2N-1}\right) - \left(R^2 - R^{2N-2}\right)\right] \Big[ \left(3 - \tau\beta\left(t_{N-\frac{3}{2}}\right)\right)(I - R^{2N}) \\ & + \left(-4 - \tau\beta\left(t_{N-\frac{5}{2}}\right)\right)\left(R - R^{2N-1}\right) + \left(1 - \tau\beta\left(t_{N-\frac{7}{2}}\right) + \tau\beta\left(t_{N-\frac{3}{2}}\right)\right)\left(R^2 - R^{2N-2}\right) \\ & + \tau\beta\left(t_{\frac{3}{2}}\right)\left(R^{N-1} - R^{N+1}\right) + \sum_{i=2}^{N-3} \tau \left[\beta\left(t_{i+\frac{1}{2}}\right) - \beta\left(t_{i-\frac{3}{2}}\right)\right]\left(R^{N-i} - R^{N+i}\right) \Big] \\ & - \left[R^{N-1} - R^{N+1} - R^{N-2} + R^{N+2}\right] \Big[ -\left(4 + \tau\beta\left(t_{N-\frac{5}{2}}\right)\right)\left(R^{N-1} - R^{N+1}\right) \\ & + \left(1 - \tau\beta\left(t_{N-\frac{7}{2}}\right) + \tau\beta\left(t_{N-\frac{3}{2}}\right)\right)\left(R^{N-2} - R^{N+2}\right) + \sum_{i=2}^{N-3} \tau \left[\beta\left(t_{i+\frac{1}{2}}\right) - \beta\left(t_{i-\frac{3}{2}}\right)\right]\left(R^{i} - R^{2N-i}\right) \\ & + \tau\beta\left(t_{\frac{3}{2}}\right)\left(R - R^{2N-1}\right) + \tau\beta\left(t_{\frac{1}{2}}\right)\left(I - R^{2N}\right) \Big] \end{aligned} \tag{5}$$

has an inverse $G_2^{-1}$ and its norm is bounded, i.e.

$$\| G_2^{-1} \|_{H \to H} \leq M(\delta). \tag{6}$$

In the paper [8] , for given $v_0$ and $v_N$,the solution of difference scheme

$$-\tau^{-2}(v_{i+1} - 2v_i + v_{i-1}) + Av_i = f_i, \quad 1 \leq i \leq N-1 \tag{7}$$

was represented by formula

$$\begin{aligned} v_i = & P\left[\left(R^i - R^{2N-i}\right) v_0 + \left(R^{N-i} - R^{N+i}\right) v_N\right] - P\left(R^{N-i} - R^{N+i}\right) D \\ & \times \sum_{j=1}^{N-1} \left(R^{N-j} - R^{N+j}\right) f_j \tau + D \sum_{j=1}^{N-1} \left(R^{|i-j|} - R^{i+j}\right) f_j \tau, \ 1 \leq i \leq N-1. \end{aligned} \tag{8}$$

Let $\alpha \in (0,1)$ is a given number. Introduce notations for $C_\tau(H), C_\tau^\alpha(H)$, and $C_\tau^{\alpha,\alpha}(H)$, the Banach spaces of $H$-valued grid functions $w_\tau = \{w_k\}_{k=1}^{N-1}$ with the corresponding norms,

$$\|w_\tau\|_{C_\tau(H)} = \max_{1\le k\le N-1} \|w_k\|_H\,,\ \|w_\tau\|_{C_\tau^\alpha(H)} = \sup_{1\le k<k+n\le N-1} (n\tau)^{-\alpha}\,\|w_{k+n}-w_k\|_H + \|w_\tau\|_{C_\tau(H)}\,,$$
$$\|w_\tau\|_{C_\tau^{\alpha,\alpha}(H)} = \|w_\tau\|_{C_\tau(H)} + \sup_{1\le k<k+n\le N-1} (1-k\tau)^\alpha\,(n\tau)^{-\alpha}\,(k\tau+n\tau)^\alpha \|w_{k+n}-w_k\|_H.$$

In the current study, we construct the second order accuracy difference scheme (ADS) for approximately solution of inverse problem (2) and study well-posedness of difference problem. Then, we discuss the second order ADS for SIP (3).

*The second order of ADS for SIP* (3)

Now, we study second order of ADS

$$\begin{cases} -\tau^{-2}(u_{k+1}-2u_k+u_{k-1}) + Au_k = f_k + p,\ \ f_k = f(t_k), 1\le k\le N-1, \\ -3u_0+4u_1-u_2 = 2\tau\phi, 3u_N - 4u_{N-1} + u_{N-2} = \sum\limits_{i=1}^{N-3} \tau\beta\left(t_{i-\frac{1}{2}}\right)(u_{i+1}-u_{i-1}) + 2\tau\eta, \\ u_l + \mu(u_{l+1}-u_l) = \zeta\ \ \left(\mu = \frac{\lambda_0}{\tau} - l\right) \end{cases} \tag{9}$$

for approximate solution inverse problem (2).

*Theorem 1.* Let us $\phi,\eta,\zeta \in D(A)$, and $f_\tau \in C_\tau(H)$ and inequality (1) is satisfied. Then, solution $\left(\{u_k\}_{k=1}^{N-1}, p\right)$ of difference problem (9) exists in $C_\tau(H)\times H$ and the next stability estimates for solution

$$\left\|\{u_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)} \le M(\delta)\left(\|\phi\|_H + \|\zeta\|_H + \|\eta\|_H + \|f_\tau\|_{C_\tau(H)}\right), \tag{10}$$
$$\left\|A^{-1}p\right\|_H \le M(\delta)\left(\|\phi\|_H + \|\zeta\|_H + \|\eta\|_H + \|f_\tau\|_{C_\tau(H)}\right) \tag{11}$$

are fulfilled.

*Proof.* Firstly, by using

$$u_k = v_k + A^{-1}p, \tag{12}$$

we get auxiliary difference problem for unknowns $\{v_k\}_{k=0}^N$ :

$$\begin{cases} -\tau^{-2}(v_{k+1}-2v_k+v_{k-1}) + Av_k = f_k\,, \quad 1\le k\le N-1, \\ -3v_0+4v_1-v_2 = 2\tau\phi, \left(3-\tau\beta\left(t_{N-\frac{3}{2}}\right)\right)v_N + \left(-4-\tau\beta\left(t_{N-\frac{5}{2}}\right)\right)v_{N-1} \\ +\left(1-\tau\beta\left(t_{N-\frac{7}{2}}\right)+\tau\beta\left(t_{N-\frac{3}{2}}\right)\right)v_{N-2} + \sum\limits_{i=2}^{N-3}\tau\left[\beta\left(t_{i+\frac{1}{2}}\right)-\beta\left(t_{i-\frac{3}{2}}\right)\right]v_i \\ +\tau\beta\left(t_{\frac{3}{2}}\right)v_1 + \tau\beta\left(t_{\frac{1}{2}}\right)v_0 = 2\tau\eta. \end{cases} \tag{13}$$

We seek solution of (13) by (8). By using (8), from first condition of difference problem (13), we get equation

$$\begin{array}{l} \left[-3(I-R^{2N}) + 4\left(R-R^{2N-1}\right) - \left(R^2-R^{2N-2}\right)\right]v_0 \\ +\left[4\left(R^{N-1}-R^{N+1}\right) - \left(R^{N-2}-R^{N+2}\right)\right]v_N = F_1, \end{array} \tag{14}$$

for unknowns $v_0$ and $v_N$, where

$$F_1 = 2\tau(I-R^{2N})\phi + 4\left(R^{N-1}-R^{N+1}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau - 4(I-R^{2N})D$$
$$\times\sum_{j=1}^{N-1}\left(R^{|1-j|}-R^{1+j}\right)f_j\tau - \left(R^{N-2}-R^{N+2}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau$$
$$+(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|2-j|}-R^{2+j}\right)f_j\tau.$$

From integral condition follows the next equation

$$\begin{aligned}&\left(3-\tau\beta\left(t_{N-\frac{3}{2}}\right)\right)(I-R^{2N})v_N+\left(-4-\tau\beta\left(t_{N-\frac{5}{2}}\right)\right)\left[\left(R^{N-1}-R^{N+1}\right)v_0+\left(R-R^{2N-1}\right)v_N\right]\\&+\left(1-\tau\beta\left(t_{N-\frac{7}{2}}\right)+\tau\beta\left(t_{N-\frac{3}{2}}\right)\right)\left[\left(R^{N-2}-R^{N+2}\right)v_0+\left(R^2-R^{2N-2}\right)v_N\right]\\&+\sum_{i=2}^{N-3}\tau\left[\beta\left(t_{i+\frac{1}{2}}\right)-\beta\left(t_{i-\frac{3}{2}}\right)\right]\left[\left(R^i-R^{2N-i}\right)v_0+\left(R^{N-i}-R^{N+i}\right)v_N\right]\\&+\tau\beta\left(t_{\frac{3}{2}}\right)\left[\left(R-R^{2N-1}\right)v_0+\left(R^{N-1}-R^{N+1}\right)v_N\right]+\tau\beta\left(t_{\frac{1}{2}}\right)(I-R^{2N})v_0=F_2\end{aligned}\tag{15}$$

for unknowns $v_0$ and $v_N$, where

$$\begin{aligned}F_2=&\left(-4-\tau\beta\left(t_{N-\frac{5}{2}}\right)\right)\left[\left(R-R^{2N-1}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau-(I-R^{2N})D\right.\\&\left.\times\sum_{j=1}^{N-1}\left(R^{|N-1-j|}-R^{N-1+j}\right)f_j\tau\right]+\left(1-\tau\beta\left(t_{N-\frac{7}{2}}\right)+\tau\beta\left(t_{N-\frac{3}{2}}\right)\right)\\&\times\left[\left(R^2-R^{2N-2}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|N-2-j|}-R^{N-2+j}\right)f_j\tau\right]\\&-\sum_{i=2}^{N-3}\tau\left[\beta\left(t_{i+\frac{1}{2}}\right)-\beta\left(t_{i-\frac{3}{2}}\right)\right]\left[\left(R^{N-i}-R^{N+i}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau\right.\\&\left.-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|i-j|}-R^{i+j}\right)f_j\tau\right]-\tau\beta\left(t_{\frac{3}{2}}\right)\left[\left(R^{N-1}-R^{N+1}\right)D\right.\\&\left.\times\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|1-j|}-R^{1+j}\right)f_j\tau+2\tau(I-R^{2N})\eta\right].\end{aligned}$$

Thus, determinant operator $G_2$ of linear system equatıon (14), (15) has bounded inverse $G_2^{-1}$. Therefore solution of linear system equatıon (14), (15) is defined by

$$\begin{aligned}v_0=&G_2^{-1}\left\{\left[\left(3-\tau\beta\left(t_{N-\frac{3}{2}}\right)\right)(I-R^{2N})+\left(-4-\tau\beta\left(t_{N-2}-\frac{\tau}{2}\right)\right)\left(R-R^{2N-1}\right)\right.\right.\\&+\left(1-\tau\beta\left(t_{N-\frac{7}{2}}\right)+\tau\beta\left(t_{N-\frac{3}{2}}\right)\right)\left(R^2-R^{2N-2}\right)\\&\left.+\sum_{i=2}^{N-3}\tau\left[\beta\left(t_{i+\frac{1}{2}}\right)-\beta\left(t_{i-\frac{3}{2}}\right)\right]\left(R^{N-i}-R^{N+i}\right)+\tau\beta\left(t_{\frac{3}{2}}\right)\left(R^{N-1}-R^{N+1}\right)\right]\\&\times\left[2\tau(I-R^{2N})\phi+4\left(R^{N-1}-R^{N+1}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau-4(I-R^{2N})D\right.\\&\times\sum_{j=1}^{N-1}\left(R^{|1-j|}-R^{1+j}\right)f_j\tau-\left(R^{N-2}-R^{N+2}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau\\&\left.+(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|2-j|}-R^{2+j}\right)f_j\tau\right]-\left(R^{N-1}-R^{N+1}-R^{N-2}+R^{N+2}\right)\\&\times\left\{2\tau(I-R^{2N})\eta+\left(-4-\tau\beta\left(t_{N-\frac{5}{2}}\right)\right)\left[\left(R-R^{2N-1}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau\right.\right.\\&\left.\times-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|N-1-j|}-R^{N-1+j}\right)f_j\tau\right]+\left(1-\tau\beta\left(t_{N-\frac{7}{2}}\right)+\tau\beta\left(t_{N-\frac{3}{2}}\right)\right)\\&\times\left[\left(R^2-R^{2N-2}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|N-2-j|}-R^{N-2+j}\right)f_j\tau\right]\end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=2}^{N-3}\tau\left[\alpha\left(t_{i+1}-\tfrac{\tau}{2}\right)-\alpha\left(t_{i-1}-\tfrac{\tau}{2}\right)\right]\Bigg[\left(R^{N-i}-R^{N+i}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau\\
&-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|i-j|}-R^{i+j}\right)f_j\tau\Bigg]-\tau\alpha\left(t_2-\tfrac{\tau}{2}\right)\left[\left(R^{N-1}-R^{N+1}\right)D\right.\\
&\left.\times\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|1-j|}-R^{1+j}\right)f_j\tau\right]\Bigg]\Bigg\}\Bigg\},
\end{aligned}
\tag{16}
$$

and

$$
\begin{aligned}
&v_N=G_2^{-1}\left\{\left[-3(I-R^{2N})+4\left(R-R^{2N-1}\right)-\left(R^2-R^{2N-2}\right)\right]2\tau(I-R^{2N})\eta\right.\\
&+\left(-4-\tau\beta\left(t_{N-2}-\tfrac{\tau}{2}\right)\right)\Bigg[\left(R-R^{2N-1}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau\\
&-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|N-1-j|}-R^{N-1+j}\right)f_j\tau\Bigg]+\left(1-\tau\beta\left(t_{N-3}-\tfrac{\tau}{2}\right)+\tau\beta\left(t_{N-1}-\tfrac{\tau}{2}\right)\right)\\
&\times\Bigg[\left(R^2-R^{2N-2}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|N-2-j|}-R^{N-2+j}\right)f_j\tau\Bigg]\\
&-\sum_{i=2}^{N-3}\tau\left[\beta\left(t_{i+\frac{1}{2}}\right)-\beta\left(t_{i-\frac{3}{2}}\right)\right]\\
&\Bigg[\left(R^{N-i}-R^{N+i}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|i-j|}-R^{i+j}\right)f_j\tau\Bigg]\\
&-\tau\beta\left(t_{\frac{3}{2}}\right)\Bigg[\left(R^{N-1}-R^{N+1}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|1-j|}-R^{1+j}\right)f_j\tau\Bigg]\\
&-\Bigg[-\left(4+\tau\beta\left(t_{N-\frac{5}{2}}\right)\right)\left(R^{N-1}-R^{N+1}\right)\\
&+\left(1-\tau\beta\left(t_{N-\frac{7}{2}}\right)+\tau\beta\left(t_{N-\frac{3}{2}}\right)\right)\left(R^{N-2}-R^{N+2}\right)+\tau\beta\left(t_{\frac{1}{2}}\right)(I-R^{2N})\\
&+\sum_{i=2}^{N-3}\tau\left[\beta\left(t_{i+\frac{1}{2}}\right)-\beta\left(t_{i-\frac{3}{2}}\right)\right]\left(R^i-R^{2N-i}\right)+\tau\beta\left(t_{\frac{3}{2}}\right)\left(R-R^{2N-1}\right)\Bigg]\\
&\times\Bigg[2\tau(I-R^{2N})\phi+4\left(R^{N-1}-R^{N+1}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau\\
&-4(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|1-j|}-R^{1+j}\right)f_j\tau\\
&\left.-\left(R^{N-2}-R^{N+2}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau+(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|2-j|}-R^{2+j}\right)f_j\tau\Bigg]\right\}.
\end{aligned}
\tag{17}
$$

Thus solution of difference problem (13) exists and it is defined by (8) with the corresponding $v_0$ and $v_N$ via (16) and (17). From (8), (16),(17), estimates (4), (6), it follows that for solution of difference problem (13) stability estimates

$$
\left\|\{v_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\le M(\delta)\left(\|\phi\|_H+\|\zeta\|_H+\|\eta\|_H+\|f_\tau\|_{C_\tau(H)}\right),
\tag{18}
$$

$$
\begin{aligned}
&\left\|\{Av_k\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)}+\left\|\left\{\frac{v_{k+1}-2v_k+v_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)}\\
&\le M(\delta)\left(\frac{1}{\alpha(1-\alpha)}\|f_\tau\|_{C_\tau^{\alpha,\alpha}(H)}+\|A\zeta\|_H+\|A\phi\|_H+\|A\eta\|_H\right).
\end{aligned}
\tag{19}
$$

are fulfilled. (12) and estimates (18) permit us to get estimates estimates (11) (10) and (19).

*Theorem 2.* Let us $f_\tau \in C_\tau^{\alpha,\alpha}(H)$, and $\phi,\zeta,\eta \in D(A)$ and inequality (1) is satisfied. Then, for solution $\left(\{u_k\}_{k=1}^{N-1}, p\right)$ of difference problem (9) the coercive stability inequality

$$\begin{array}{l}\left\|\left\{\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)} + \left\|\{Au_k\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)} + \|p\|_H \\ \leq M(\delta)\left(\frac{1}{\alpha(1-\alpha)}\|f_\tau\|_{C_\tau^{\alpha,\alpha}(H)} + \|A\zeta\|_H + \|A\phi\|_H + \|A\eta\|_H\right)\end{array} \quad (20)$$

is valid.

The proof of inequality (20) is based on formulas (8), (12), (16), (17), and (19).

*Approximation of (3)*

Denote by

$$\widetilde{\Omega}_h = \left\{x = (h_1 m_1, ..., h_n m_n); m = (m_1, ..., m_n), m_i = \overline{0, M_i},\ h_i M_i = 1, i = \overline{1, n}\ \right\},$$

$$\Omega_h = \widetilde{\Omega}_h \cap \Omega, S_h = \widetilde{\Omega}_h \cap S$$

and by $A_h^x$ difference operator

$$A_h^x u^h(x) = -\sum_{i=1}^n \left(a_i(x) u_{\bar{x}_i}^h(x)\right)_{x_i, j_i} + \sigma u^h(x)$$

acting in the space of grid functions $u^h(x)$, satisfying boundary condition $u^h(x) = 0$ for all $x \in S_h$.

In the beginning, by using approximation in variable $x$ and later by approximation in variable $t$, one can get the following difference scheme for approximately solution of SIP (3):

$$\begin{array}{l} -\tau^{-2}\left(u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)\right) + Au_k^h(x) = f_k^h(x) + p^h(x), \quad 1 \leq k \leq N-1, x \in \Omega_h \\ -3u_0^h(x) + 4u_1^h(x) - u_2^h(x) = \tau\phi^h(x), u_l^h(x) + \mu(u_{l+1}^h(x) - u_l^h(x)) = \zeta^h(x) \\ 3u_N^h(x) - 4u_{N-1}^h(x) + u_{N-2}^h(x) = \sum\limits_{i=1}^{N-1} \tau\alpha\left(t_i - \frac{\tau}{2}\right)\left(u_{i+1}^h(x) - u_i^h(x)\right) + 2\tau\eta^h(x), x \in \widetilde{\Omega}_h. \\ . \end{array} \quad (21)$$

Let $L_{2h} = L_2(\widetilde{\Omega}_h)$ and $W_{2h}^2 = W_2^2(\widetilde{\Omega}_h)$, the Banach spaces of the grid functions $u^h(x) = \{u(h_1 m_1, \cdots, h_n m_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the corresponding norms

$$\begin{array}{l}\left\|u^h\right\|_{L_{2h}} = (\sum_{x\in\widetilde{\Omega}_h} |u^h(x)|^2 h_1 \cdots h_n)^{1/2}, \\ \left\|u^h\right\|_{W_{2h}^2} = \left\|u^h\right\|_{L_{2h}} + +(\sum_{x\in\widetilde{\Omega}_h}\sum_{i=1}^n \left|(u^h(x))_{x_i\overline{x_i},\, m_i}\right|^2 h_1\cdots h_n)^{1/2}.\end{array}$$

*Theorem 3.* Assume that (1) is valid, $f_\tau \in C_\tau^{\alpha,\alpha}(L_{2h})$, and $\phi^h, \eta^h, \zeta^h \in D(A_h^x) \cap L_{2h}$. Then, the solution of difference problem (21) exists and for solution the stability estimates hold:

$$\begin{array}{l}\left\|\{u_k^h\}_1^{N-1}\right\|_{C_\tau(L_{2h})} \leq M(\delta)\,[\left(\left\|\phi^h\right\|_{L_{2h}} + \left\|\eta^h\right\|_{L_{2h}} + \left\|\zeta^h\right\|_{L_{2h}} + \|f_\tau\|_{C_\tau(L_{2h})}\right), \\ \left\|p^h\right\|_{L_{2h}} \leq M(\delta)\left(\left\|\zeta^h\right\|_{W_{2h}^2} + \left\|\eta^h\right\|_{W_{2h}^2} + \left\|\phi^h\right\|_{W_{2h}^2} + \frac{1}{\alpha(1-\alpha)}\|f_\tau\|_{C_\tau^{\alpha,\alpha}(L_{2h})}\right).\end{array}$$

*Theorem 4.* Assume that (1) is true, $f_\tau \in C_\tau^{\alpha,\alpha}(W_{2h}^2)$, and $\phi^h, \eta^h, \zeta^h \in D(A_h^x) \cap W_{2h}^2$. Then, for the solution of difference problem (21) the coercive stability estimate obeys

$$\begin{array}{l}\left\|\left\{\frac{u_{k+1}^h - 2u_k^h + u_{k-1}^k}{\tau^2})\right\}_1^{N-1}\right\|_{C_\tau(L_{2h})} + \left\|\{u_k^h\}_1^{N-1}\right\|_{C_\tau(W_{2h}^2)} + \left\|p^h\right\|_{L_{2h}} \\ \leq M(\delta)\,[\left(\left\|\zeta^h\right\|_{W_{2h}^2} + \left\|\eta^h\right\|_{W_{2h}^2} + \left\|\phi^h\right\|_{W_{2h}^2} + \frac{1}{\alpha(1-\alpha)}\|f_\tau\|_{C_\tau^{\alpha,\alpha}(W_{2h}^2)}\right).\end{array}$$

The proofs of Theorems 3 and 4 are based on the symmetry property of the operator $A_h^x$ in the Hilbert space $L_{2h}$ and the corresponding theorem in [20] on the coercivity stability inequality for the solution of the elliptic difference problem in $L_{2h}$ with first kind boundary condition.

*Test examples*

In the present section, we illustrate computed results for twodimensional and threedimentional examples of inverse elliptic problem with Neumann-type overdetermination and integral condition. All computed results are carried out by using MATLAB.

*2D example*

Notice that pair functions $(p(x), u(t,x)) = \left(\left(\pi^2+1\right) \sin(\pi x), \left(e^{-t} + t + 1\right)\sin(\pi x)\right)$ is exact solution of the following 2D overdetermined elliptic problem with integral boundary condition:

$$\begin{cases} -u_{tt}(t,x) - u_{xx}(t,x) + u(t,x) = f(t,x) + p(x), \quad t, x \in (0,1), \\ u_t(0,x) = 0, \ u(0.3,x) = \zeta(x), u_t(1,x) = \int\limits_0^1 e^{-\lambda} u_\lambda(\lambda) d\lambda + \eta(x), \ x \in [0,1], \\ u(t,0) = 0, u(t,1) = 0, \ t \in [0,1], \end{cases} \tag{22}$$

where

$$f(t,x) = \left[-e^{-t} + \left(\pi^2+1\right)\left(e^{-t}+t\right)\right]\sin(\pi x), \zeta(x) = \left(e^{-0.3} + 1.3\right)\sin(\pi x)$$
$$\eta(x) = \left[\tfrac{1}{2} - \tfrac{1}{2}e^{-2}\right] \sin(\pi x).$$

The notation $[0,1]_\tau \times [0,1]_h$ means the set of grid points

$$[0,1]_\tau \times [0,1]_h = \{(t_i, x_n) : t_i = i\tau, \ i = \overline{0,N}, \ x_n = nh, \ n = \overline{0,M}\},$$

which depends on the small parameters $\tau$ and $h$ such that $N\tau = 1, Mh = 1$. Let us

$$l_0 = \left[0.3\tau^{-1}\right], \mu_0 = 0.3\tau^{-1} - l_0; \phi_n = 0, \eta_n = \eta(x_n), \zeta_n = \zeta(x_n), \ n = \overline{0,M};$$
$$f_n^k = f(t_k, x_n), k = \overline{0,N}, \ n = \overline{0,M}.$$

To approximately soving (22), we use algorithm which contains three stages. Firstly, we find approximately solution of auxiliary NBVP

$$\begin{cases} \tau^{-2}\left(v_n^{k+1} - 2v_n^k + v_n^{k-1}\right) + h^{-2}\left(v_{n+1}^k - 2v_n^k + v_{n-1}^k\right) - v_n^k = -f(t_k, x_n), \\ k = \overline{1, N-1}, \ n = \overline{1, M-1}, \\ v_0^k = v_M^k = 0, \ k = \overline{0,N}, \ -3v_n^0 + 4v_n^1 - v_n^2 = 0, \\ 3v_n^N - 4v_n^{N-1} + v_n^{N-2} = \sum\limits_{j=1}^{N-1} \frac{\tau}{2} e^{-\left(t_j - \frac{\tau}{2}\right)} \left(v_n^{j+1} - v_n^{j-1} + v_n^j - v_n^{j-2}\right) + 2\tau\eta_n, \ n = \overline{0,M}. \end{cases} \tag{23}$$

Secondly, we find $p_n$. It is caried out by

$$p_n = -\tfrac{1}{h^2}[(\zeta_{n+1} - (\mu_0 v_{n+1}^{l_0+1} - (\mu_0 - 1)v_{n+1}^{l_0})) - 2(\zeta_n - (\mu_0 v_n^{l_0+1} - (\mu_0-1)v_n^{l_0}))$$
$$+ (\zeta_{n-1} - (\mu_0 v_{n-1}^{l_0+1} - (\mu_0-1)v_{n-1}^{l_0}))] + \zeta_n - (\mu_0 v_n^{l_0+1} - (\mu_0-1)v_n^{l_0}), \quad , n = \overline{1, M-1}.$$

Difference problem (23) can be rewritten in the matrix form

$$\begin{aligned} &Av_{n+1} + Bv_n + Cv_{n-1} = Ig^{(n)}, \ n = \overline{1, M-1}, \\ &v_0 = \overrightarrow{0}, v_M = \overrightarrow{0}. \end{aligned} \tag{24}$$

Here, $A, B, C, I$ are $(N+1)\times(N+1)$ square matrices, and $I$ is identity matrix, $v_s, s=n-1,n,n+1, g^{(n)}$ are column matrices with $(N+1)$ rows, $v_s=\left[\begin{array}{ccc} v_s^0 & \dots & v_s^N \end{array}\right]^t$. Denote by

$$a=\frac{1}{h^2}, c=\frac{1}{h^2},\ q=-\frac{2}{h^2}-\frac{2}{\tau^2}-1, r=\frac{1}{\tau^2}.$$

Then,

$$A_n=diag(0,a,a,\dots,a,0), C_n=A_n, g_k^{(n)}=-f(t_k,x_n),\ k=\overline{1,N-1},\ n=\overline{1,M-1},$$

$$b_{i,i}=q, b_{i-1,i}=r, b_{i,i-1}=r, i=\overline{2,N}, b_{1,1}=-3, b_{1,2}=4, b_{1,3}=-1,$$
$$b_{N+1,N+1}=2\tau\left(\frac{e^{t_{N-\frac{3}{2}}}}{4}+e^{-t_{N-\frac{3}{2}}}\right)-3,\ b_{N+1,N}=2\tau\left(\frac{e^{-t_{N-5}}}{4}+\frac{e^{-t_{N-\frac{3}{2}}}}{4}-e^{-t_{N-\frac{1}{2}}}\right)+4,$$
$$b_{N+1,N-1}=\frac{\tau e^{-t_{N-\frac{7}{2}}}}{2}+\frac{\tau e^{-t_{N-\frac{5}{2}}}}{2}-\frac{\tau e^{-t_{N-\frac{3}{2}}}}{2}-1,$$

$$b_{N+1,1}=2\tau\left(-\frac{e^{-t_{\frac{3}{2}}}}{4}-e^{-t_{\frac{1}{2}}}\right),\ b_{N+1,2}=2\tau\left(-\frac{e^{-t_{\frac{3}{2}}}}{4}-\frac{e^{-t_{\frac{5}{2}}}}{4}+e^{-t_{\frac{1}{2}}}\right),$$

$$b_{N+1,3}=\frac{\tau e^{-t_{\frac{3}{2}}}}{2}-\frac{\tau e^{-t_{\frac{5}{2}}}}{2}-\frac{\tau e^{-t_{\frac{7}{2}}}}{2},$$

$$b_{N+1,j}=\frac{\tau}{2}\left(e^{-t_{j-\frac{3}{2}}}+e^{-t_{j-\frac{1}{2}}}-e^{-t_{j+\frac{1}{2}}}-e^{-t_{j+\frac{3}{2}}}\right)\ ,j=4,...,N-2;$$

$$b_{i,j}=0,\ \text{for other } i \text{ and } j; g_n^0=2\tau\phi_n,\ g_n^N=2\tau\eta_n,\ n=\overline{1,M-1}.$$

To solve (24), we use modified Gauss elimination method.

Thirdly, we define $\left\{u_n^k\right\}$ by $u_n^k=v_n^k+\zeta_n-\left(\mu_0 v_n^{l_0+1}-(\mu_0-1)\,v_n^{l_0}\right)$.

Errors are presented in Tables 1-3 for second order ADS in case N=M=10,20, 40, 80, 160 and 320. It can be seen from Tables 1-3 when N, M are increased two times that errors are decreased with approximately ratio $\frac{1}{4}$.

Table 1

**Test example (22) - error $v$**

| DS \ $(N,M)$ | $(10,10)$ | $(20,20)$ | $(40,40)$ | $(80,80)$ | $(160,160)$ | $(320,320)$ |
|---|---|---|---|---|---|---|
| 2nd order of ADS | $6.29\times10^{-3}$ | $1.57\times10^{-3}$ | $3.93\times10^{-4}$ | $9.84\times10^{-5}$ | $2.46\times10^{-5}$ | $6.15\times10^{-6}$ |

Table 2

**Test example (22) - error $u$**

| DS \ $(N,M)$ | $(10,10)$ | $(20,20)$ | $(40,40)$ | $(80,80)$ | $(160,160)$ | $(320,320)$ |
|---|---|---|---|---|---|---|
| 2nd order of ADS | $3.13\times10^{-4}$ | $7.95\times10^{-5}$ | $2.02\times10^{-5}$ | $5.10\times10^{-6}$ | $1.28\times10^{-6}$ | $3.22\times10^{-7}$ |

Table 3

**Test example (22) - error $p$**

| Appr. \ $(N,M)$ | $(10,10)$ | $(20,20)$ | $(40,40)$ | $(80,80)$ | $(160,160)$ | $(320,320)$ |
|---|---|---|---|---|---|---|
| 2nd order | $5.03\times10^{-3}$ | $1.28\times10^{-3}$ | $3.21\times10^{-4}$ | $8.06\times10^{-5}$ | $2.02\times10^{-5}$ | $5.05\times10^{-6}$ |

*3D example*

Now, consider the three dimensional inverse elliptic problem with integral condition

$$\begin{cases} -u_{tt}(t,x,y) - u_{xx}(t,x,y) - u_{yy}(t,x,y) + u(t,x,y) = f(t,x,y) + p(x,y), x, y, \ t \in (0,1), \\ u(t,0,y) = u(t,1,y) = 0, \ y,t \in [0,1], u(t,x,0) = u(t,x,1) = 0, \ x,t \in [0,1], \\ u_t(0,x,y) = \phi(x,y), u(0.6,x,y) = \zeta(x,y), \\ u_t(1,x,y) - \int\limits_0^1 e^{-\lambda} u_\lambda(\lambda,x,y)d\lambda \ = \eta(x,y), x,y \in [0,1], \end{cases} \tag{25}$$

where

$$f(t,x,y) = 2\pi^2 e^{-t} q(x,y), \phi(x,y) = -q(x,y), \eta(x,y) = \left[-e^{-1} + \tfrac{1}{3}\left(e^{-0.6} + e^{-1.2}\right)\right] q(x,y),$$
$$\zeta(x,y) = \left(e^{-\frac{3}{5}} + 1\right) q(x,y), q(x,y) = \sin(\pi x)\sin(\pi y)$$

It is clear that pair funcions $p(x,y) = \left(2\pi^2 + 1\right) q(x,y)$ and $u(t,x,y) = \left(e^{-t} + 1\right) q(x,y)$ is exact solution of (25).

Denote by $[0,1]_\tau \times [0,1]_h \times [0,1]_h$ set of grid points depending on the small parameters $\tau$ and $h$

$$[0,1]_\tau \times [0,1]_h^2 \ = \{(t_i, x_n, y_m) : t_i = i\tau, \ i = \overline{0,N}, \ x_n = nh, \ n = \overline{0,M},$$
$$y_m = mh, \ m = \overline{0,M}, \ \tau N = 1, hM = 1\}.$$

Let us

$$l_0 = \left[0.3\tau^{-1}\right], \mu_0 = 0.3\tau^{-1} - l_0 \ , \phi_{m,n} = \phi\left(x_n, y_m\right), \eta_{m,n} = \eta\left(x_n, y_m\right), \zeta_{m,n} = \xi\left(x_n, y_m\right),$$
$$n = \overline{0,M}, \ m = \overline{0,M}; f_{m,n}^i = f(t_i, x_n, y_m), \ i = \overline{0,N}, \ n = \overline{0,M}, m = \overline{0,M}.$$

Firstly, difference scheme for approximate solution of NBVP can be written in the following form:

$$\begin{cases} -\tau^{-2}\left(v_{m,n}^{k+1} - 2v_{m,n}^k + v_{m,n}^{k-1}\right) - h^{-2}\left(v_{m,n+1}^k - 2v_{m,n}^k + v_{m,n-1}^k\right) \\ -h^{-2}\left(v_{m+1,n}^k - 2v_{m,n}^k + v_{m-1,n}^k\right) + v_{m,n}^k = f_{m,n}^k, \\ k = \overline{1,N-1}, \ n = \overline{1,M-1}, \ m = \overline{1,M-1}, \\ v_{0,n}^k = v_{M,n}^k = v_{m,n}^k = v_{m,M}^k = 0, \ k = 0,\cdots,N, \ n = \overline{1,M-1}, \ m = \overline{1,M-1}, \\ -3v_{m,n}^0 + 4v_{m,n}^1 - v_{m,n}^2 = 2\tau\phi_{m,n}, \quad 3v_{m,n}^N - 4v_{m,n}^{N-1} + v_{m,n}^{N-2} \\ = \sum\limits_{j=1}^{N-1} \frac{\tau}{2} e^{-\left(t_j - \frac{\tau}{2}\right)} \left(v_{m,n}^{j+1} - v_{m,n}^{j-1} + v_{m,n}^j - v_{m,n}^{j-2}\right) + 2\tau\eta_{m,n}, \\ n = \overline{1,M-1}, n = \overline{1,M-1}. \end{cases} \tag{26}$$

Secondly, calculation of $p_n$ $\left(n = \overline{1,M-1}, m = \overline{1,M-1}\right)$ is caried out by

$$p_{m,n} = -\tfrac{1}{h^2}\left\{\left[\zeta_{m,n+1} - \left(\mu_0 v_{m,n+1}^{l_0+1} - \left(\mu_0 - 1\right) v_{m,n+1}^{l_0}\right)\right] - 2\left[\zeta_{m,n} - \left(\mu_0 v_{m,n}^{l_0+1} - \left(\mu_0 - 1\right) v_{m,n}^{l_0}\right)\right]\right.$$
$$\left. + \left[\zeta_{m,n-1} - \left(\mu_0 v_{m,n-1}^{l_0+1} - \left(\mu_0 - 1\right) v_{m,n-1}^{l_0}\right)\right]\right\} - \tfrac{1}{h^2}\left\{\left[\zeta_{m+1,n} - \left(\mu_0 v_{m+1,n}^{l_0+1} - \left(\mu_0 - 1\right) v_{m+1,n}^{l_0}\right)\right]\right.$$
$$\left. -2\left[\zeta_{m,n} - \left(\mu_0 v_{m,n}^{l_0+1} - \left(\mu_0 - 1\right) v_{m,n}^{l_0}\right)\right] + \left[\zeta_{m-1,n} - \left(\mu_0 v_{m-1,n}^{l_0+1} - \left(\mu_0 - 1\right) v_{m-1,n}^{l_0}\right)\right]\right\}.$$

Thirdly, we calculate $\left\{u_n^k\right\}$ by

$$u_{m,n}^k = v_{m,n}^k + \zeta_{m,n} - \left(\mu_0 v_{m,n}^{l_0+1} - \left(\mu_0 - 1\right) v_{m,n}^{l_0}\right).$$

Difference problem (26) can be rewritten in the matrix form (24). In this case, $g_n$ is a column matrix with $(N+1)(M+1)$ elements, $A$, $B$, $C, I$ are square matrices with $(N+1)(M+1)$ rows and columns, and $I$ is the identity matrix, $v_s$ is column matrix with $(N+1)(M+1)$ elements such that

$$v_s = \left[\begin{array}{ccccccccc} v_{0,s}^0 & \cdots & v_{0,s}^N & v_{1,s}^0 & \cdots & v_{1,s}^N & \cdots & v_{M,s}^0 & \cdots & v_{M,s}^N \end{array}\right]^t, s = n-1, n, n+1.$$

Denote by

$$a = \frac{1}{h^2}, q = 1 + \frac{2}{\tau^2} + \frac{4}{h^2}, r = \frac{1}{\tau^2}.$$

Then,

$$A = C = \begin{bmatrix} O & O & \cdots & O & O \\ O & E & \cdots & O & O \\ \dots & \dots & \ddots & \dots & \dots \\ O & O & \cdots & E & O \\ O & O & \cdots & O & O \end{bmatrix}, \quad B = \begin{bmatrix} Q & O & \cdots & O & O \\ O & D & \cdots & O & O \\ \dots & \dots & \ddots & \dots & \dots \\ O & O & \cdots & D & O \\ O & O & \cdots & O & Q \end{bmatrix},$$

$$E = diag(0, a, a, \dots, a, 0), Q = I_{(N+1)\times(N+1)}, O = O_{(N+1)\times(N+1)},$$
$$g^k_{m,n} = -f(t_k, x_n, y_m), \ k = \overline{1, N-1}, \ n = \overline{1, M-1}, m = \overline{1, M-1},$$

$$d_{i,i} = q, d_{i-1,i} = r, d_{i,i-1} = r, i = \overline{2, N}, d_{1,1} = -3, d_{1,2} = 4, d_{1,3} = -1,$$
$$d_{N+1,N+1} = 2\tau\left(\frac{e^{-\left(t_{N-1}-\frac{\tau}{2}\right)}}{4} + e^{-\left(t_N-\frac{\tau}{2}\right)}\right) - 3, \ d_{N+1,N} = 2\tau\left(\frac{e^{-\left(t_{N-2}-\frac{\tau}{2}\right)}}{4} + \frac{e^{-\left(t_{N-1}-\frac{\tau}{2}\right)}}{4} - e^{-\left(t_N-\frac{\tau}{2}\right)}\right) + 4,$$
$$d_{N+1,N-1} = \frac{\tau e^{-\left(t_{N-3}-\frac{\tau}{2}\right)}}{2} + \frac{\tau e^{-\left(t_{N-2}-\frac{\tau}{2}\right)}}{2} - \frac{\tau e^{-\left(t_{N-1}-\frac{\tau}{2}\right)}}{2} - 1, \ d_{N+1,1} = 2\tau\left(-\frac{e^{-\left(t_2-\frac{\tau}{2}\right)}}{4} - e^{-\left(t_1-\frac{\tau}{2}\right)}\right),$$

$$d_{N+1,2} = 2\tau\left(-\frac{e^{-\left(t_2-\frac{\tau}{2}\right)}}{4} - \frac{e^{-\left(t_3-\frac{\tau}{2}\right)}}{4} + e^{-\left(t_1-\frac{\tau}{2}\right)}\right), \ d_{N+1,3} = \frac{\tau e^{-\left(t_2-\frac{\tau}{2}\right)}}{2} - \frac{\tau e^{-\left(t_3-\frac{\tau}{2}\right)}}{2} - \frac{\tau e^{-\left(t_4-\frac{\tau}{2}\right)}}{2},$$
$$d_{N+1,j} = \frac{\tau}{2}\left(e^{-\left(t_{j-1}-\frac{\tau}{2}\right)} + e^{-\left(t_j-\frac{\tau}{2}\right)} - e^{-\left(t_{j+1}-\frac{\tau}{2}\right)} - e^{-\left(t_{j+2}-\frac{\tau}{2}\right)}\right), j = 4, ..., N-2;$$
$$d_{i,j} = 0, \text{ for other } i \text{ and } j; g^0_{m,n} = 2\tau\phi_{m,n}, \ g^N_{m,n} = 2\tau\eta_{m,n}, \ n = \overline{1, M-1}, m = \overline{1, M-1}.$$

In Tables 4-6, errors approximations in case N=M=10, 20, 40 for $u, v$ and $p$ are displayed. It can be seen from Tables 4-6 when N, M are increased two times that errors are decreased with approximately ratio $\frac{1}{4}$.

Table 4

**Test example (25) - error $v$**

| DS $\backslash (N, M)$ | $(10, 10)$ | $(20, 20)$ | $(40, 40)$ |
|---|---|---|---|
| 2nd order of ADS | $4.75 \times 10^{-3}$ | $1.18 \times 10^{-3}$ | $2.97 \times 10^{-4}$ |

Table 5

**Test example (25) - error $u$**

| DS $\backslash (N, M)$ | $(10, 10)$ | $(20, 20)$ | $(40, 40)$ |
|---|---|---|---|
| 2nd order of ADS | $2.43 \times 10^{-4}$ | $6.23 \times 10^{-5}$ | $1.56 \times 10^{-5}$ |

Table 6

**Test example (25) - error $p$**

| Approximation $\backslash (N, M)$ | $(10, 10)$ | $(20, 20)$ | $(40, 40)$ |
|---|---|---|---|
| 2nd order | $3.42 \times 10^{-4}$ | $1.15 \times 10^{-4}$ | $3.07 \times 10^{-5}$ |

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## Интегралдық шарты бар және қайта анықталған Нейман типті эллипстік кері есептің сандық есептеуі

Әртүрлі нақты процестерді модельдеу кезінде дербес туындылы дифференциалдық теңдеу үшін дереккөздерді сәйкестендіру есептерін шешу әдістері маңызды рөл атқарады. Мақала интегралдық шарты бар туынды үшін белгілі бір есептің эллипстік аппроксимациясына арналған. Алғашқыда, кері есеп туынды үшін интегралдық шарттары бар бейлокаль қандай да бір көмекші шеттік есептерге әкеледі. Теңдеудің параметрі бейлокаль көмекші есепті шешкен соң анықталады. Абстрактілі анықталған эллипстік есепті жуықтап шешу үшін екінші дәлдіктің айырымдық схемасы ұсынылған. Оператор тәсілін қолдана отырып, айырымдық есептің шешімінің бар екендігі дәлелденді. Салынған айырмашылық сызбасын шешу үшін тұрақтылық пен мәжбүрлік тұрақтылық бағалануы орнатылған. Кейін алынған абстрактілі нәтижелер интегралды шарттары бар Нейман типіндегі эллипстік көп өлшемді айырымдық есептерінің шешімінің орнықтылық бағамын алу үшін қолданылады. Қорытындылай келе, MATLAB бағдарламасын қолдана отырып, екі өлшемді және үш өлшемді тестік мысалдарын қысқаша түсіндірмесімен және сандық нәтижесін ұсынамыз.

*Кілт сөздер:* айырымдық схема, эллипстік кері есеп, қайта анықталған, дереккөзді сәйкестендіру мәселесі, орнықтылық, мәжбүрлі тұрақтылық, бағамы.

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## Численное решение эллиптической обратной задачи с интегральным условием и переопределением типа Неймана

При моделировании различных реальных процессов важную роль играют методы решения задачи идентификации источника для уравнения в частных производных. Настоящая статья посвящена аппроксимации эллиптической переопределенной задачи с интегральным условием для производных. Вначале обратная задача сводится к некоторой вспомогательной нелокальной краевой задаче с интегральным граничным условием для производных. Параметр уравнения определяется после решения этой вспомогательной нелокальной задачи. Предложена разностная схема второго порядка точности для приближенного решения абстрактной переопределенной эллиптической задачи. С помощью операторного подхода доказано существование решения разностной задачи. Для решения построенной разностной схемы установлены оценки устойчивости и коэрцитивной устойчивости. Позднее полученные абстрактные результаты применяются для получения оценок устойчивости решения переопределенных эллиптических многомерных разностных задач типа Неймана с интегральными условиями. Кроме того, используя программу MATLAB, авторами представлены численные результаты для двух- и трехмерных тестовых примеров с кратким объяснением реализации на компьютере.

*Ключевые слова:* разностная схема, обратная эллиптическая задача, переопределение, проблема идентификации источника, устойчивость, коэрцитивная устойчивость, оценка.

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