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Ali Çakmak[1], Yusuf Yaylı[2]

[1]*Department of Mathematics, Faculty of Sciences and Arts, Bitlis Eren University, Bitlis 13000, Turkey;*
[2]*Department of Mathematics, Faculty of Science, Ankara University, Ankara 06100, Turkey*
(E-mail: alicakmak@yahoo.com)

On the parallel surfaces of the non-developable surfaces

In the differential geometry of curves and surfaces, the curvatures of curves and surfaces are often calculated and results are given. In particular, the results given by using the apparatus of the curve-surface pair are important in terms of what kind of surface the surface indicates. In this study, some relationships between curvatures of the parallel surface pair (X, X^r) via structure functions of non-developable ruled surface $X(u, v) = a(u) + vb(u)$ are established such that $a(u)$ is striction curve of non-developable surface and $b(u)$ is a unit spherical curve in E^3. Especially, it is examined whether the non-developable surface X^r is minimal surface, linear Weingarten surface and Weingarten surface. X and its parallel X^r are expressed on the Helicoid surface sample. It is indicated on the figure with the help of SWP. Moreover, curvatures of curve-surface pairs (X, a) and (X^r, β) are investigated and some conclusions are obtained.

Keywords: parallel surfaces, non-developable ruled surface, striction line, Gaussian curvature, mean curvature, curvatures of curve-surface pair.

Introduction

The parallel surfaces have an important place in the theory of surfaces. A parallel surface can be defined as the locus of points at a non-zero constant distance throughout normal of surface from a regular surface [1].

A surface composed by a singly infinite system of straight lines is called a ruled surface. A developable ruled surface is a special ruled surface with the property that it has the same tangent plane at all points on one and the same straight line. We know that a ruled surface is a developable ruled surface if and only if its Gaussian curvature K is zero [2; 89]. If $K \neq 0$, the ruled surface is non-developable [3; 32]. In 3-dimensional Euclidean space, a regular curve is defined by its curvature κ and torsion τ and also a curve-surface pair is defined by its curvatures κ_g, κ_n and τ_g, where κ_g, κ_n and τ_g are geodesic curvature, asymptotic curvature and geodesic torsion, respectively. The relations between the curvatures of a curve-surface pair and the curvatures of the curve can be seen in many papers [4–8].

We denote a regular parameter surface with the parameters u and v in E^3 by $X(u, v)$ and a non-developable ruled surface by

$$X(u, v) = a(u) + vb(u), \tag{1}$$

where $b^2(u) = 1$ and the parameter u is the arc length parameter of $b(u)$ as a unit spherical curve in E^3. Here, if $a'(u).b'(u) = 0$, $a(u)$ is striction line of ruled surface [9-10].

In this paper, firstly, we obtain the parallel surface $X^r(u, v)$ of the non-developable ruled surface $X(u, v)$. Then, we calculate Gaussian and mean curvature of the parallel surface $X^r(u, v)$. Later, we determine relations between Gaussian and mean curvatures of parallel surface pair by means of structure functions of non-developable surface. Furthermore, we show that $X^r(u, v)$ is a Weingarten surface. Finally, we give some theorems and results by calculating geodesic curvature, asymptotic curvature and geodesic torsion of curve-surface pairs (X, a) and(X^r, β).

Preliminaries

Let $a : I \to X$ be a unit speed curve lying on X such that X is a regular surface in Euclidean 3-space. We know that the Frenet frame $\{T, N, B\}$ correspond at each point of the curve $a(u)$ because $a(u)$ is a space curve, where u is arc length parameter. Other than this frame, we can talk about frame called Darboux frame of $a(u)$ in E^3. The Darboux frame is denoted by $\{T, Y, n\}$ under the conditions that T is the unit tangent vector of $a(u)$, n is the unit normal of X and $Y = n \times T$.

Definition 2.1. The Darboux derivative formulas can be defined using the following matrix:

$$\begin{pmatrix} T' \\ Y' \\ n' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} T \\ Y \\ n \end{pmatrix},$$

where κ_g is defined as geodesic curvature, κ_n is defined as normal curvature and τ_g is defined as geodesic torsion. Furthermore, it is known that [11; 248]

$$\kappa_g = \left\langle a''(u), Y \right\rangle, \tag{2}$$

$$\kappa_n = \left\langle a''(u), n \right\rangle, \tag{3}$$

$$\tau_g = -\left\langle n', Y \right\rangle, \tag{4}$$

and also

a) $a(u)$ is a geodesic curve $\Leftrightarrow \kappa_g = 0$.

b) $a(u)$ is a asymptotic curve $\Leftrightarrow \kappa_n = 0$.

c) $a(u)$ is a line of curvature $\Leftrightarrow \tau_g = 0$.

Definition 2.2. Let X be a surface in E^3 with unit normal n. Parallel surface X^r of X is given by $X^r = \{P + rn_P : P \in X, r \in R \text{ and } r = constant\}$, where for $r \in R$, $f(P) = P + rn_P$ defines a new surface X^r. For all P on X, $n^{f(p)} = n^p$ [1] *Theorem 2.3.* Let (X, X^r) be a parallel surface pair. Suppose that the Gauss curvatures of X and X^r be denoted by K and K^r and the mean curvature of X and X^r be denoted by H and H^r, respectively. Then, we can write [12; 212]

$$K^r = \frac{K}{1 - 2rH + r^2K}, \tag{5}$$

$$H^r = \frac{H - rK}{1 - 2rH + r^2K}. \tag{6}$$

Definition 2.4. A ruled surface in E^3 may therefore be represented in the form

$$X(a, b) : I \times E \to E^3,$$

$$(u, v) \to X(a, b)(u, v) = a(u) + vb(u)$$

such that $a : I \to E^3$, $b : I \to E^3$ are differentiable transformations. Here, $a(u)$ is called base curve and $b(u)$ is called the director curve [13; 190]

As stated previously, Gauss curvature is zero, the ruled surface is developable ruled surface. Otherwise, the surface is non-developable [14].

Definition 2.5. Suppose that the non-developable ruled surface $X(u, v)$ is given by the equation (1) in E^3. Let $a(u)$ be the striction line of the $X(u, v)$ and $b(u)$ be a unit spherical curve, where u is the arc length parameter of $b(u)$. Then, if we write as $x(u) = b(u)$, $x'(u) = \alpha(u)$, and $y(u) = \alpha(u) \times x(u)$, the spherical Frenet formulas of the curve $b(u)$ can be given by

$$x'(u) = \alpha(u),$$

$$\alpha'(u) = -x(u) + k_g(u)y(u),$$

$$y'(u) = -k_g(u)\alpha(u),$$

where $k_g(u)$ is called the spherical curvature function and $\{x(u), \alpha(u), y(u)\}$ is called the spherical Frenet frame of $b(u)$ [9-10].

Definition 2.6. Suppose that $X(u,v)$ is given by the equation (1) in E^3 and $a(u)$ is the striction line of $X(u,v)$ under the condition $a'(u) = \lambda(u)x(u) + \mu(u)y(u)$. Then, the surface $X(u,v)$ can be given by the triple $\{k_g(u), \lambda(u), \mu(u)\}$ in E^3. Here, $k_g(u)$, $\lambda(u)$ and $\mu(u)$ are defined as structure functions of the surface $X(u,v)$ in E^3 [9-10].

Definition 2.7. Suppose that $X(u,v)$ is given by the equation (1) in E^3 and $a\,(u)$ is the striction line of $X(u,v)$ under the condition $a'(u) = \lambda(u)x(u) + \mu(u)y(u)$. Here, $\{\alpha(u), x(u) = b(u), y(u)\}$ is the spherical Frenet frame of $b(u)$. If $\lambda(u) \neq 0$, $X(u,v)$ is described as pitched ruled surface [9–10].

Let $X(u,v)$ be given by equation (1). In this case, the coefficients of the first fundamental form of $X(u,v)$

$$E = \lambda^2(u) + \mu^2(u) + \nu^2,$$

$$F = \lambda(u),$$

$$G = 1.$$

The unit normal of the surface $X(u,v)$ is

$$n = \frac{-\mu(u)\alpha(u) + vy(u)}{\sqrt{\mu^2(u) + v^2}}.$$

The second fundamental quantities of $X(u,v)$ are

$$e = \frac{-\left(\lambda(u) - k_g(u)\mu(u)\right)\mu(u) + \left(\mu'(u) + k_g(u)v\right)v}{\sqrt{\mu^2\,(u) + v^2}},$$

$$f = \frac{-\mu(u)}{\sqrt{\mu^2(u) + v^2}},$$

$$g = 0.$$

As a result of these calculations,

$$K(u,v) = \frac{-\mu^2(u)}{\left(\mu^2(u) + v^2\right)^2} \tag{7}$$

and

$$H(u,v) = \frac{k_g(u)v^2 + \mu'(u)v + k_g(u)\mu^2(u) + \lambda(u)\mu(u)}{2\sqrt{\left(\mu^2(u) + v^2\right)^3}}, \tag{8}$$

where K and H are the Gauss and mean curvature of $X(u,v)$, respectively [9-10].

Proposition 2.8. Suppose that the surface $X(u,v)$ is given by (1) such that $\lambda(u)$, $\mu(u)$ and $k_g(u)$ are the structure functions of $X(u,v)$. If λ, μ and k_g are constants, the surface $X(u,v)$ is a Weingarten surface [10]

The curvatures of the parallel surface pairs

Suppose that $X\,(u,v)$ is given by (1) in E^3. By definition of parallel surface, we obtain

$$X^r\,(u,v) = X\,(u,v) + r\,n\,(u,v)\,,$$

$$X^r(u,v) = a(u) + \nu b(u) + r\left[\frac{-\mu(u)\,\alpha(u) + vy(u)}{\sqrt{\mu^2(u) + v^2}}\right].$$

Theorem 3.1. Suppose that $X(u,v)$ is given by (1) in E^3 and $X^r(u,v)$ is parallel surface of the surface $X(u,v)$. Then, the relationships between Gauss and mean curvature of $X(u,v)$ and $X^r(u,v)$ are given, respectively, by

$$K^r = \frac{-\mu^2}{\left(\mu^2+v^2\right)^2 - r\sqrt{\mu^2+v^2}\left(k_g v^2 + \mu' v + k_g\mu^2 + \lambda\mu\right) - r^2\mu^2} \tag{9}$$

and

$$H^r = \frac{\sqrt{\mu^2+v^2}\left(k_g v^2 + \mu' v + k_g\mu^2 + \lambda\mu\right) + 2r\mu^2}{2(\mu^2+v^2)^2 - 2r\sqrt{\mu^2+v^2}\begin{pmatrix} k_g v^2 + \mu' v \\ + k_g\mu^2 + \lambda\mu \end{pmatrix} - 2r^2\mu^2}. \tag{10}$$

Proof. Combining the equations (5)-(8), we can easily obtain the equations (9)-(10).

Corollary 3.2. If $k_g = \lambda = 0$ and μ is a constant, $k_g v^2 + \mu' v + k_g\mu^2 + \lambda\mu = 0$. This mean that, $H = 0$. In this case, X is minimal but X^r is not minimal surface. Because, $H^r \neq 0$ under the conditions that $k_g = \lambda = 0$ and μ is a constant.

Corollary 3.3. X is not a linear Weingarten surface. Because, $a \neq 0$ and $b \neq 0$ are not constants satisfying $aH + bK = 1$. Similarly, X^r is also not linear Weingarten surface.

Example 3.4. Let us consider that $X(u,v) = a(u) + vb(u) = (v\cos u, v\sin u, u)$ is a helicoid surface. Here, we choose $a(u) = (0,0,u), b(u) = (\cos u, \sin u, 0) = x(u)$. Hence, we obtain

$$b'(u) = x'(u) = (-\sin u, \cos u, 0) = \alpha(u),$$

$$y(u) = \alpha(u) \times x(u) = (0,0,-1),$$

$$\alpha'(u) = (-\cos u, -\sin u, 0) = -x(u) + k_g y(u), \tag{11}$$

$$y'(u) = 0 = k_g\alpha(u). \tag{12}$$

From the equations (11)-(12), we find $k_g = 0$. Moreover,

$$a'(u) = (0,0,1) = \lambda x + \mu y = \lambda(\cos u, \sin u, 0) + \mu(0,0,-1) \tag{13}$$

and from the equation (13), we find $\lambda = 0$ and $\mu = -1$. Hence, we obtain unit surface normal as follows:

$$n = \frac{1}{\sqrt{1+v^2}}(-\sin u, \cos u, -v)$$

In this case, we can write

$$X^r(u,v) = (v\cos u, v\sin u, u) + \frac{r}{\sqrt{1+v^2}}(-\sin u, \cos u, -v),$$

where $X^r(u,v)$ is parallel surface of $X(u,v)$. For $r = 1$, the images of $X(u,v)$ and $X^r(u,v)$ are shown in Figure 1.

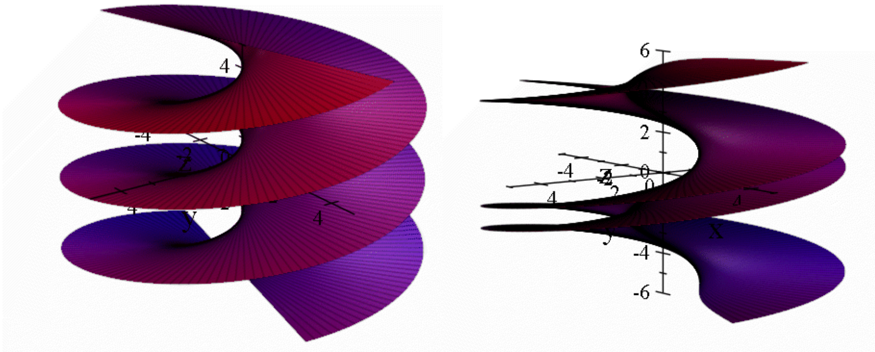

Figure 1. $X(u;v)$ and $X^r(u;v)$.

Theorem 3.5. Suppose that any non-developable ruled surface is given by (1) in E^3 such that $a'(u) = \lambda(u)x(u) + \mu(u)y(u)$, Then, $X^r(u,v)$ is a Weingarten surface if and only if λ, μ and k_g are constants.

Proof. It is known that if $X(u,v)$ is a Weingarten surface,

$$K_u H_v = K_v H_u, \tag{14}$$

where K and H are Gauss and mean curvature of $X(u,v)$, respectively. In this case, from the equations (9)–(10), we find the following partial derivatives with respect to u and v:

$$K_u^r = \frac{-\mu\left(\begin{array}{c}\left(2v^4-2\mu^4\right)\mu'\sqrt{\mu^2+v^2} \\ +r\left(\begin{array}{c}(v^3\mu+v\mu^3)\mu''+\left(-2v^3-v\mu^2\right)(\mu')^2 \\ +\left(\begin{array}{c}-2v^4k_g-v^2k_g\mu^2 \\ +k_g\mu^4-v^2\mu\lambda\end{array}\right)\mu' \\ +\left(\mu^2+v^2\right)\left(\left(\mu^2+v^2\right)k_g'+\lambda'\mu\right)\mu\end{array}\right)\end{array}\right)}{\sqrt{\mu^2+v^2}\left(r\sqrt{\mu^2+v^2}w-\mu^4+\left(r^2-2v^2\right)\mu^2-v^4\right)^2}, \tag{15}$$

$$K_v^r = \frac{-\mu^2\left(\begin{array}{c}3k_g\mu^2v+3k_gv^3-4\mu^2\sqrt{\mu^2+v^2}v \\ -4\sqrt{\mu^2+v^2}v^3+\mu'\mu^2+2\mu'v^2+\lambda\mu v\end{array}\right)}{\sqrt{\mu^2+v^2}\left(\sqrt{\mu^2+v^2}w-\mu^4+\left(r^2-2v^2\right)\mu^2-v^4\right)^2}, \tag{16}$$

$$H_u^r = \frac{1}{2}\left(\begin{array}{c}\left(\begin{array}{c}\sqrt{\mu^2+v^2}w \\ +2r\mu^2\end{array}\right)\left(\begin{array}{c}-r\sqrt{\mu^2+v^2}t-\frac{rw\mu\mu'}{\sqrt{\mu^2+v^2}} \\ +4\left(\mu^2+v^2\right)\mu\mu'-2r^2\mu\mu'\end{array}\right) \\ +\left(\begin{array}{c}\left(\frac{w\mu\mu'}{\sqrt{\mu^2+v^2}}+\sqrt{\mu^2+v^2}t+4r\mu\mu'\right) \\ \left(\left(\mu^2+v^2\right)^2-r\sqrt{\mu^2+v^2}w-r^2\mu^2\right)\end{array}\right)\end{array}\right), \tag{17}$$

$$H_v^r = \frac{1}{2}\left(\begin{array}{c}\left(\begin{array}{c}\frac{wv}{\sqrt{\mu^2+v^2}} \\ +\sqrt{\mu^2+v^2}\left(2k_gv+\mu'\right)\end{array}\right)\left(\begin{array}{c}\left(\mu^2+v^2\right)^2 \\ -r\sqrt{\mu^2+v^2}w-r^2\mu^2\end{array}\right) \\ +\left(\sqrt{\mu^2+v^2}w+2r\mu^2\right)\left(\begin{array}{c}4\left(\mu^2+v^2\right)v-\frac{rwv}{\sqrt{\mu^2+v^2}} \\ -r\sqrt{\mu^2+v^2}\left(2k_gv+\mu'\right)\end{array}\right)\end{array}\right), \tag{18}$$

where $w = \lambda\mu + v^2k_g + \mu^2k_g + v\mu'$, $t = \mu\lambda' + \lambda\mu' + 2\mu k_g\mu' + v^2{k_g}' + \mu^2{k_g}' + v\mu''$. From the equation (14), $K_u^rH_v^r - K_v^rH_u^r = 0$. Here, if we use the equations (15)-(18) and make the necessary calculations, we obtain that all the structure functions of $X(u,v)$ are constants.

Hence, from the Proposition 2.8. , we can write the following result:

Corollary 3.6. $X^r(u,v)$ is a Weingarten surface if and only if $X(u,v)$ is a Weingarten surface.

The curvatures of the parallel curve-surface pairs

The striction line $a(u)$ on $X(u,v)$ generates a Darboux frame by the vector fields $\{T,n,Y\}$, the unit tangent, the principal normal and their cross product, respectively. Hence, for $n=\frac{-\mu\alpha+\nu y}{\sqrt{\mu^2+v^2}}$

$$T=\lambda x+\mu y,$$

$$Y=n\times T=\mu^2 x-\lambda v\alpha-\lambda\mu y.$$

Using the equations (2)-(4) geodesic curvature of curve-surface pair (a,X)

$$\kappa_g=\left\langle a'',Y\right\rangle=\left\langle \lambda' x+(\lambda-\mu k_g)\,\alpha+\mu' y,\mu^2 x-\lambda\mu y\right\rangle,$$

$$\kappa_g=\mu\left(\lambda'\mu-\lambda\mu'\right), \tag{19}$$

asymptotic curvature of curve-surface pair (a,X)

$$\kappa_n=\left\langle a'',n\right\rangle=\left\langle \lambda' x+(\lambda-\mu k_g)\alpha+\mu' y,-\alpha\right\rangle$$

$$\kappa_n=\mu k_g-\lambda, \tag{20}$$

and geodesic torsion of curve-surface pair (a,X)

$$\tau_g=-\left\langle n',Y\right\rangle=-\left\langle x-k_g y,\mu^2 x-\lambda\mu y\right\rangle,$$

$$\tau_g=-\mu(\mu+\lambda k_g). \tag{21}$$

From the equations (19)–(21) and Definition 2.1, the following theorems can be written:

Theorem 4.1. Suppose that $X(u,v)$ is given by (1) in E^3 such that $a'(u)=\lambda x+\mu y$. $a(u)$ is geodesic curve if and only if $\frac{\lambda}{\mu}$ is a constant, where λ and μ are the structure functions of $X(u,v)$.

Proof. If the equation (19) equals to zero, $a(u)$ is a geodesic curve. In this case, we get

$$\kappa_g=\mu\left(\lambda'\mu-\lambda\mu'\right)=0.$$

Here, since $\mu\neq 0$, $\lambda'\mu-\lambda\mu'=0$. If we solve this differential equation, we obtain that $\frac{\lambda}{\mu}$ is a constant. This finishes the proof.

Theorem 4.2. Suppose that $X(u,v)$ is given by (1) in E^3 such that $a'(u)=\lambda x+\mu y$. $a(u)$ is asymptotic curve if and only if $k_g=\frac{\lambda}{\mu}$, where λ, μ and k_g are the structure functions of $X(u,v)$.

Proof. From the Definition 2.1., if the equation (20) equals to zero, $a(u)$ is asymptotic curve. In this case, we obtain

$$\kappa_n=\mu k_g-\lambda=0.$$

From here, we can easily get $k_g=\frac{\lambda}{\mu}$.

Theorem 4.3. Suppose that $X(u,v)$ is given by (1) in E^3 such that $a'(u) = \lambda x + \mu y$. $a(u)$ is line of curvature if and only if $k_g = -\frac{\mu}{\lambda}$, where λ, μ and k_g are the structure functions of $X(u,v)$.

Proof. From the Definition 2.1., if the equation (21) equals to zero, $a(u)$ is line of curvature. In this case, we can write

$$\tau_g = -\mu\,(\mu + \lambda k_g) = 0.$$

Since $\mu \neq 0$ in this last equation, we get $k_g = -\frac{\mu}{\lambda}$.

Now, the above calculations will be found for the parallel surface. By considering definition of parallel surface, image on parallel surface of the striction line $a(u)$ can be given by

$$\beta(u) = a(u) + r\,n.$$

In this case, we write Darboux frame elements T^r, Y^r, n^r of the parallel curve-surface pair

$$T^r = \frac{(1 + r(\lambda - \mu k_g))\,T + r(\mu + \lambda k_g)Y}{\sqrt{(1 + r(\lambda - \mu k_g))^2 + (r(\mu + k_g\lambda))^2}},$$

$$Y^r = \frac{(-r(\mu + \lambda k_g))\,T + (1 + r(\lambda - \mu k_g))\,Y}{\sqrt{(1 + r(\lambda - \mu k_g))^2 + (r(\mu + k_g\lambda))^2}},$$

$$n^r = n,$$

and also we obtain geodesic curvature, asymptotic curvature and geodesic torsion, respectively as following:

$$\kappa_g^r = \frac{\begin{array}{l} -\,r^2(\mu + k_g\lambda)\left[(\lambda' - \mu' k_g - \mu k'_g) - (\lambda'\mu - \mu'\lambda)(\mu + k_g\lambda)\right] \\ -\,[1 + r(\lambda - \mu k_g)]^2(\lambda'\mu - \lambda\mu') \\ -\,r(1 + r(\lambda - \mu k_g))(\mu' + k'_g\lambda + k_g\lambda') \end{array}}{\sqrt{(1 + r(\lambda - \mu k_g))^2 + r^2(\mu + k_g\lambda)^2}},$$

$$\kappa_n^r = -(\lambda - \mu k_g)\,(1 + r(\lambda - \mu k_g)) - r(\mu + k_g\lambda)^2,$$

$$\tau_g^r = \frac{-(\mu + k_g\lambda)}{\sqrt{(1 + r\,(\lambda - \mu k_g))^2 + r^2(\mu + k_g\lambda)^2}}.$$

Then, we have the following theorems:

Theorem 4.4. Suppose that $X(u,v)$ is given by (1) in E^3 such that $a'(u) = \lambda x + \mu y$. $a(u)$ is line of curvature if and only if the image on parallel surface of $a(u)$ is line of curvature $\left(\tau_g = 0 \Leftrightarrow \tau_g^r = 0\right)$.

Theorem 4.5. Suppose that $X(u,v)$ is given by (1) in E^3 such that $a'(u) = \lambda x + \mu y$. If $a(u)$ is an asymptotic curve, then

$$\kappa_g^r = \frac{(\lambda'\mu - \lambda\mu')\left(1 + r^2(\mu + \lambda k_g)^2\right) - r(\mu' + \lambda k'_g + k_g\lambda')}{\sqrt{1 + r^2(\mu + \lambda k_g)^2}},$$

$$\kappa_n^r = -r(\mu + \lambda k_g)^2,$$

$$\tau_g^r = \frac{-(\mu + \lambda k_g)}{\sqrt{1 + r^2(\mu + \lambda k_g)^2}}.$$

Theorem 4.6. Suppose that $X(u,v)$ is given by (1) in E^3 such that $a'(u) = \lambda x + \mu y$. If $a(u)$ is line of curvature, then,

$$\kappa_g^r = \pm(\lambda'\mu - \lambda\mu')\,(1 + r(\lambda - \mu k_g))\,,$$

$$\kappa_n^r = (-\lambda + \mu k_g)(1 + r\lambda - r k_g\mu),$$

$$\tau_g^r = \tau_g.$$

Theorem 4.6. immediately gives the following results:

Corollary 4.7. Let a non-developable ruled surface be given by (1) such that λ, μ and k_g are the structure functions and the curve $a(u)$ be both the striction line and the line of curvature. Then, the image on parallel surface of $a(u)$ is asymptotic curve if and only if $k_g = \frac{\lambda}{\mu}(\kappa_n = 0 \Leftrightarrow \kappa_n^r = 0)$.

Corollary 4.8. Let a non-developable ruled surface be given by (1) such that λ, μ and k_g are the structure functions and the curve $a(u)$ be both the striction line and the line of curvature. Then, the image on parallel surface of $a(u)$ is a geodesic curve if and only if $a(u)$ is a geodesic curve ($\frac{\lambda}{\mu} = constant$).

Theorem 4.9. Suppose that $X(u, v)$ is given by (1) in E^3 such that $a'(u) = \lambda x + \mu y$. If $\kappa_n = \frac{1}{r}$, then,

$$\kappa_g^r = \mp r(\lambda'\mu - \lambda\mu')(\mu + \lambda k_g),$$

$$\kappa_n^r = -r(\mu + \lambda k_g)^2,$$

$$\tau_g^r = -\frac{1}{r}.$$

Corollary 4.10. Let a non-developable ruled surface be given by (1) and λ, μ and k_g be the structure functions of this surface. For $\kappa_n = \frac{1}{r}$, the image on parallel surface of $a(u)$ is geodesic curve if and only if $a(u)$ is geodesic curve ($\frac{\lambda}{\mu} = constant$).

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А. Сакмак, Ю. Яйли

Жаймаланбайтын беттерге параллель беттер жайлы

Қисықтар мен беттердің дифференциалдық геометриясында қисықтар мен беттердің қисаюы көп қарастырылған және нәтижелер келтірілген. Атап айтқанда, қисық бет жұбының аппаратын қолдану арқылы алынған нәтижелер беттің қандай түрін нұсқайтындығында маңызды. Зерттеуде $X(u,v) = a(u) + vb(u)$ кеңейтілген сызықтық бетінің құрылымдық функциялары арқылы (X, X^r) жұптар беттерінің параллельдік қисықтықтарының арасындағы қандай да бір өзара байланыс $a(u)$ жаймаланбайтын беттің үйкелу қисығы, ал $b(u)$ E^3 В-тегі бірлік сфералық қисық болатындай етіп қойылған. Дербес жағдайда, X^r жаймаланбайтын беті минималды бет, Вейнгартен сызықтық беті және Вейнгартен беті бола ала ма, осы жағдай зерттелген. X және оның X^r параллелі геликоид бетінің образында келтірілген. Суретте бұл (SWP) беттік толқындарды қолдайтын плазма көмегімен келтірілген. Сонымен қатар, (X, a) және (X^r, β) қисық-бет жұбының қисықтықтары зерттелген және нәтижелер алынған.

Кілт сөздер: параллель беттер, жаймаланбайтын сызықтық бет, үйкелу сызығы, Гаусс қисығы, орташа қисықтық, қисық-бет жұбының қисықтығы.

А. Сакмак, Ю. Яйли

О параллельных поверхностях неразвертывающихся поверхностей

В дифференциальной геометрии кривых и поверхностей искривления кривых и поверхностей часто рассчитываются и даются результаты. В частности, результаты, полученные с использованием аппарата пары кривая — поверхность, важны с точки зрения того, на какого рода поверхность указывает поверхность. В этом исследовании некоторые взаимосвязи между кривизной параллельной поверхности пары (X, X^r) через структурные функции неразвертывающейся линейчатой поверхности $X(u,v) = a(u) + vb(u)$ устанавливаются таким образом, что $a(u)$ является кривой трения неразвертывающейся поверхности, а $b(u)$ — единичной сферической кривой в E^3. В частности, исследуется, является ли неразвертывающаяся поверхность X^r минимальной поверхностью, линейной поверхностью Вейнгартена и поверхностью Вейнгартена. X и ее параллель X^r выражены на образце поверхности геликоида. На рисунке это показано с помощью плазмы, поддерживаемой поверхностными волнами (SWP). Кроме того, исследованы кривизны пар кривая — поверхность (X, a) и (X^r, β) и получены некоторые результаты.

Ключевые слова: параллельные поверхности, неразвертывающаяся линейчатая поверхность, линия трения, кривизна Гаусса, средняя кривизна, кривизна пары кривая – поверхность.

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