DOI 10.31489/2020M2/40-45

UDC 517.51

S. Bitimkhan, D.T. Alibieva

*Ye.A. Buketov Karaganda State University, Kazakhstan*
*(E-mail: bsamat10@mail.ru)*

# Generalization of one theorem of F. Riesz to some other spaces

It is known from the analysis course that in order a function to serve as an undefined integral of a summable function, it is necessary and sufficient that it be absolutely continuous. Therefore, it is natural to raise the question of the characteristic of a function which is an undefined integral of the function included in $L_p, p > 1$ . The answer is well known theorem of F.Riesz concerning the conditions of representability of a given function in the form of an integral with variable upper limit on the functions of Lebesgue spaces. In the one-dimensional and multi-dimensional case, many mathematicians have generalized this theorem for Lebesgue and Orlicz spaces. In this work we will prove theorem of F.Riesz for other functional spaces. Generalization of the theorem of F.Riesz to the case when subintegral function from the weighted Lebesgue spaces is obtained. Also, we prove a necessary condition for the above representation of a function $f \in L_p\varphi(L_p)$.

*Keywords:* function, functional spaces, integral, theorem of F.Riesz, weighted Lebesgue space.

## *1 Introduction*

In the theory of functions, the following theorem of F.Riesz is known (see, for example, [1], page 225): for the function $F(x)$ $(a \leq x \leq b)$ to be representable as

$$F(x) = C + \int_a^x f(t)dt,$$

where $f(t) \in L_p(p > 1)$, it is necessary and sufficient that for every subdivision $[a;b]$ by points $a = x_0 < x_1 < < ... < x_n = b$ the inequality was executed

$$\sum_{i=1}^{n-1} \frac{|F(x_{i+1}) - F(x_i)|^p}{(x_{i+1} - x_i)^{p-1}} \leq K < \infty,$$

where $K$ does not depend on the way $[a;b]$ is subdivided.

In the future, a number of authors have proposed various generalizations of this theorem [2–5]. We will prove this theorem for spaces, which are defined below.

Let $W(x)$ is a non-negative function. Through $L_{p,W}[a;b]$ we will designate the space of all measurable by Lebesgue on $[a;b]$ functions $f$, for which

$$\|f\|_{p,W} = \left(\int_a^b |f(x)|^p W(x)dx\right)^{\frac{1}{p}} < +\infty, 1 \leq p < +\infty.$$

We assume that the function $W(x)$ satisfies $A_p$-condition [6] (or $W \in A_p$), if

$$\sup_{I \subset [a;b]} \left[\frac{1}{|I|}\int_I W(x)dx\right]^{\frac{1}{p}} \cdot \left[\frac{1}{|I|}\int_I (W(x))^{-\frac{1}{p-1}}dx\right]^{\frac{1}{p'}} < +\infty, \frac{1}{p} + \frac{1}{p'} = 1.$$

Let the function $\varphi(t)$ satisfies the following conditions [7]:
a) $\varphi(t)$ is an even, non-negative, non-decreasing on $[0, +\infty)$;
b) $\varphi(t^2) \leq C\varphi(t)$, $t \in [0, \infty)$, $C \geq 1$;
c) $\frac{\varphi(t)}{t^{\varepsilon}} \downarrow$ on $(0, +\infty)$ for some $\varepsilon > 0$.

Measurable on $[a;b]$ function $f \in L_p\varphi(L_p)$, if

$$\int\limits_a^b |f(x)|^p \cdot \varphi\left(|f(x)|^p\right) dx < +\infty.$$

*2 The results and their proofs*

We proved the following theorem.

*Theorem 1.* For the function $F(x)(a \leq x \leq b)$ to be representable as:

$$F(x) = C + \int\limits_a^x f(t)dt, \tag{1}$$

where $f(t) \in L_{p,W}[a;b](p > 1)$, it is necessary and sufficient that for every subdivision $[a;b]$ by points $a = x_0 < x_1 < ... < x_n = b$ the inequality was executed

$$\sum_{k=0}^{n-1} \frac{|F(x_{k+1}) - F(x_k)|^p}{\left(\int\limits_{x_k}^{x_{k+1}} W^{\frac{-p'}{p}}(t)dt\right)^{p-1}} \leq K, \tag{2}$$

where $K$ does not depend on the way $[a;b]$ is subdivided.

*Proof.* Let's prove the necessity of theorem. Suppose inequality (1) holds. Then by Holder's inequality:

$$|F(x_{k+1}) - F(x_k)| = \left|\int\limits_{x_k}^{x_{k+1}} f(t)dt\right| = \left|\int\limits_{x_k}^{x_{k+1}} f(t)W^{\frac{1}{p}}(t)W^{\frac{-1}{p}}(t)dt\right| \leq$$

$$\leq \left(\int\limits_{x_k}^{x_{k+1}} |f(t)|^p W(t)dt\right)^{\frac{1}{p}} \left(\int\limits_{x_k}^{x_{k+1}} W^{\frac{-p'}{p}}(t)dt\right)^{\frac{1}{p'}}, \forall k = 0, ..., n-1,$$

where $p' = \frac{p}{p-1}$.

We obtain:

$$\frac{|F(x_{k+1}) - F(x_k)|^p}{\left(\int\limits_{x_k}^{x_{k+1}} W^{\frac{-p'}{p}}(t)dt\right)^{p-1}} \leq \int\limits_{x_k}^{x_{k+1}} |f(t)|^p W(t)dt, k = 0, ..., n-1.$$

Therefore, folding of these inequalities, we will get:

$$\sum_{k=0}^{n-1} \frac{|F(x_{k+1}) - F(x_k)|^p}{\left(\int\limits_{x_k}^{x_{k+1}} W^{\frac{-p'}{p}}(t)dt\right)^{p-1}} \leq \int\limits_a^b |f(t)|^p W(t)dt = K.$$

The necessity of condition (2) is proved.

Now we prove sufficiency of the conditions (2). First of all, note that inequality (2) can only increase if we discard some components of its left part. Therefore, for any finite system of mutually not impose intervals $(a_k, b_k), (k = 1, 2, ..., n)$ contained in $[a, b]$ will be

$$\sum_{k=1}^{n} \frac{|F(b_k) - F(a_k)|^p}{\left(\int\limits_{a_k}^{b_k} W^{\frac{-p'}{p}}(t)dt\right)^{p-1}} \leq K.$$

But because of the Holder's inequality holds:

$$\sum_{k=1}^{n}|F(b_k)-F(a_k)|=\sum_{k=1}^{n}\frac{|F(b_k)-F(a_k)|}{\left(\int\limits_{a_k}^{b_k}W^{\frac{-p'}{p}}(t)dt\right)^{\frac{p-1}{p}}}\cdot\left(\int\limits_{a_k}^{b_k}W^{\frac{-p'}{p}}(t)dt\right)^{\frac{p-1}{p}}\leq$$

$$\leq\left(\sum_{k=1}^{n}\frac{|F(b_k)-F(a_k)|^p}{\left(\int\limits_{a_k}^{b_k}W^{\frac{-p'}{p}}(t)dt\right)^{p-1}}\right)^{\frac{1}{p}}\cdot\left(\sum_{k=1}^{n}\int\limits_{a_k}^{b_k}W^{\frac{-p'}{p}}(t)dt\right)^{\frac{1}{p'}}\leq\sqrt[p]{K}\cdot\left(\int\limits_{a}^{b}W^{\frac{-p'}{p}}(t)dt\right)^{\frac{1}{p'}}.$$

From the last inequality implies absolute continuity of the function $F(x)$. Then this function is representable in the form (1), where $f$ is some summable function. It remains to prove that $f(x)\in L_{p,W}[a;b]$.

With this goal in expanding the segment $[a;b]$ into equal parts by the points $x_k^{(n)}=a+\frac{k}{n}(b-a), k=0,1,...,n$ let us introduce the function $f_n(t)$, believing:

$$f_n(t)=\frac{F(x_{k+1}^{(n)})-F(x_k^{(n)})}{x_{k+1}^{(n)}-x_k^{(n)}}\cdot\chi_k(t),$$

where $\chi_k(t)$ - a characteristic function of the interval $\left(x_k^{(n)},x_{k+1}^{(n)}\right)$.

At the division points we believe $f_n\left(x_k^{(n)}\right)=0, k=0,1,...,n$.

It is easy to see that almost everywhere will be:

$$\lim_{n\to\infty}|f_n(t)|^p\,W(t)=|f(t)|^p\,W(t).$$

Hence, by Fatou's theorem:

$$\int\limits_{a}^{b}|f(t)|^p\,W(t)dt\leq\sup_n\left\{\int\limits_{a}^{b}|f_n(t)|^p\,W(t)dt\right\}.$$

For $f_n(t)$, since $W\in A_p$ we get the following inequality:

$$\int\limits_{a}^{b}|f_n(t)|^p\,W(t)dt=\sum_{k=0}^{n-1}\int\limits_{x_k{}^{(n)}}^{x_{k+1}^{(n)}}\frac{\left|F(x_{k+1}^{(n)})-F(x_k^{(n)})\right|^p}{\left(x_{k+1}^{(n)}-x_k^{(n)}\right)^p}\cdot W(t)dt=$$

$$=\sum_{k=0}^{n-1}\frac{\left|F(x_{k+1}^{(n)})-F(x_k^{(n)})\right|^p}{\left(x_{k+1}^{(n)}-x_k^{(n)}\right)^p}\cdot\int\limits_{x_k{}^{(n)}}^{x_{k+1}^{(n)}}W(t)dt\leq$$

$$\leq C\cdot\sum_{k=0}^{n-1}\frac{\left|F(x_{k+1}^{(n)})-F(x_k^{(n)})\right|^p}{\left(x_{k+1}^{(n)}-x_k^{(n)}\right)^p}\cdot\frac{\left(x_{k+1}^{(n)}-x_k^{(n)}\right)^p}{\left(\int\limits_{x_k{}^{(n)}}^{x_{k+1}^{(n)}}W^{\frac{-p'}{p}}(t)dt\right)^{\frac{p}{p'}}}=$$

$$= C \cdot \sum_{k=0}^{n-1} \frac{\left|F(x_{k+1}^{(n)}) - F(x_k^{(n)})\right|^p}{\left(\int\limits_{x_k{}^{(n)}}^{x_{k+1}^{(n)}} W^{\frac{-p'}{p}}(t)dt\right)^{p-1}} \leq C \cdot K.$$

And it became,

$$\int\limits_a^b |f(t)|^p \cdot W(t)dt < +\infty,$$

i.e. $f(t) \in L_{p,W}[a;b]$.

The theorem is proved.

*Remark.* In the case of $W(t) \equiv 1$, the theorem of F.Riesz follows from the proved theorem.

Now we will prove the necessary condition of representation (1), from the function of $L_p\varphi(L_p)$ space.

*Theorem 2.* If $F(x)$ can be represented as

$$F(x) = C + \int\limits_a^x f(t)dt,$$

where $f \in L_p\varphi(L_p)$, then for every subdivision of $[a;b]$ by points $a = x_0 < x_1 < ... < x_n = b$ the following inequality holds:

$$\sum_{k=0}^{n-1} \frac{|F(x_{k+1}) - F(x_k)|^p}{\left(\int\limits_{x_k}^{x_{k+1}} \varphi^{\frac{-1}{p-1}}\left(|f(t)|^p\right)dt\right)^{p-1}} \leq K.$$

*Proof.* Let $F(x)$ represented as

$$F(x) = C + \int\limits_a^x f(t)dt.$$

Then by Holder's inequality we get:

$$|F(x_{k+1}) - F(x_k)| = \left|\int\limits_{x_k}^{x_{k+1}} f(t)dt\right| = \left|\int\limits_{x_k}^{x_{k+1}} f(t)\varphi^{\frac{1}{p}}\left(|f(t)|^p\right)\varphi^{\frac{-1}{p}}\left(|f(t)|^p\right)dt\right| \leq$$

$$\leq \left(\int\limits_{x_k}^{x_{k+1}} |f(t)|^p \varphi\left(|f(t)|^p\right)dt\right)^{\frac{1}{p}} \left(\int\limits_{x_k}^{x_{k+1}} \varphi^{\frac{-1}{p-1}}\left(|f(t)|^p\right)dt\right)^{\frac{p-1}{p}}, \forall k = 0, ..., n-1.$$

Hence, for all $k = 0, ..., n-1$ we obtain

$$\frac{|F(x_{k+1}) - F(x_k)|^p}{\left(\int\limits_{x_k}^{x_{k+1}} \varphi^{\frac{-1}{p-1}}\left(|f(t)|^p\right)dt\right)^{p-1}} \leq \int\limits_{x_k}^{x_{k+1}} \varphi^{\frac{-1}{p-1}}\left(|f(t)|^p\right)dt.$$

Now adding up these inequalities, because $f \in L_p\varphi(L_p)$ we get

$$\sum_{k=0}^{n-1} \frac{|F(x_{k+1}) - F(x_k)|^p}{\left(\int\limits_{x_k}^{x_{k+1}} \varphi^{\frac{-1}{p-1}}\left(|f(t)|^p\right)dt\right)^{p-1}} \leq K.$$

*Remark.* In the case of $\varphi(t) \equiv 1$, the necessary part of the theorem of F.Riesz follows from the proved theorem.

References


1 Натансон И.П. Теория функций вещественной переменной / И.П. Натансон. — М.: Гостехиздат, 1950. — 480 с.

2 Медведев Ю.Т. Обобщение одной теоремы Ф. Рисса / Ю.Т. Медведев // Успехи математических наук. — 1953. — Т. VIII. — Вып. 6 (58). — С. 115-118.

3 Кудрявцев Л.Д. О $p$-вариации отображений и суммируемости степеней производной Радона-Никодима / Л.Д. Кудрявцев // Успехи математических наук. — 1955. — Т. 10, № 2. — С. 167-174.

4 Fuglede B. On a theorem of F.Riesz / B. Fuglede // Mathematica Scandinavica. — 1955. — Vol. 3. — P.283-302.

5 Kakochashvili G. On the theorem of F. Riesz in variable Lebesgue space / G. Kakochashvili, Sh. Zviadadze // Transactions of A.Razmadze Mathematical Institute. — 2016. — Vol. 170, Issue 1. — P. 56-61.

6 Muckenhoupt B. Weighted norm inequalities for the Hardy maximal functions / B. Muckenhoupt // Transactions of the American Mathematical Society. — 1972. — Vol. 162. — P. 207–226.

7 Bitimkhan S. Hardy-Littlewood theorem for series with general monotone coefficients / S. Bitimkhan // Bulletin of the Karaganda university-mathematics. — 2018. — 2(90). — P. 43-48.


С. Битимхан, Д.Т. Алибиева

## Ф. Рисстің бір теоремасын кейбір басқа кеңістіктерге жалпылау

Функцияның қосындыланатын функциядан анықталмаған интеграл түрінде болуы үшін оның абсолютті үзіліссіз болуы қажетті және жеткілікті екені анализ курсынан белгілі. Осыған байланысты $L_p, p > 1$ кеңістігіне кіретін функцияның анықталмаған интегралы болатын функцияның сипаттамалық белгісі туралы сұрақ қойылуы заңды. Жауабы Ф.Рисстің берілген функцияның Лебег кеңістігі функциясынан алынған жоғарғы шегі айнымалы интеграл арқылы жазылу шартына қатысты теоремасында. Бірөлшемді және көпөлшемді жағдайларда көптеген математиктер бұл теореманы Лебег және Орлич кеңістіктерінде талдады. Осы жұмыста авторлар Ф. Рисс теоремасын басқа функционалдық кеңістіктер үшін дәлелдеген. Ф.Рисс теоремасының жалпыламасы интеграл астындағы функция салмақты Лебег кеңістігінен болған жағдайға алынды. Сонымен бірге, жоғарыдағы интегралдық жазылудың қажетті шарты $f \in L_p\varphi(L_p)$ функциясы үшін дәлелденді.

*Кілт сөздер:* функция, функциялық кеңістіктер, интеграл, Ф. Рисс теоремасы, салмақты Лебег кеңістігі.

С. Битимхан, Д.Т. Алибиева

## Обобщение одной теоремы Ф. Рисса на некоторые другие пространства

Из курса анализа известно, что для того чтобы функция служила неопределенным интегралом суммируемой функции, необходимо и достаточно, чтобы она была абсолютно непрерывна. В связи с этим естественно поставить вопрос о характеристическом признаке функции, являющейся неопределенным интегралом функции, входящей в $L_p, p > 1$. Ответом служит известная теорема Ф.Рисса, касающаяся условий представимости заданной функции в виде интеграла с переменным верхним пределом от функции пространства Лебега. В одномерном и многомерном случаях многие математики обобщили эту теорему для пространств Лебега и Орлича. В настоящей работе авторами предпринята попытка доказать теорему Ф.Рисса для других функциональных пространств. Получено обобщение теоремы Ф.Рисса на случай, когда подынтегральная функция из весового пространства Лебега. Также доказано необходимое условие сказанного выше представления от функции $f \in L_p\varphi(L_p)$.

*Ключевые слова:* функция, функциональные пространства, интеграл, теорема Ф.Рисса, весовое пространство Лебега.

References

1 Natanson, I.P. (1950). *Funktsii veshchestvennoi peremennoi [The theory of functions of a real variable]*. Moscow: Hostekhizdat [in Russian].

2 Medvedev, Yu.T. (1953). Obobshchenie odnoi teoremy F.Rissa [Generalization of one theorem of F. Riesz]. *Uspekhi matematicheskikh nauk — Successes of mathematical Sciences, VIII, 6(58)*, 115–118 [in Russian].

3 Kudriavtsev, L.D. (1955). O $p$-variatsii otobrazhenii i summiruemosti stepenei proizvodnoi Radona-Nikodima [About $p$-variation mappings and the summability of degree of the Radon-Nikodim derivative]. *Uspekhi matematicheskikh nauk —Successes of mathematical Sciences, 10, 2*, 167-174 [in Russian].

4 Fuglede, B. (1955). On a theorem of F.Riesz. *Mathematica Scandinavica, 3*, 283-302.

5 Kakochashvili, G.,& Zviadadze, Sh. (2016). On the theorem of F. Riesz in variable Lebesgue space. *Transactions of A. Razmadze Mathematical Institute, 170, 1*, 56-61.

6 Muckenhoupt, B. (1972). Weighted norm inequalities for the Hardy maximal functions. *Transactions of the American Mathematical Society, 162*, 207-226.

7 Bitimkhan, S. (2018). Hardy-Littlewood theorem for series with general monotone coefficients. *Bulletin of the Karaganda University.Mathematics series, 2(90)*, 43-48.