DOI 10.31489/2020M2/84-99

UDC 517.956

A.B. Keldibekova

Ye.A. Buketov Karaganda State University, Kazakhstan
(E-mail: Keldibekova_ a_ b@mail.ru)

Solvability of a semi-periodic boundary value problem for a third order differential equation with mixed derivative

This article is devoted to the study of the solvability of a semi-periodic boundary value problem for an evolution equation of the pseudoparabolic type. Nonlocal problems for high order partial differential equations have been investigated by many authors [1–4]. A certain interest in the study of these problems is caused in connection with their applied values. These problems include highly porous media with a complex topology, and first of all, soil and ground. Such equations can also describe long waves in dispersed systems. To solve this problem, new functions are introduced in the work and the method of a parameterizations applied [5]. Then the boundary value problem for a third order differential equation is reduced to a periodic boundary value problem for a family of systems of ordinary differential equations [6–18]. New constructive algorithms for finding an approximate solution are proposed and in terms of the initial data, coefficient-like signs of the unique solvability of the problem under study are obtained.

Keywords: partial differential equation, third-order pseudoparabolic equation, algorithm, approximate solution.

Introduction

On $\Omega = [0, \omega] \times [0, T]$ we consider the semi-periodic boundary value problem

$$\frac{\partial^3 u}{\partial x^2 \partial t} = A(x,t)\frac{\partial^2 u}{\partial x^2} + B(x,t)\frac{\partial u}{\partial t} + C(x,t)u + f(x,t), (x,t) \in \Omega, \quad (1)$$

$$u(x,0) = u(x,T), \quad x \in [0, \omega], \quad (2)$$

$$u(0,t) + \frac{\partial u(0,t)}{\partial x}x = \varphi(x,t), \quad (x,t) \in \Omega, \quad (3)$$

where $(n \times n)$ - are the matrices $A(x,t), B(x,t), C(x,t)$, n-vector functions $f(x,t), \varphi(x,t)$ continuous on Ω, here $\|u(x,t)\| = \max\limits_{i=\overline{1,n}} |u_i(x,t)|, \quad \|A(x,t)\| = \max\limits_{i=\overline{1,n}} \sum\limits_{j=1}^{n} |a_{ij}(x,t)|$.

Let $C(\Omega, R^n)$ - be the function space $u : \Omega \to R^n$ continuous on Ω, with the norm

$$\|u\|_0 = \max_{(x,t)\in\overline{\Omega}} \|u(x,t)\|.$$

A function $u(x,t) \in C(\Omega, R^n)$, having partial derivatives

$$\frac{\partial u^2(x,t)}{\partial x^2} \in C(\Omega, R^n), \quad \frac{\partial u(x,t)}{\partial t} \in C(\Omega, R^n), \quad \frac{\partial^3 u(x,t)}{\partial x^2 \partial t} \in C(\Omega, R^n)$$

is called a solution to problem (1)–(3) if it satisfies system (1) for all $(x,t) \in \Omega$, and conditions (2), (3).

To find a solution, we introduce the functions $v(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}$, $w(x,t) = \frac{\partial u(x,t)}{\partial t}$ and reduce problem (1)–(3) to a family of periodic boundary value problems for a system of ordinary differential equations of the form

$$\frac{\partial v}{\partial t} = A(x,t)v + B(x,t)w + C(x,t)u + f(x,t), \quad (x,t) \in \Omega, \quad (4)$$

$$v(x,0) = v(x,T), \quad x \in [0, \omega], \quad (5)$$

functional relationships

$$w(x,t)=\varphi_t'(x,t)+\int\limits_0^x\int\limits_0^\xi\frac{\partial v(\xi_1,t)}{\partial t}d\xi_1 d\xi, \tag{6}$$

$$u(x,t)=\varphi(x,t)+\int\limits_0^x\int\limits_0^\xi v(\xi_1,t)d\xi_1 d\xi. \tag{7}$$

To solve problem (4)–(7) we apply the method of a parametrization.

By the step $h>0: Nh=T$ we make fragmentation $[0,T)=\bigcup\limits_{r=1}^{N}[(r-1)h,rh),\ N=1,2,\ldots.$ Moreover, the area Ω is divided into N parts. By $v_r(x,t)$, $u_r(x,t)$ we denote, respectively, the restriction of the function $v(x,t)$, $u(x,t)$ in $\Omega_r=[0,\omega]\times[(r-1)h,rh)$, $\quad r=\overline{1,N}$.

By $\lambda_r(x)$ we denote the value of the function $v_r(x,t)$ at $t=(r-1)h$, i.e. $\lambda_r(x)=v_r(x,(r-1)h)$ and make the replacement $\widetilde{v}_r(x,t)=v_r(x,t)-\lambda_r(x), r=\overline{1,N}$. We obtain an equivalent boundary value problem with unknown functions $\lambda_r(x)$:

$$\frac{\partial\widetilde{v}_r}{\partial t}=A(x,t)\widetilde{v}_r+A(x,t)\lambda_r(x)+B(x,t)w_r+(x,t)u_r+f(x,t), \tag{8}$$

$$\widetilde{v}_r(x,(r-1)h)=0,\quad x\in[0,\omega],\quad r=\overline{1,N}, \tag{9}$$

$$\lambda_1(x)-\lambda_N(x)-\lim_{t\to T-0}\widetilde{v}_N(x,t)=0,\quad x\in[0,\omega], \tag{10}$$

$$\lambda_s(x)+\lim_{t\to sh-0}\widetilde{v}_s(x,t)-\lambda_{s+1}(x)=0,\quad x\in[0,\omega],\quad s=\overline{1,N-1}. \tag{11}$$

$$w_r(x,t)=\varphi_t'(x,t)+\int\limits_0^x\int\limits_0^\xi\frac{\partial\widetilde{v}_r(\xi_1,t)}{\partial t}d\xi_1 d\xi, \tag{12}$$

$$u_r(x,t)=\varphi(x,t)+\int\limits_0^x\int\limits_0^\xi\lambda_r(\xi_1)d\xi_1 d\xi+\int\limits_0^x\int\limits_0^\xi\widetilde{v}_r(\xi_1,t)d\xi_1 d\xi,\quad (x,t)\in\Omega_r,\quad r=\overline{1,N}, \tag{13}$$

where (11) is the condition for combining functions in the internal lines of the partition. Problem (8), (9) for fixed $\lambda_r(x), w_r(x,t), u_r(x,t)$ is a one-parameter family of Cauchy problems for systems of ordinary differential equations, where $x\in[0,\omega]$, and is equivalent to the integral equation

$$\widetilde{v}_r(x,t)=\int\limits_{(r-1)h}^{t}A(x,\tau)\widetilde{v}_r(x,\tau)d\tau+\int\limits_{(r-1)h}^{t}A(x,\tau)d\tau\cdot\lambda_r(x)+\int\limits_{(r-1)h}^{t}F(x,\tau,w_r,u_r)d\tau, \tag{14}$$

where

$$\int\limits_{(r-1)h}^{t}F(x,\tau,w_r,u_r)d\tau=\int\limits_{(r-1)h}^{t}B(x,\tau)w_r(x,\tau)d\tau+\int\limits_{(r-1)h}^{t}C(x,\tau)u_r(x,\tau)d\tau+\int\limits_{(r-1)h}^{t}f(x,\tau)d\tau.$$

Instead of $\widetilde{v}_r(x,\tau)$ we substitute the corresponding right-handed part of (14) and by repeating this process ν ($\nu=1,2,\ldots$) times we obtain

$$\widetilde{v}_r(x,t)=D_{\nu r}(x,t)\lambda_r(x)+F_{\nu r}(x,t,w_r,u_r)+G_{\nu r}(x,t,\widetilde{v}_r),\quad r=\overline{1,N}, \tag{15}$$

where

$$D_{\nu r}(x,t)=\sum_{j=0}^{\nu-1}\int\limits_{(r-1)h}^{t}A(x,\tau_1)d\tau_1\dots\int\limits_{(r-1)h}^{\tau_j}A(x,\tau_{j+1})d\tau_{j+1}\dots d\tau_1,$$

$$F_{\nu r}(x,t,w_r,u_r)=\int\limits_{(r-1)h}^{t}\big[B(x,\tau_1)w_r(x,\tau_1)+C(x,\tau_1)u_r(x,\tau_1)+f(x,\tau_1)\big]d\tau_1+$$

$$+\sum_{j=1}^{\nu-1}\int\limits_{(r-1)h}^{t}A(x,\tau_1)\dots\int\limits_{(r-1)h}^{\tau_{j-1}}A(x,\tau_j)\int\limits_{(r-1)h}^{\tau_j}\big[w_r(x,\tau_{j+1})+C(x,\tau_{j+1})u_r(x,\tau_{j+1})+f(x,\tau_{j+1})\big]d\tau_{j+1}d\tau_j\dots d\tau_1,$$

$$G_{\nu r}(x,t,\widetilde{v}_r)=\int\limits_{(r-1)h}^{t}A(x,\tau_1)\dots\int\limits_{(r-1)h}^{\tau_{\nu-2}}A(x,\tau_{\nu-1})\int\limits_{(r-1)h}^{\tau_{\nu-1}}A(x,\tau_\nu)\widetilde{v}_r(x,\tau_\nu)d\tau_\nu d\tau_{\nu-1}\dots d\tau_1,$$

$\tau_0=t, r=\overline{1,N}$. Passing to the limit as $t\to rh-0$ in (15) we have

$$\lim_{t\to rh-0}\widetilde{v}_r(x,t)=D_{\nu r}(x,rh)\lambda_r(x)+F_{\nu r}(x,rh,w_r,u_r)+G_{\nu r}(x,rh,\widetilde{v}_r),$$

$x\in[0,\omega], r=\overline{1,N}$. Substituting in (10), (11) instead of $\lim\limits_{t\to rh-0}\widetilde{v}_r(x,t), r=\overline{1,N}$, the corresponding right-handed parts for unknown functions $\lambda_r(x), r=\overline{1,N}$, we obtain the system of functional equations:

$$Q_\nu(x,h)\lambda(x)=-F_\nu(x,h,w,u)-G_\nu(x,h,\widetilde{v}), \tag{16}$$

where $Q_\nu(x,h)=$

$$=\begin{vmatrix} I & 0 & \dots & 0 & -[I+D_{\nu N}(x,Nh)] \\ I+D_{\nu 1}(x,h) & -I & \dots & 0 & 0 \\ 0 & I+D_{\nu 2}(x,2h) & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I+D_{\nu,N-1}(x,(N-1)h) & -I \end{vmatrix},$$

$$F_\nu(x,h,w,u)=(-F_{\nu N}(x,Nh,w_N,u_N),F_{\nu 1}(x,h,w_1,u_1),\dots,F_{\nu,N-1}(x,(N-1)h,w_{N-1},u_{N-1})),$$

$$G_\nu(x,h,\widetilde{v})=(-G_{\nu N}(x,Nh,\widetilde{v}_N),G_{\nu 1}(x,h,\widetilde{v}_1),\dots,G_{\nu,N-1}(x,(N-1)h,\widetilde{v}_{N-1})),$$

$I-$ is the unit matrix of dimension n.

To find a system of five functions $\{\lambda_r(x),\widetilde{v}_r(x,t),w_r(x,t),\ u_r(x,t)\}$, $r=\overline{1,N}$, we have a closed system consisting of equations (16), (15), (12) and (13).

Assuming the invertibility of the matrix $Q_\nu(x,h)$ for all $x\in[0,\omega]$, from equation (16), where

$$\widetilde{v}_r(x,t)=0, u_r(x,t)=\varphi(x,t), w_r(x,t)=\varphi'_t(x,t),$$

we find $\lambda^{(0)}(x)=(\lambda_1^{(0)}(x),\lambda_2^{(0)}(x),\dots,\lambda_N^{(0)}(x))'$:

$$\lambda^{(0)}(x)=-[Q_\nu(x,h)]^{-1}\big\{F_\nu(x,h,\varphi'_t,\varphi)+G_\nu(x,h,0)\big\}.$$

Using equation (15), for $\lambda_r(x)=\lambda_r^{(0)}(x)$ we find the functions $\{\widetilde{v}_r^{(0)}(x,t)\}$, $r=\overline{1,N}$, i.e.

$$\widetilde{v}_r^{(0)}(x,t)=D_{\nu r}(x,t)\lambda_r^{(0)}(x)+F_{\nu r}(x,t,\varphi'_t,\varphi)+G_{\nu r}(x,t,0).$$

The functions $w_r^{(0)}(x,t), u_r^{(0)}(x,t), r=\overline{1,N}$, are determined from the relations

$$w_r^{(0)}(x,t)=\varphi_t'(x,t)+\int\limits_0^x\int\limits_0^\xi \frac{\partial\widetilde{v}_r^{(0)}(\xi_1,t)}{\partial t}d\xi_1 d\xi,$$

$$u_r^{(0)}(x,t)=\varphi(x,t)+\int\limits_0^x\int\limits_0^\xi \lambda_r^{(0)}(\xi_1)d\xi_1 d\xi+\int\limits_0^x\int\limits_0^\xi \widetilde{v}_r^{(0)}(\xi_1,t)d\xi_1 d\xi, \quad (x,t)\in\Omega_r.$$

For the initial approximation of the problem (8)-(13) we take the system $(\lambda_r^{(0)}(x),\ \widetilde{v}_r^{(0)}(x,t),\ u_r^{(0)}(x,t)),\ r=\overline{1,N}$ and successive approximations are constructed according to the following algorithm:

Step 1. A) Assuming that $w_r(x,t)=w_r^{(0)}(x,t), \quad u_r(x,t)=u_r^{(0)}(x,t), \quad r=\overline{1,N}$, are first approximations in $\lambda_r(x), \widetilde{v}_r(x,t), r=\overline{1,N}$, we find by solving problem (8)-(11). By taking

$$\lambda_r^{(1,0)}(x)=\lambda_r^{(0)}(x), \quad \widetilde{v}_r^{(1,0)}(x,t)=\widetilde{v}_r^{(0)}(x,t),$$

the system couple $\{\lambda_r^{(1)}(x),\widetilde{v}_r^{(1)}(x,t)\}, r=\overline{1,N}$, we find as the limit of the sequence $\lambda_r^{(1,m)}(x),\widetilde{v}_r^{(1,m)}(x,t)$, are defined the next way:

Step 1.1. Assuming the invertibility of the matrix $Q_\nu(x,h)$, $x\in[0,\omega]$, equation (16), where

$$\widetilde{v}_r(x,t)=\widetilde{v}_r^{(1,0)}(x,t),$$

we find $\lambda^{(1,1)}(x)=(\lambda_1^{(1,1)}(x),\lambda_2^{(1,1)}(x),\ldots,\lambda_N^{(1,1)}(x))'$:

$$\lambda^{(1,1)}(x)=-[Q_\nu(x,h)]^{-1}\Big\{F_\nu(x,h,w^{(0)},u^{(0)})+G_\nu(x,h,\widetilde{v}^{(1,0)})\Big\}.$$

Substituting the found $\lambda_r^{(1,1)}(x)$, $r=\overline{1,N}$, in (15) we find

$$\widetilde{v}_r^{(1,1)}(x,t)=D_{\nu r}(x,t)\lambda_r^{(1,1)}(x)+F_{\nu r}(x,t,w^{(0)},u^{(0)})+G_{\nu r}(x,t,\widetilde{v}^{(1,0)}).$$

Step 1.2 From equation (16), where $\widetilde{v}_r(x,t)=\widetilde{v}_r^{(1,1)}(x,t)$, we define

$$\lambda^{(1,2)}(x)=-[Q_\nu(x,h)]^{-1}\Big\{F_\nu(x,h,w^{(0)},u^{(0)})+G_\nu(x,h,\widetilde{v}^{(1,1)})\Big\}.$$

By using expression (15), again, we find the functions $\{\widetilde{v}_r^{(1,2)}(x,t)\}$, $r=\overline{1,N}$,

$$\widetilde{v}_r^{(1,2)}(x,t)=D_{\nu r}(x,t)\lambda_r^{(1,2)}(x)+F_{\nu r}(x,t,w^{(0)},u^{(0)})+G_{\nu r}(x,t,\widetilde{v}^{(1,1)}).$$

At the $(1,m)$ step, we obtain the system of couple $\{\lambda_r^{(1,m)}(x),\widetilde{v}_r^{(1,m)}(x,t)\}, r=\overline{1,N}$.

Let's suppose that the solution of problem (8)–(11) is a sequence of systems of couples $\{\lambda_r^{(1,m)}(x),\ \widetilde{v}_r^{(1,m)}(x,t)\}$ is defined and for $m\to\infty$ converges to continuous, respectively, on $x\in[0,\omega]$, $(x,t)\in\Omega_r$ functions $\lambda_r^{(1)}(x), \quad \widetilde{v}_r^{(1)}(x,t) \quad , r=\overline{1,N}$.

B) The functions $w_r^{(1)}(x,t), u_r^{(1)}(x,t), r=\overline{1,N}$, are determined from the relations

$$w_r^{(1)}(x,t)=\varphi_t'(x,t)+\int\limits_0^x\int\limits_0^\xi \frac{\partial\widetilde{v}_r^{(1)}(\xi_1,t)}{\partial t}d\xi_1 d\xi,$$

$$u_r^{(1)}(x,t)=\varphi(x,t)+\int\limits_0^x\int\limits_0^\xi \lambda_r^{(1)}(\xi_1)d\xi_1 d\xi+\int\limits_0^x\int\limits_0^\xi \widetilde{v}_r^{(1)}(\xi_1,t)d\xi_1 d\xi, \quad (x,t)\in\Omega_r.$$

Step 2. A) Assuming that

$$w_r(x,t)=w_r^{(1)}(x,t),\quad u_r(x,t)=u_r^{(1)}(x,t),\quad r=\overline{1,N},$$

are the second approximations in $\lambda_r(x),\widetilde{v}_r(x,t),r=\overline{1,N}$, we find solving problem (8)–(11). Taking

$$\lambda_r^{(2,0)}(x)=\lambda_r^{(1)}(x),\quad \widetilde{v}_r^{(2,0)}(x,t)=\widetilde{v}_r^{(1)}(x,t),$$

the system of couples $\{\lambda_r^{(2)}(x),\ \widetilde{v}_r^{(2)}(x,t)\}$, $r=\overline{1,N}$, we find as the limit of the sequence $\lambda_r^{(2,m)}(x)$, $\widetilde{v}_r^{(2,m)}(x,t)$, that defines in the following way:

Step 2.1 Assuming the matrix $Q_\nu(x,h)$, $x\in[0,\omega]$, is invertible, from equation (16), where

$$\widetilde{v}_r(x,t)=\widetilde{v}_r^{(2,0)}(x,t),$$

we find $\lambda^{(2,1)}(x)=(\lambda_1^{(2,1)}(x),\lambda_2^{(2,1)}(x),...,\lambda_N^{(2,1)}(x))'$:

$$\lambda^{(2,1)}(x)=-[Q_\nu(x,h)]^{-1}\Big\{F_\nu(x,h,w^{(1)},u^{(1)})+G_\nu(x,h,\widetilde{v}^{(2,0)})\Big\}.$$

By substituting the found $\lambda_r^{(2,1)}(x)$, $r=\overline{1,N}$, in (15) we find

$$\widetilde{v}_r^{(2,1)}(x,t)=D_{\nu r}(x,t)\lambda_r^{(2,1)}(x)+F_{\nu r}(x,t,w^{(1)},u^{(1)})+G_{\nu r}(x,t,\widetilde{v}^{(2,0)}).$$

Step 2.2 From equation (16), where

$$\widetilde{v}_r(x,t)=\widetilde{v}_r^{(2,1)}(x,t),$$

we define

$$\lambda^{(2,2)}(x)=-[Q_\nu(x,h)]^{-1}\Big\{F_\nu(x,h,w^{(1)},u^{(1)})+G_\nu(x,h,\widetilde{v}^{(2,1)})\Big\}.$$

Using expression (15), again, we find the functions $\{\widetilde{v}_r^{(2,2)}(x,t)\}$, $r=\overline{1,N}$:

$$\widetilde{v}_r^{(2,2)}(x,t)=D_{\nu r}(x,t)\lambda_r^{(2,2)}(x)+F_{\nu r}(x,t,w^{(1)},u^{(1)})+G_{\nu r}(x,t,\widetilde{v}^{(2,1)}).$$

At the $(2,m)$ step, we obtain the system of couples

$$\{\lambda_r^{(2,m)}(x),\widetilde{v}_r^{(2,m)}(x,t)\},r=\overline{1,N}.$$

Let's suppose that the solution to problem (8)–(11) is a sequence of systems of couples $\{\lambda_r^{(2,m)}(x)$, $\widetilde{v}_r^{(2,m)}(x,t)\}$ are defined and at $m\to\infty$ converges to $\{\lambda_r^{(2)}(x),\ \widetilde{v}_r^{(2)}(x,t)\}$, $\quad r=\overline{1,N}$.

B) The functions $w_r^{(2)}(x,t)$, $\quad u_r^{(2)}(x,t)$, $\quad r=\overline{1,N}$, are determined from the ratios

$$w_r^{(2)}(x,t)=\varphi_t'(x,t)+\int\limits_0^x\int\limits_0^\xi\frac{\partial\widetilde{v}_r^{(2)}(\xi_1,t)}{\partial t}d\xi_1d\xi,$$

$$u_r^{(2)}(x,t)=\varphi(x,t)+\int\limits_0^x\int\limits_0^\xi\lambda_r^{(2)}(\xi_1)d\xi_1d\xi+\int\limits_0^x\int\limits_0^\xi\widetilde{v}_r^{(2)}(\xi_1,t)d\xi_1d\xi,\quad (x,t)\in\Omega_r.$$

By continuing the process, at the step k we obtain the system $\{\lambda_r^{(k)}(x),\ \widetilde{v}_r^{(k)}(x,t),\ w_r^{(k)}(x,t),\ u_r^{(k)}(x,t)\}$, $r=\overline{1,N}$.

The conditions of the following statement provide feasibility and convergence of the proposed algorithm, as well as unique solvability problems (8)–(13).

Theorem 1. Let's suppose that for some $h>0: Nh=T, N=1,2,...,$ and ν, $\nu\in\mathbb{N}$, $(nN\times nN)$ the matrix $Q_\nu(x,h)$ is invertible for all $x\in[0,\omega]$ and the inequalities are carried out
1) $\|[Q_\nu(x,h)]^{-1}\|\leq\gamma_\nu(x,h)$;
2) $q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\leq\mu<1$, where $q_\nu(x,h)=1+\gamma_\nu(x,h)\sum\limits_{j=1}^{\nu}\frac{(\alpha(x)h)^j}{j!}$.
Then there is a unique solution of problem (8)–(13) and the estimates are valid

$$a)\max\Big\{\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\Big\|\frac{\partial\widetilde{v}_r^*(x,t)}{\partial t}-\frac{\partial\widetilde{v}_r^{(k)}(x,t)}{\partial t}\Big\|,$$

$$\max_{r=\overline{1,N}}\|\lambda_r^*(x)-\lambda_r^{(k)}(x)\|+\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+p)}(x,t)-\widetilde{v}_r^{(k)}(x,t)\|\Big\}\leq$$

$$\leq d_0(x)\sum_{j=k-1}^{\infty}\frac{1}{j!}\Big(\int_0^x d_0(\xi)d\xi\Big)^j\int_0^x\int_0^\xi\int_0^{\xi_1}\max\Big\{d_2(\xi_2),d_1(\xi_2)\Big\}d\xi_2d\xi_1d\xi\max\Big\{\|\varphi_t'\|_0,\|\varphi\|_0,\|f\|_0\Big\},$$

$$b)\max\Big\{\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|w_r^*(x,t)-w_r^{(k)}(x,t)\|,\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^*(x,t)-u_r^{(k)}(x,t)\|\Big\}\leq$$

$$\leq\int_0^x\int_0^\xi\max\Big\{\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\Big\|\frac{\partial\widetilde{v}_r^*(\xi_1,t)}{\partial t}-\frac{\partial\widetilde{v}_r^{(k)}(\xi_1,t)}{\partial t}\Big\|,$$

$$\max_{r=\overline{1,N}}\|\lambda_r^*(\xi_1)-\lambda_r^{(k)}(\xi_1)\|+\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^*(\widetilde{\xi},t)-\widetilde{v}_r^{(k)}(\widetilde{\xi},t)\|\Big\}d\xi,\quad k=1,2,\ldots$$

where

$$\alpha(x)=\max_{t\in[0,T]}\|A(x,t)\|,\quad\beta(x)=\max_{t\in[0,T]}\|B(x,t)\|,\quad\sigma(x)=\max_{t\in[0,T]}\|C(x,t)\|,$$

$$\rho_1(x)=\beta(x)+\sigma(x)+1,\quad\rho_2(x)=\alpha(\xi)\Big(1+q_\nu(\xi,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(\xi)h)^j}{j!}\Big)+1,$$

$$\rho_3(x)=\frac{(q_\nu(x,h)+\gamma_\nu(x,h))h\sum\limits_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}}{1-q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}},$$

$$d_0(x)=\max\Big\{\int_0^x\rho_2(\xi)[\beta(\xi)+\sigma(\xi)]d\xi,\rho_3(x)[\beta(x)+\sigma(x)]\Big\},$$

$$d_1(x)=\frac{1+\gamma_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}}{1-q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}}q_\nu(x,h)+\gamma_\nu(x,h)\Big]\Big[h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\beta(x)\int_0^x\rho_1(\xi)\rho_2(\xi)d\xi+$$

$$+h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\sigma(x)\int_0^x\int_0^\xi\rho(\xi_1)\theta_\nu(\xi_1,h)d\xi_1d\xi+\frac{(\alpha(x)h)^\nu}{\nu!}\rho_1(x)q_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!},$$

$$d_2(x)=\int_0^x\rho_2(\xi)\Big[\beta(\xi)\int_0^\xi\rho_2(\xi_1)\rho_1(\xi_1)d\xi_1+\sigma(\xi_1)\int_0^\xi\int_0^{\xi_1}\rho(\xi_2)\theta_\nu(\xi_2,h)d\xi_2d\xi_1\Big]d\xi.$$

Proof. We have the following inequality

$$\|F_\nu(x,h,w,u)\| \le h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\big[\beta(x)\|w_r(x,t)\|+\sigma(x)\|u_r(x,t)\|+\|f(x,t)\|\big],$$

$$\|G_\nu(x,h,\widetilde{v})\| \le \frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r(x,t)\|,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|D_{\nu r}(x,t)\| \le \sum_{j=1}^{\nu}\frac{(\alpha(x)h)^j}{j!}.$$

The following estimates follow from the zero step of the algorithm:

$$\max_{r=\overline{1,N}}\|\lambda_r{}^{(0)}(x)\| \le \rho_1(x)\gamma_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\max\Big\{\|\varphi_t'\|_0,\|\varphi\|_0,\|f\|_0\Big\},$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(0)}(x,t)\| \le$$

$$\le \sum_{j=1}^{\nu}\frac{(\alpha(x)h)^j}{j!}\max_{r=\overline{1,N}}\|\lambda_r{}^{(0)}(x)\| + [\beta(x)+\sigma(x)+1]h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\max\Big\{\|\varphi_t'\|_0,\|\varphi\|_0,\|f\|_0\Big\}\le$$

$$\le \rho_1(x)q_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\max\Big\{\|\varphi_t'\|_0,\|\varphi\|_0,\|f\|_0\Big\},$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|w_r^{(0)}(x,t)-\varphi_t'(x,t)\| \le \int_0^x \rho_2(\xi)\rho_1(\xi)d\xi\max\Big\{\|\varphi_t'\|_0,\|\varphi\|_0,\|f\|_0\Big\},$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(0)}(x,t)-\varphi(x,t)\| \le \int_0^x\int_0^\xi \rho(\xi_1)\theta_\nu(\xi_1,h)d\xi_1 d\xi\max\Big\{\|\varphi_t'\|_0,\|\varphi\|_0,\|f\|_0\Big\}.$$

The following estimates are valid:

$$\max_{r=\overline{1,N}}\|\lambda_r{}^{(1,1)}(x)-\lambda_r{}^{(1,0)}(x)\| \le$$

$$\le \gamma_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\beta(x)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|w_r^{(0)}(x,t)-\varphi_t'(x,t)\|+$$

$$+\gamma_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\sigma(x)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(0)}(x,t)-\varphi(x,t)\|+$$

$$+\gamma_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(0)}(x,t)\|,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\| \le$$

$$\le q_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\beta(x)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|w_r^{(0)}(x,t)-\varphi_t'(x,t)\|+$$

$$+q_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\sigma(x)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(0)}(x,t)-\varphi(x,t)\|+$$

$$+q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(0)}(x,t)\|.$$

Select the inequality

$$\Delta^{(1,1)}(x)=\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|+\max_{r=\overline{1,N}}\|\lambda_r{}^{(1,1)}(x)-\lambda_r{}^{(1,0)}(x)\|\leq$$

$$\leq\theta_\nu(x,h)\beta(x)\int_0^x\rho_1(\xi)\rho_2(\xi)d\xi\max\Big\{\|\varphi_t'\|_0,\|\varphi\|_0,\|f\|_0\Big\}+$$

$$+\theta_\nu(x,h)\sigma(x)\int_0^x\int_0^\xi\rho_1(\xi_1)\theta_\nu(\xi_1,h)d\xi_1d\xi\max\Big\{\|\varphi_t'\|_0,\|\varphi\|_0,\|f\|_0\Big\}+$$

$$+\theta_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\rho_1(x)q_\nu(x,h)\max\Big\{\|\varphi_t'\|_0,\|\varphi\|_0,\|f\|_0\Big\}.$$

Thus,

$$\max_{r=\overline{1,N}}\|\lambda_r{}^{(1,m+1)}(x)-\lambda_r{}^{(1,m)}(x)\|\leq$$

$$\leq\gamma_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,m)}(x,t)-\widetilde{v}_r^{(1,m-1)}(x,t)\|,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,m+1)}(x,t)-\widetilde{v}_r^{(1,m)}(x,t)\|\leq$$

$$\leq q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,m)}(x,t)-\widetilde{v}_r^{(1,m-1)}(x,t)\|.$$

Due to the inequality $q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}<1$ follows the uniform convergence $v_r^{(1,\,m+1)}(x,\,t)$, at $(x,t)\in\Omega_r$, to $v_r^{(1)}(x,t)$ and the convergence of a sequence of systems of functions $\lambda_r^{(1,m+1)}(x)$ to continuous $x\in[0,\omega]$ functions $\lambda_r^{(1)}(x)$ for all $r=\overline{1,N}$:

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,m+1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|\leq$$

$$\leq\sum_{j=0}^m\left[q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\right]^j\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|.$$

$$\max_{r=\overline{1,N}}\|\lambda_r{}^{(1,m+1)}(x)-\lambda_r{}^{(1,0)}(x)\|\leq$$

$$\leq\sum_{j=0}^m\left[q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\right]^j\gamma_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|+$$

$$+\max_{r=\overline{1,N}}\|\lambda_r{}^{(1,1)}(x)-\lambda_r{}^{(1,0)}(x)\|.$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,m+1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|+\max_{r=\overline{1,N}}\|\lambda_r{}^{(1,m+1)}(x)-\lambda_r{}^{(1,0)}(x)\|\leq$$

$$\leq \sum_{j=0}^{m} \left[q_\nu(x,h) \frac{(\alpha(x)h)^\nu}{\nu!} \right]^j \left[1 + \gamma_\nu(x,h) \frac{(\alpha(x)h)^\nu}{\nu!} \right] \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(1,1)}(x,t) - \widetilde{v}_r^{(1,0)}(x,t)\| +$$

$$+ \max_{r=\overline{1,N}} \|\lambda_r{}^{(1,1)}(x) - \lambda_r{}^{(1,0)}(x)\|.$$

Moving to the limit at $m \to \infty$, we obtain estimates:

$$\Delta^{(1)}(x) = \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(1)}(x,t) - \widetilde{v}_r^{(0)}(x,t)\| + \max_{r=\overline{1,N}} \|\lambda_r{}^{(1)}(x) - \lambda_r{}^{(0)}(x)\| \leq$$

$$\leq d_1(x) \max\left\{ \|\varphi_t'\|_0, \|\varphi\|_0, \|f\|_0 \right\}.$$

$$\widetilde{\Delta}^{(1)}(x) = \max_{r=\overline{1,N}} \sup \left\| \frac{\partial \tilde{v}_r^{(1)}(\xi,t)}{\partial t} - \frac{\partial \tilde{v}_r^{(0)}(\xi,t)}{\partial t} \right\| \leq d_2(x) \max\left\{ \|\varphi_t'\|_0, \|\varphi\|_0, \|f\|_0 \right\}.$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|w_r^{(1)}(x,t) - w_r^{(0)}(x,t)\| \leq \int_0^x \int_0^\xi \widetilde{\Delta}^{(1)}(\xi_1) d\xi_1 d\xi,$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^{(1)}(x,t) - u_r^{(0)}(x,t)\| \leq \int_0^x \int_0^\xi \Delta^{(1)}(\xi_1) d\xi_1 d\xi.$$

For difference systems $\lambda_r{}^{(k+1)}(x) - \lambda_r{}^{(k)}(x)$, $\widetilde{v}_r^{(k+1)}(x,t) - \widetilde{v}_r^{(k)}(x,t)$, $w_r^{(k+1)}(x,t) - w_r^{(k)}(x,t)$, $u_r^{(k+1)}(x,t) - -u_r^{(k)}(x,t)$, $r = \overline{1,N}$, $k = 1,2,\ldots$ valid estimates:

$$\max_{r=\overline{1,N}} \|\lambda_r{}^{(k+1,1)}(x) - \lambda_r{}^{(k+1,0)}(x)\| \leq$$

$$\leq \gamma_\nu(x,h) h \sum_{j=0}^{\nu-1} \frac{(\alpha(x)h)^j}{j!} \beta(x) \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|w_r^{(k)}(x,t) - w_r^{(k-1)}(x,t)\| +$$

$$+ \gamma_\nu(x,h) h \sum_{j=0}^{\nu-1} \frac{(\alpha(x)h)^j}{j!} \sigma(x) \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^{(k)}(x,t) - u_r^{(k-1)}(x,t)\|,$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(k+1,1)}(x,t) - \widetilde{v}_r^{(k+1,0)}(x,t)\| \leq$$

$$\leq q_\nu(x,h) h \sum_{j=0}^{\nu-1} \frac{(\alpha(x)h)^j}{j!} \beta(x) \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|w_r^{(k)}(x,t) - w_r^{(k-1)}(x,t)\| +$$

$$+ q_\nu(x,h) h \sum_{j=0}^{\nu-1} \frac{(\alpha(x)h)^j}{j!} \sigma(x) \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^{(k)}(x,t) - u_r^{(k-1)}(x,t)\|,$$

$$\max_{r=\overline{1,N}} \|\lambda_r{}^{(k+1,m+1)}(x) - \lambda_r{}^{(k+1,m)}(x)\| \leq$$

$$\leq \gamma_\nu(x,h) \frac{(\alpha(x)h)^\nu}{\nu!} \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(k+1,m)}(x,t) - \widetilde{v}_r^{(k+1,m-1)}(x,t)\|,$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(k+1,m+1)}(x,t) - \widetilde{v}_r^{(k+1,m)}(x,t)\| \leq$$

$$\leq q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,m)}(x,t)-\widetilde{v}_r^{(k+1,m-1)}(x,t)\|.$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,m+1)}(x,t)-\widetilde{v}_r^{(k+1,0)}(x,t)\|\leq$$

$$\leq\sum_{j=0}^{m}\Big[q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\Big]^j\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,1)}(x,t)-\widetilde{v}_r^{(k+1,0)}(x,t)\|.$$

$$\max_{r=\overline{1,N}}\|\lambda_r{}^{(k+1,m+1)}(x)-\lambda_r{}^{(k+1,0)}(x)\|\leq$$

$$\leq\sum_{j=0}^{m-1}\Big[q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\Big]^j\gamma_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1,1)}(x,t)-\widetilde{v}_r^{(k+1,0)}(x,t)\|+$$

$$+\max_{r=\overline{1,N}}\|\lambda_r{}^{(k+1,1)}(x)-\lambda_r{}^{(k+1,0)}(x)\|.$$

Moving to the limit at $m\to\infty$, we obtain estimates:

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1)}(x,t)-\widetilde{v}_r^{(k)}(x,t)\|\leq$$

$$\leq\frac{q_\nu(x,h)h\sum\limits_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\beta(x)}{1-q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|w_r^{(k)}(x,t)-w_r^{(k-1)}(x,t)\|+$$

$$+\frac{q_\nu(x,h)h\sum\limits_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\sigma(x)}{1-q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k)}(x,t)-u_r^{(k-1)}(x,t)\|, \qquad (17)$$

$$\max_{r=\overline{1,N}}\|\lambda_r{}^{(k+1)}(x)-\lambda_r{}^{(k)}(x)\|\leq$$

$$\leq\frac{\gamma_\nu(x,h)h\sum\limits_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\beta(x)}{1-q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|w_r^{(k)}(x,t)-w_r^{(k-1)}(x,t)\|+$$

$$+\frac{\gamma_\nu(x,h)h\sum\limits_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\sigma(x)}{1-q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k)}(x,t)-u_r^{(k-1)}(x,t)\|. \qquad (18)$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|w_r^{(k+1)}(x,t)-w_r^{(k)}(x,t)\|\leq\int\limits_0^x\int\limits_0^\xi\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\left\|\frac{\partial v_r^{(k-1)}(\xi_1,t)}{\partial t}-\frac{\partial v_r^{(k)}(\xi_1,t)}{\partial t}\right\|d\xi_1d\xi,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k+1)}(x,t)-u_r^{(k)}(x,t)\|\leq\int\limits_0^x\int\limits_0^\xi\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|v_r^{(k+1)}(\xi_1,t)-z_r^{(k)}(\xi_1,t)\|d\xi_1d\xi,$$

Summing, respectively, the left and right parts of inequalities (17), (18) we have

$$\Delta^{(k+1)}(x)=\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+1)}(x,t)-\widetilde{v}_r^{(k)}(x,t)\|+\max_{r=\overline{1,N}}\|\lambda_r{}^{(k+1)}(x)-\lambda_r{}^{(k)}(x)\|\leq$$

$$\leq \rho_3(x)\beta(x) \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|w_r^{(k)}(x,t) - w_r^{(k-1)}(x,t)\| +$$

$$+\rho_3(x)\sigma(x) \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^{(k)}(x,t) - u_r^{(k-1)}(x,t)\|. \quad (19)$$

$$\widetilde{\Delta}^{(k+1)}(x) = \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \left\| \frac{\partial \tilde{v}_r^{(1)}(\xi,t)}{\partial t} - \frac{\partial \tilde{v}_r^{(0)}(\xi,t)}{\partial t} \right\| \leq$$

$$\leq \int_0^x \rho_2(\xi)\beta(\xi) \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|w_r^{(k)}(\xi,t) - w_r^{(k-1)}(\xi,t)\| d\xi +$$

$$+ \int_0^x \rho_2(\xi)\sigma(\xi) \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^{(k)}(\xi,t) - u_r^{(k-1)}(\xi,t)\| d\xi. \quad (20)$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|w_r^{(k+1)}(x,t) - w_r^{(k)}(x,t)\| \leq \int_0^x \int_0^\xi \widetilde{\Delta}^{(k+1)}(\xi_1) d\xi_1 d\xi,$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^{(k+1)}(x,t) - u_r^{(k)}(x,t)\| \leq \int_0^x \int_0^\xi \Delta^{(k+1)}(\xi_1) d\xi_1 d\xi.$$

For the function $\left\{\widetilde{\Delta}^{(k+1)}(x), \Delta^{(k+1)}(x)\right\}$ based on (19), (20) we establish the inequality

$$\max\{\widetilde{\Delta}^{(k+1)}(x), \Delta^{(k+1)}(x)\} \leq$$

$$\leq \max\left\{ \int_0^x \rho_2(\xi)[\beta(\xi)+\sigma(\xi)]d\xi, \rho_3(x)[\beta(x)+\sigma(x)] \right\} \int_0^x \int_0^\xi \int_0^{\xi_1} \max\{\widetilde{\Delta}^{(k)}(\xi_2), \Delta^{(k)}(\xi_2)\} d\xi_2 d\xi_1 d\xi \leq$$

$$\leq d_0(x) \int_0^x \int_0^\xi \int_0^{\xi_1} \max\{\widetilde{\Delta}^{(k)}(\xi_2), \Delta^{(k)}(\xi_2)\} d\xi_2 d\xi_1 d\xi, \quad (21)$$

$$\max\left\{\widetilde{\Delta}^{(k+1)}(x), \Delta^{(k)}(x)\right\} \leq \frac{d_0(x)}{(k-1)!} \left(\int_0^x d_0(\xi) d\xi \right)^{k-1} \int_0^x \int_0^\xi \int_0^{\xi_1} \max\{\widetilde{\Delta}^{(1)}(\xi_2), \Delta^{(1)}(\xi_2)\} d\xi_2 d\xi_1 d\xi.$$

Establish inequalities

$$\max\Bigg\{ \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \left\| \frac{\partial \widetilde{v}_r^{(k+p)}(x,t)}{\partial t} - \frac{\partial \widetilde{v}_r^{(k)}(x,t)}{\partial t} \right\|,$$

$$\max_{r=\overline{1,N}} \|\lambda_r^{(k+p)}(x) - \lambda_r^{(k)}(x)\| + \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(k+p)}(x,t) - \widetilde{v}_r^{(k)}(x,t)\| \Bigg\} \leq$$

$$\leq \max\left\{\widetilde{\Delta}^{(k+p)}(x), \Delta^{(k)}(x)\right\} + \max\left\{\widetilde{\Delta}^{(k+p-1)}(x), \Delta^{(k+p-1)}(x)\right\} + ... + \max\left\{\widetilde{\Delta}^{(1)}(x), \Delta^{(1)}(x)\right\} \leq$$

$$\leq d_0(x)\sum_{j=k-1}^{k+p-2}\frac{1}{j!}\Big(\int_0^x d_0(\xi)d\xi\Big)^j\int_0^x\int_0^\xi\int_0^{\xi_1}\max\{\widetilde{\Delta}^{(1)}(\xi_2),\Delta^{(1)}(\xi_2)\}d\xi_2 d\xi_1 d\xi\leq$$

$$\leq d_0(x)\sum_{j=k-1}^{k+p-2}\frac{1}{j!}\Big(\int_0^x d_0(\xi)d\xi\Big)^j\int_0^x\int_0^\xi\int_0^{\xi_1}\max\Big\{d_2(\xi_2),d_1(\xi_2)\Big\}d\xi_2 d\xi_1 d\xi\max\Big\{\|\varphi_t'\|_0,\|\varphi\|_0,\|f\|_0\Big\},$$

$$\max\Big\{\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|w_r^{(k+p)}(x,t)-w_r^{(k)}(x,t)\|,\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k+p)}(x,t)-u_r^{(k)}(x,t)\|\Big\}\leq$$

$$\leq\int_0^x\int_0^\xi\max\Big\{\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\Big\|\frac{\partial\widetilde{v}_r^{(k+p)}(\xi_1,t)}{\partial t}-\frac{\partial\widetilde{v}_r^{(k)}(\xi_1,t)}{\partial t}\Big\|,$$

$$\max_{r=\overline{1,N}}\|\lambda_r^{(k+p)}(\xi_1)-\lambda_r^{(k)}(\xi_1)\|+\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k+p)}(\xi_1,t)-\widetilde{v}_r^{(k)}(\xi_1,t)\|\Big\}d\xi_1 d\xi,$$

proceeding to the limit for $p\to\infty$, for all $(x,t)\in\Omega_r$, $r=\overline{1,N}$, obtain the estimates of Theorem 1.

We prove uniqueness. Let there be $(\lambda_r^{**}(x)+\widetilde{v}_r^{**}(x,t), w_r^{**}(x,t), u_r^{**}(x,t))$, $r=\overline{1,N}$, another solution to the boundary value problem (8)–(13).

Similarly to relation (21) for the differences $\lambda_r{}^*(x)-\lambda_r{}^{**}(x)$, $\widetilde{v}_r^*(x,t)-\widetilde{v}_r^{**}(x,t)$, $\dfrac{\partial\widetilde{v}_r^*(x,t)}{\partial t}-\dfrac{\partial\widetilde{v}_r^{**}(x,t)}{\partial t}$, $r=\overline{1,N}$, for all $(x,t)\in\overline{\Omega}$ we get:

$$\max\Big\{\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\Big\|\frac{\partial\widetilde{v}_r^*(x,t)}{\partial t}-\frac{\partial\widetilde{v}_r^{**}(x,t)}{\partial t}\Big\|,$$

$$\max_{r=\overline{1,N}}\|\lambda_r^*(x)-\lambda_r^{**}(x)\|+\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^*(x,t)-\widetilde{v}_r^{**}(x,t)\|,\Big\}\leq$$

$$\leq d_0(x)\int_0^x\int_0^\xi\int_0^{\xi_1}\max\Big\{\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\Big\|\frac{\partial\widetilde{v}_r^*(\xi_2,t)}{\partial t}-\frac{\partial\widetilde{v}_r^{**}(\xi_2,t)}{\partial t}\Big\|,$$

$$\max_{r=\overline{1,N}}\|\lambda_r^*(\xi_2)-\lambda_r^{**}(\xi_2)\|+\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^*(\xi_2,t)-\widetilde{v}_r^{**}(\xi_2,t)\|\Big\}d\xi_2 d\xi_1 d\xi\leq$$

$$\leq d_0(x)\frac{x^3}{6}\int_0^x\max\Big\{\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\Big\|\frac{\partial v_r^*(\xi,t)}{\partial t}-\frac{\partial\widetilde{v}_r^{**}(\xi,t)}{\partial t}\Big\|,$$

$$\max_{r=\overline{1,N}}\|\lambda_r^*(\xi)-\lambda_r^{**}(\xi)\|+\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^*(\xi,t)-\widetilde{v}_r^{**}(\xi,t)\|\Big\}d\xi.$$

Using the Bellman-Gronwall inequality [19] we have

$$\max\Big\{\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\Big\|\frac{\partial\widetilde{v}_r^*(x,t)}{\partial t}-\frac{\partial\widetilde{v}_r^{**}(x,t)}{\partial t}\Big\|,$$

$$\max_{r=\overline{1,N}}\|\lambda_r^*(x)-\lambda_r^{**}(x)\|+\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^*(x,t)-\widetilde{v}_r^{**}(x,t)\|,\Big\}=0.$$

Whence it follows that $\widetilde{v}_r^*(x,t) = \widetilde{v}_r^{**}(x,t)$, $\lambda_r{}^*(x) = \lambda_r{}^{**}(x)$, $\dfrac{\partial \widetilde{v}_r^*(x,t)}{\partial t} = \dfrac{\partial \widetilde{v}_r^{**}(x,t)}{\partial t}$, $r = \overline{1,N}$. From the inequalities

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|w_r^*(x,t) - w_r^{**}(x,t)\| \leq \int\limits_0^x \int\limits_0^\xi \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \Big\| \frac{\partial \widetilde{v}_r^*(\xi_1,t)}{\partial t} - \frac{\partial \widetilde{v}_r^{**}(\xi_1,t)}{\partial t} \Big\| d\xi_1 d\xi,$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^*(x,t) - u_r^{**}(x,t)\| \leq$$

$$\leq \int\limits_0^x \int\limits_0^\xi \Big(\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^*(\xi_1,t) - \widetilde{v}_r^{**}(\xi_1,t)\| + \max_{r=\overline{1,N}} \|\lambda_r^*(\xi_1) - \lambda_r^{**}(\xi_1)\| \Big) d\xi_1 d\xi$$

we have $w_r^*(x,t) = w_r^{**}(x,t)$, $u_r^*(x,t) = u_r^{**}(x,t)$, $r = \overline{1,N}$, for all $(x,t) \in \Omega_r$. Theorem 1 is proved.

Because of the equivalence of problems (1)–(3) and (8)–(13) Theorem 1 implies

Theorem 2. Let's suppose that the conditions of Theorem 1 are satisfied. Then problem (1)–(3) has a unique solution $u^*(x,t)$ and the estimate are valid

$$\max\Big\{ \Big\| \frac{\partial u^*(x,t)}{\partial t} - \frac{\partial u^{(k)}(x,t)}{\partial t} \Big\|_0, \|u^*(x,t) - u^{(k)}(x,t)\|_0 \Big\} \leq$$

$$\leq \int\limits_0^x \int\limits_0^\xi d_0(\xi_1) \sum_{j=k-1}^{\infty} \frac{1}{j!} \Big(\int\limits_0^{\xi_1} d_0(\xi_2) d\xi_2 \Big)^j \int\limits_0^{\xi_1} \max\Big\{ d_2(\xi_2), d_1(\xi_2) \Big\} d\xi_2 d\xi_1 d\xi \max\Big\{ \|\varphi_t'\|_0, \|\varphi\|_0, \|f\|_0 \Big\}.$$

References

1 Нахушев А.М. Уравнения математической биологии/ А.М. Нахушев. — М.: Высш. шк., 1995. — 301 с. Zbl 0991.35500.

2 Kozhanov A.I. Composite Type Equations and Inverse Problems (Inverse and Iii-Posed Problems) / A.I. Kozhanov // VSP: Utrecht, 1999. — 171 p. DOI: 10.1515/9783110943276.

3 Kozhanov A.I. Pseudoparabolic and pseudohyperbolic equations in noncylindrical time domains / A.I. Kozhanov, G.A. Lukina // Mathematical notes of NEFU. — 2019. — 26. — No. 3. — P. 15-30. DOI: 10.25587/SVFU.2019.17.12.002.

4 Kozhanov A.I. Boundary-value problems for some higher-order nonclassical differential equations / A.I. Kozhanov, N.R. Pinigina // Mathematical notes. — 2017. — 101. — P. 467–474. DOI: 10.1134/S00014 34617030087.

5 Джумабаев Д.С. Признаки однозначной разрешимости линейной краевой задачи для обыкновенных дифференциальных уравнений / Д.С. Джумабаев // Журн. вычисл. мат. и мат. физики. — 1989. — № 1(29). — С. 34–46. DOI: 10.1016/0041-5553(89)90038-4.

6 Dzhumabaev D.S. On one approach to solve the linear boundary value problems for Fredholm integro-differential equations / D.S. Dzhumabaev // Journal of Computational and Applied Mathematics. — 2016. — 294. — P. 342–357. DOI: 10.1016/j.cam.2015.08.023.

7 Dzhumabaev D.S. Well-posedness of nonlocal boundary value problem for a system of loaded hyperbolic equations and an algorithm for finding its solution / D.S. Dzhumabaev // Journal of Mathematical analysis and Applications. — 2018. — 461. — No. 1. — P. 817–836. DOI: 10.1016/j.jmaa.2017.12.005.

8 Assanova A.T. On a Nonlocal Problem with Integral Conditions for the System of Hyperbolic Equations / A. T. Assanova//Differential equations. — 2018. — 54. — No.2. — P.201–214. DOI: 10.1134/S0012266118020076.

9 Орумбаева Н.Т. Об одном алгоритме нахождения решения периодической краевой задачи для системы квазилинейных гиперболических уравнений / Н.Т. Орумбаева // Сибир. электрон. мат. изв. — 2013. — 10. — С. 464–474. DOI: 10.17377/semi.2013.10.036.

10 Orumbayeva N. On a method of finding a solution of semi-periodic boundary value problem for hyperbolic equations / N. Orumbayeva, B. Shayakhmetova // AIP Conference Proceedings. — 2016. — Vol. 1759. — P. 020121. DOI: 10.1063/1.4959735.

11 Orumbaeva N.T. On solvability of non-linear semi-periodic boundary-value problem for system of hyperbolic equations / N.T. Orumbaeva//Russian Mathematics (Iz. VUZ). — 2016. — 60. — P. 23–37. DOI: 10.3103/ S1066369X16090036.

12 Orumbaeva N.T. Algorithms for finding a solution to the initial boundary value problems for differential equations in partial derivatives / N.T. Orumbaeva // Bulletin of the Karaganda University. Series Mathematics. — 2016. — 2. — No. 82. — P.107–112. ISSN 2518-7929 (Print), ISSN 2663–5011 (Online).

13 Orumbaeva N.T. On a Solution of a Nonlinear Semi-periodic Boundary Value Problem for a Differential Equation with Arbitrary Functions / N.T. Orumbayeva, G. Sabitbekova // Springer Proceedings in Mathematics & Statistics — 2017. — 216. — P. 158–163.

14 Orumbaeva N.T. A boundary value problem for nonlinear differential equation with arbitrary functions / N.T. Orumbayeva, G. Sabitbekova // Bulletin of the Karaganda University. Series Mathematics. — 2017. — 1. — No. 85. — P. 71–76. DOI: 10.31489/2017M1/71-76.

15 Kosmakova M.T. Problems of heat conduction with different boundary conditions in noncylindrical domains [Electronic resource] / M.T. Kosmakova, N.T. Orumbayeva, N.K. Medeubaev and Zh.M. Tuleutaeva // AIP Conference Proceedings. — 2018. — Vol.1997. — P.020071. — Access mode: https://aip. scitation.org/doi/abs/10.1063/1.5049065.

16 Assanova A.T. Well-posedness of a periodic boundary value problem for the system of hyperbolic equations with delayed argument/ A.T. Assanova, N.B. Iskakova and N.T. Orumbayeva.//Bulletin of the Karaganda University-Mathematics. — 2018. — 1. No. 89. — P.8–14. DOI: 10.31489 /2018M1/8-14.

17 Orumbayeva N.T. On the solvability of the duo-periodic problem for the hyperbolic equation system with a mixed derivative, N.T. Orumbayeva and A.B. Keldibekova // Bulletin of the Karaganda University. Series Mathematics. — 2019. — 1. — No. 93. — P. 59–71. DOI: 10.31489 /2019M1/59-71.

18 Assanova A.T. On the well-posedness of periodic problems for the system of hyperbolic equations with finite time delay, A.T. Assanova, N.B. Iskakova and N.T. Orumbayeva // Mathematical Methods in the Applied Sciences. — 2020. — 43. — No. 2. — P. 881–902. DOI: 10.1002/mma.5970.

19 Треногин В.А. Функциональный анализ: учеб. пос. для студ. вузов / В.А. Треногин. — М.: Наука, 1980. — 495 с.

А.Б. Кельдибекова

Аралас туындысы бар үшінші ретті дифференциалдық теңдеу үшін жартылай периодты шеттік есептің шешілуі

Мақала псевдопараболалық типтегі эволюциялық теңдеу үшін жартылай периодты шеттік есептің шешілуін зерттеуге арналған. Жоғары ретті дербес туындылы дифференциалдық теңдеулер үшін локальды емес есептерді көптеген авторлар зерттеген [1–4]. Мұндай есептерді зерттеуге деген қызығушылық олардың қолданбалы мағынасына байланысты туындады. Осындай есептерге күрделі топологиясы бар қатты кеуекті орта, бірінші кезекте топырақ және жер жатады. Сонымен бірге, мұндай теңдеулер дисперсиялық жүйелердегі ұзын толқындарды да сипаттауы мүмкін. Мақалада осындай есептерді шешу үшін жаңа функциялар енгізілген және параметризациялау әдісі қолданылған [5]. Онда үшінші реттік дифференциалдық теңдеу үшін шеттік есеп, қарапайым дифференциалдық теңдеулер жүйелер тобы үшін периодтық шеттік есебіне келтіріледі [6–18]. Жуық шешімді табудың жаңа конструктивтік алгоритмі ұсынылған және бастапқы есеп терминдері негізінде зерттеліп отырған есептің бірмәнді шешілуінің коэффициенттік белгілері алынған.

Кілт сөздер: дербес туынды теңдеу, үшінші ретті псевдопараболалық теңдеу, алгоритм, жуық шешім.

А.Б. Кельдибекова

Разрешимость полупериодической краевой задачи для дифференциального уравнения третьего порядка со смешанной производной

Статья посвящена исследованию разрешимости полупериодической краевой задачи для эволюционного уравнения типа псевдопараболических. Нелокальные задачи для дифференциальных уравнений с частными производными высокого порядка были исследованы многими авторами [1–4]. Определенный интерес к изучению данных задач вызван в связи с их прикладными значениями. К таким задачам относятся сильно пористые среды со сложной топологией, и, в первую очередь, почва и почвогрунт. Также такие уравнения могут описывать длинные волны в дисперсных системах. Для решения данной задачи в работе введены новые функции и применен метод параметризации [5]. Тогда краевая задача для дифференциального уравнения третьего порядка сводится к периодической краевой задаче для семейства систем обыкновенных дифференциальных уравнений [6–18]. Предложены новые конструктивные алгоритмы нахождения приближенного решения, и в терминах исходных данных получены коэффициентные признаки однозначной разрешимости исследуемой задачи.

Ключевые слова: уравнение в частных производных, псевдопараболическое уравнение третьего порядка, алгоритм, приближенное решение.

References

1 Nahushev, A.M. (1995). *Uravneniia matematicheskoi biolohii [Equations of mathematical biology]*. Moscow: Vysshaia shkola. Zbl 0991.35500 [in Russian].

2 Kozhanov, A.I. (1999). Composite Type Equations and Inverse Problems (Inverse and Iii-Posed Problems). *VSP: Utrecht*. DOI: 10.1515/9783110943276

3 Kozhanov, A.I. & Lukina, G.A. (2019). Pseudoparabolic and pseudohyperbolic equations in noncylindrical time domains. *Mathematical notes of NEFU, 3*, 26, 15–30. DOI: 10.25587/SVFU. 2019.17.12.002.

4 Kozhanov, A.I. & Pinigina, N.R. (2017). Boundary-value problems for some higher-order nonclassical differential equations. *Mathematical notes,* 101, 467–474. DOI: 10.1134/S0001434617030087.

5 Dzhumabaev, D.S. (1989). Priznaki odnoznachnoi razreshimosti lineinoi kraevoi zadachi dlia obyknovennykh differentsialnykh uravnenii [Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation]. *Zhurnal vychislitelnoi matematiki i matematicheskoi fiziki — Computational Mathematics and Mathematical Physics, 29, 1,* 34–46. DOI: 10.1016/0041-5553(89)90038-4 [in Russian].

6 Dzhumabaev, D.S. (2016). On one approach to solve the linear boundary value problems for Fredholm integro-differential equations. *Journal of Computational and Applied Mathematics, 294*, 342–357. DOI: 10.1016/j.cam.2015.08.023.

7 Dzhumabaev, D.S. (2018). Well-posedness of nonlocal boundary value problem for a system of loaded hyperbolic equations and an algorithm for finding its solution. *Journal of Mathematical analysis and Applications, 461*, 1, 817–836. DOI: 10.1016/j.jmaa.2017.12.005.

8 Assanova, A.T. (2018). On a Nonlocal Problem with Integral Conditions for the System of Hyperbolic Equations. *Differential equations, 54*, 2, 201–214. DOI: 10.1134/S0012266118020076.

9 Orumbaeva, N.T. (2013). Ob odnom alhoritme nakhozhdeniia resheniia periodicheskoi kraevoi zadachi dlia sistemy kvazilineinykh hiperbolicheskikh uravnenii [On an algorithm of finding periodical boundary value problem for system of the quasi-linear of hyperbolic equations]. *Siberian Electronic Mathematical Reports, 10*, 464–474 [in Rusian]. DOI: 10.17377/semi.2013.10.036.

10 Orumbaeva, N. & Shayakhmetova, B. (2016). On a method of finding a solution of semi-periodic boundary value problem for hyperbolic equations. *AIP Conference Proceedings, 1759* 020121. DOI: 10.1063/1.4959735.

11 Orumbaeva, N.T. (2016). On solvability of non-linear semi-periodic boundary-value problem for system of hyperbolic equations. *Russian Mathematics (Iz. VUZ), 9*, 26–41. DOI: 10.3103/S106 6369X16090036.

12 Orumbaeva, N.T. (2016). Algorithms for finding a solution to the initial boundary value problems for differential equations in partial derivatives. *Bulletin of the Karaganda University. Series Mathematics, 2*, 82, 107–112. ISSN 2518-7929 (Print), ISSN 2663–5011 (Online).

13 Orumbaeva, N.T. & Sabitbekova, G. (2017). On a Solution of a Nonlinear Semi-periodic Boundary Value Problem for a Differential Equation with Arbitrary Functions. *Springer Proceedings in Mathematics & Statistics, 216*, 158–163. Retrieved from https://doi.org/10.1007/978-3-319-67053-9_14

14 Orumbaeva, N.T. & Sabitbekova, G. (2017). A boundary value problem for nonlinear differential equation with arbitrary functions, *Bulletin of the Karaganda University. Series Mathematics, 1, 85,* 71–76. DOI: 10.31489/2017M1/71-76.

15 Kosmakova, M.T., Orumbayeva, N.T., Medeubaev N.K. & Tuleutaeva Zh.M. (2018). Problems of heat conduction with different boundary conditions in noncylindrical domains. textitAIP Conference Proceedings, V, 1997, 020071. Retrieved from https://aip.scitation.org/doi/abs/10.1063/1.5049065.

16 Assanova, A.T., Iskakova, N.B. & Orumbayeva, N.T. (2018). Well-posedness of a periodic boundary value problem for the system of hyperbolic equations with delayed argument. *Bulletin of the Karaganda University. Serise Mathematics, 1*, 89, 8—14. DOI: 10.31489/2018M1/8-14.

17 Orumbayeva, N.T. & Keldibekova, A.B. (2019). On the solvability of the duo-periodic problem for the hyperbolic equation system with a mixed derivative. *Bulletin of the Karaganda University. Series Mathematics, 1*, 93, 59—71. DOI:10.31489/2019M1/59-71.

18 Assanova, A.T., Iskakova, N.B. & Orumbayeva, N.T. (2020). On the well-posedness of periodic problems for the system of hyperbolic equations with finite time delay. *Mathematical Methods in the Applied Sciences, 43*, 2, 881–902 DOI: 10.1002/mma.5970.

19 Trenogin, V.A. (1980). *Funktsionalnyi analiz [Functional analysis]*. Moscow: Nauka [in Russian].