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The Cauchy problem for the Navier-Stokes equations[1]

Ch. Fefferman in his works two problems for Navier-Stokes equations are set out: one of them is the Cauchy problem and he considers «only those solutions that are infinitely smooth functions are physically meaningful». In this article, the author received positive answers for the above problem of Ch. Fefferman. He proved the uniqueness and existence of smooth solutions of the Cauchy problem for the Navier-Stokes equations. The ratio between the pressure P and the kinetic energy density E, previously established by the author. is taken as the basis. As a result of in-depth studies of the Cauchy problem for the Navier-Stokes equations, it is shown that E is a bounded, continuous function that satisfies the Laplace equation and has continuous first-order derivatives with respect to t and all kinds of second derivatives with respect to the spatial variables $\mathbf{x}$ and is a regular harmonic function in the space R_3. An explicit form of E is found with the help of which the Navier-Stokes equations are reduced to a system of linear parabolic equations and the solutions are written out by the Fourier transform that are infinitely differentiable with respect to t and $\mathbf{x}$. The systems of equations for the curl-vector are found. Proven uniqueness, the existence of infinite smoothness. An estimate is obtained linking the curl-vectors with the Reynolds number.

Keywords: The Cauchy problem for the Navier-Stokes equations, the uniqueness and existence of smooth solutions of the Navier-Stokes equations, the harmonicity of the kinetic energy density, the equations for the vortex vector, the Cauchy problem for the curl-vector equations, the uniqueness and existence of smooth solutions of the equations curl-vectors

0.1 Some introductory information

Unsolved problems in the theory of Navier-Stokes equations homogeneous liquids are given in [1–2], [3] and others.

In a number of works [4]–[6] of the author, the results of some explored. The substantiation of the simplest principle is given in [4] maximum for three-dimensional Navier-Stokes equations, which allows get a positive answer to an unresolved problem O.A. Ladyzhenskaya in [1, 2].

In [5], based on the properties of solutions of the Navier-Stokes equations, the relation between the pressure and squared modulus of the velocity vector. Based on what the uniqueness of the weak and the existence of strong solutions to a problem from a class of functions

$$C((0,T];W_2^1(G)\cup C^1((0,T];W_2^2(G))$$

for the Navier-Stokes equations in bounded domain of G in whole time $t\in[0,T],\forall T<\infty$.

The justification of the method was given in [6] splitting for solving the Navier-Stokes equations. Shown the compactness of the solution sequence, thereby the existence of strong solutions to the three-dimensional Navier-Stokes equations in whole time.

The original Navier-Stokes equations are not equations of type Cauchy-Kovalevskaya. Using ratio $(P=-|\mathbf{U}|^2)\vee(P=0)$ from [5] the system of equations (1a) can be reduced to the Cauchy-Kovalevskaya type. We will study the Navier-Stokes equations (1a) taking into account the relation $P=-|\mathbf{U}|^2$, preserving the condition of incompressibility of the fluid.

The Cauchy problem for the Navier-Stokes equations with respect to the velocity vector $\mathbf{U}=(U_1,U_2,U_3)$ in the domain $Q=(0,\infty)\times R_3$ it will be written in the form [5]:

$$\frac{\partial\mathbf{U}}{\partial t}-\mu\Delta\mathbf{U}+(\mathbf{U},\nabla)\mathbf{U}-2\nabla E=\mathbf{f}(t,\mathbf{x}),\ \nabla\cdot\mathbf{U}=0, \tag{1a}$$

[1]The work was done on the personal initiative of the author.

$$\mathbf{U}(0,\mathbf{x}) = \mathbf{\Phi}(\mathbf{x}), \tag{1b}$$

where $\mathbf{x} \in R_3$; $E = \frac{1}{2}|\mathbf{U}|^2$; $t \in (0,\infty)$.

Known [1] orthogonal decomposition $\mathbf{L}_2(Q) = \mathbf{G}(Q) \oplus \mathring{\mathbf{J}}(Q)$, moreover, the elements $\mathring{\mathbf{J}}(Q)$ at $\forall t$belong to $\mathring{\mathbf{J}}(R_3)$, and the elements $\mathbf{G}(Q)$ belong to the subspace $\mathbf{G}(R_3)$; $\mathring{\mathbf{J}}(R_3)$ — the space of solenoidal vectors, and $\mathbf{G}(R_3)$ consists of $\nabla\eta$, where η is a unique function in R_3. $\mathbf{L}_\infty(Q)-$ subspace $\mathbf{C}(Q)$. $W_p^k(B_R)$ is the Sobolev space.

In the plane $t = 0$, we introduce the ball B_R (*imaginary, of course, since in the case of the Cauchy problem a homogeneous incompressible fluid fills all spaces* R_3) of radius $R \gg 1$ with center at origin of coordinates.

Input $\mathbf{f}$ and $\mathbf{\Phi}$ problems (1) satisfy the requirements:

$$\textbf{i)}\, \mathbf{f}(t,\mathbf{x}) \in \mathbf{C}^\infty(Q) \cap \mathring{\mathbf{J}}(Q), \Big|\frac{\partial^\gamma \mathbf{f}(t,\mathbf{x})}{\partial t^\gamma}\Big| \le \mathrm{g}_{\gamma\kappa}(1+\mathrm{t})^{-\kappa} \wedge \Big|\frac{\partial^\alpha \mathbf{f}(\mathrm{t},\mathbf{x})}{\partial \mathrm{x}_1^{\alpha_1}\partial \mathrm{x}_2^{\alpha_2}\partial \mathrm{x}_3^{\alpha_3}}\Big| \le \mathrm{d}_{\alpha\kappa}(1+|\mathbf{x}|)^{-\kappa};$$

$$\textbf{ii)}\, \mathbf{\Phi}(\mathbf{x}) \in \mathbf{C}^\infty(R_3) \cap \mathring{\mathbf{J}}(R_3), \ \Big|\frac{\partial^\alpha \mathbf{\Phi}(\mathbf{x})}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}\Big| \le q_{\alpha\kappa}(1+|\mathbf{x}|)^{-\kappa}, \alpha = \alpha_1+\alpha_2+\alpha_3,$$

where $\alpha_i \in \{0,1,...,\alpha\}, \gamma$, $\kappa-$ positive integers. $\mathrm{g}_{\gamma\kappa}, d_{\alpha\kappa}, q_{\alpha\kappa}$ -positive constants. □

0.2 On the harmonically of the kinetic energy density E

Theorem 1. If the input data of the problem (1) satisfies the requirements **i)**, **ii)**, then for the solutions of the problem (1) the estimate

$$\|\mathbf{U}\|_{C(0,\infty;L_\infty(R_3))} \le q_{04}\|\mathbf{\Phi}\|_{\mathbf{L}_\infty(R_3)} + d_{2,4}\|\mathbf{f}\|_{C(0,\infty;L_\infty(R_3))} \equiv A_1, \tag{2}$$

$$\big\|E\big\|_{C(0,\infty;L_\infty(R_3))} \le A_1, \quad E = \frac{1}{2}|\mathbf{U}|^2, \ d_{2,4} = \mathrm{g}_{02}\mathrm{q}_{04}. \tag{3}$$

Proof. We write a formula from vector algebra

$$(\mathbf{U},\nabla)\mathbf{U} - \nabla E = [rot\mathbf{U},\mathbf{U}]$$

using this formula of the equation (1a) we rewrite

$$\frac{\partial \mathbf{U}}{\partial t} - \mu\Delta\mathbf{U} - \frac{1}{2}\nabla|\mathbf{U}|^2 = -[rot\mathbf{U},\mathbf{U}] + \mathbf{f}(t,\mathbf{x}). \tag{4}$$

We multiply the equation (4) by the vector function $\mathbf{U} \ne 0$, then, taking into account the property $[rot\mathbf{U},\mathbf{U}] \perp \mathbf{U}$ we get

$$\frac{\partial \mathbf{U}}{\partial t} - \mu\Delta\mathbf{U} - \nabla E = \mathbf{f}(t,\mathbf{x}). \tag{5}$$

Acting by the operator div on (5), we have

$$\Delta E = 0. \tag{6}$$

Lemma 1. There is a relation

$$-\big(\Delta\mathbf{U},\mathbf{U}\big) \ge 0. \tag{7}$$

Proof. By painting ΔE and doing a little counting, we find

$$\Delta E = div\nabla\big(\frac{1}{2}|\mathbf{U}|^2\big) = \big(\Delta\mathbf{U},\mathbf{U}\big) + \sum_{\alpha=1}^{3}\big(\nabla U_\alpha\big)^2 = 0.$$

Whence the inequality (7) follows.

Multiply the equation (4) by a vector function $p|\mathbf{U}|^{2(p-1)}\mathbf{U}$ and taking into account the property $[rot\mathbf{U},\mathbf{U}] \perp \mathbf{U}$ integrate over domain of B_R

$$\frac{1}{2}\frac{d}{dt}\int\limits_{B_R}|\mathbf{U}|^{2p}\,\mathbf{dx} - p\mu\int\limits_{B_R}\big(\Delta\mathbf{U},\mathbf{U}\big)|\mathbf{U}|^{2(p-1)}\mathbf{dx} -$$

$$-\frac{1}{2}\int\limits_{B_R}\mathbf{U}\nabla|\mathbf{U}|^{2p}\mathbf{dx} = p\int\limits_{B_R}(|\mathbf{U}|^{2(p-1)}\mathbf{U}\mathbf{f}\mathbf{dx}. \tag{8}$$

Each term (8) is simplified accordingly. When evaluating the second term in the left-hand side, we take into account (7). Third $\int\limits_{B_R} \mathbf{U}\nabla|\mathbf{U}|^{2p}\mathbf{dx} = 0$, due to the orthogonality [1] of the spaces $\mathring{\mathbf{J}}(B_R)$ and $\mathbf{G}(B_R)$. The right-hand side, estimated by Holder inequality, we get:

$$\frac{1}{2}\frac{d}{dt}\int\limits_{B_R} |\mathbf{U}|^{2p}d\mathbf{x} \le p\Big(\int\limits_{B_R} |\mathbf{U}|^{2p}d\mathbf{x}\Big)^{\frac{2p-1}{2p}}\Big(\int\limits_{B_R} |\mathbf{f}|^{2p}d\mathbf{x}\Big)^{\frac{1}{2p}}. \tag{9}$$

Both parts (9), dividing by a positive value $p\Big(\int\limits_{B_R} |\mathbf{U}|^{2p}d\mathbf{x}\Big)^{\frac{2p-1}{2p}}$, we have

$$\frac{d}{dt}\Big(\int\limits_{B_R} |\mathbf{U}|^{2p}d\mathbf{x}\Big)^{\frac{1}{2p}} \le \Big(\int\limits_{B_R} |\mathbf{f}|^{2p}d\mathbf{x}\Big)^{\frac{1}{2p}}.$$

Choosing an arbitrary $t \in (0,\infty)$ and integrating the last time inequality ranging from 0 to t, find

$$\Big(\int\limits_{B_R} |\mathbf{U}(t,\mathbf{x})|^{q}d\mathbf{x}\Big)^{\frac{1}{q}} \le \Big(\int\limits_{B_R} |\mathbf{\Phi}(\mathbf{x})|^{q}d\mathbf{x}\Big)^{\frac{1}{q}} + \int\limits_0^t\Big(\int\limits_{B_R} |\mathbf{f}(\tau,\mathbf{x})|^{q}d\mathbf{x}\Big)^{\frac{1}{q}}d\tau, \forall q = 2p, p \in N.$$

Or

$$\|\mathbf{U}(t)\|_{L_q(B_R)} \le \|\mathbf{\Phi}(\mathbf{x})\|_{L_q(B_R)} + \int\limits_0^t \|\mathbf{f}(\tau)\|_{L_q(B_R)}d\tau,$$

since it is inequality valid for any q, we put $q = \infty$ and take into account the property **i**) of the well-known vector function

$$|\mathbf{f}| \le \mathrm{g}_{0\kappa}(1+\mathrm{t})^{-\kappa},\ \kappa = 2$$

then

$$\|\mathbf{U}(t)\|_{L_\infty(B_R)} \le \|\mathbf{\Phi}\|_{L_\infty(B_R)} + \sup_{t\ge 0}\|\mathbf{f}(t)\|_{L_\infty(B_R)},\ t \in (0,\infty),$$

as you can see, the right-hand side is independent of the time t and the inequality holds for all $t \in (0,\infty)$, thereby the left side continuous in t, i.e.

$$\|\mathbf{U}(t)\|_{C(0,\infty;L_\infty(B_R))} \le \|\mathbf{\Phi}\|_{L_\infty(B_R)} + \|\mathbf{f}\|_{C(0,\infty;L_\infty(B_R))}.$$

From where, using the input properties **i**), **ii**) we have

$$\|\mathbf{U}(t)\|_{C(0,\infty;L_\infty(B_R))} \le \frac{8\pi}{3}(1 - \frac{R^2}{(1+R)^3})\Big(\|\mathbf{\Phi}\|_{L_\infty(B_R)} + \|\mathbf{f}\|_{C(0,\infty;L_\infty(B_R))}\Big).$$

Hence, for $R \to \infty$ we arrive at the proof inequalities (2), (3) of the theorem 1. □

Next, we differentiate the equations (5) with respect to time t, and initial conditions for it are found from the systems of equations themselves (5), т. е.

$$\mathbf{U}_t(0,\mathbf{x}) = \mu\Delta\mathbf{\Phi}(\mathbf{x}) + \frac{1}{2}\nabla|\mathbf{\Phi}|^2 + \mathbf{f}(0,\mathbf{x}) \equiv \mathbf{\Phi_1}(\mathbf{x}),$$

then we have the extended Cauchy problem for $\mathbf{U}_t$,

$$\frac{\partial\mathbf{U}_t}{\partial t} - \mu\Delta\mathbf{U}_t - \nabla E_t = \mathbf{f}_t(t,\mathbf{x}), \tag{10a}$$

$$\mathbf{U}_t(0,\mathbf{x}) = \mathbf{\Phi}_1(\mathbf{x}). \tag{10b}$$

Problem (10) is no different from problems (5), (1b), only in place of the vector functions $\mathbf{U}$ and the functions E stand for them corresponding derivatives $\mathbf{U}_t$ и E_t.

Multiply the equation (10a) by the vector function $p|\mathbf{U}_t|^{2(p-1)}\mathbf{U}_t$ and integrate over the ball B_R, then

$$\frac{1}{2}\frac{d}{dt}\int\limits_{B_R}|\mathbf{U}_t|^{2p}\,\mathbf{dx}-p\mu\int\limits_{B_R}\big(\Delta\mathbf{U}_t,\mathbf{U}_t\big)|\mathbf{U}_t|^{2(p-1)}\mathbf{dx}-$$

$$-\frac{1}{2}\int\limits_{B_R}\mathbf{U}_t\nabla|\mathbf{U}_t|^{2p}\mathbf{dx}=p\int\limits_{B_R}|\mathbf{U}_t|^{2(p-1)}\mathbf{U}_t\mathbf{f}_t\mathbf{dx}. \tag{11}$$

Proof. We denote $\mathbf{v}=\mathbf{U}_t$.
Lemma 2. There is an inequality (see [1])

$$-\big(\Delta\mathbf{v},\mathbf{v}\big)\geq\lambda_1(\mathbf{v},\mathbf{v}), \tag{12}$$

since the operator $-\Delta$ in the finite domain B_R positive definite, i.e. $-\big(\Delta\mathbf{v},\mathbf{v}\big)=\lambda^2(\mathbf{v},\mathbf{v})$, where $\lambda_1=\min\lambda^2$.

Each term (11) simplify accordingly. The second term in the left-hand side is estimated taking into account (12) using the following inequality chains:

$$-p\mu\int\limits_{B_R}\big(\Delta\mathbf{U}_t,\mathbf{U}_t\big)|\mathbf{U}_t|^{2(p-1)}\mathbf{dx}\geq-p\mu\sup_{t\geq0}\||\mathbf{U}_t|^{2(p-1)}\|_{L_\infty(B_R)}\int\limits_{B_R}\big(\Delta\mathbf{U}_t,\mathbf{U}_t\big)\mathbf{dx}\geq$$

$$\geq p\mu\lambda_1\sup_{t\geq0}\||\mathbf{U}_t|^{2(p-1)}\|_{L_\infty(B_R)}\int\limits_{B_R}\big(\mathbf{U}_t,\mathbf{U}_t\big)\mathbf{dx}\geq0.$$

Third term $\int\limits_{B_R}\mathbf{U}_t\nabla|\mathbf{U}_t|^{2p}\mathbf{dx}=0$ due to the orthogonality of spaces $\mathring{\mathbf{J}}(B_R)$ and $\mathbf{G}(B_R)$. The right side is estimated by inequality Holder:

$$\frac{1}{2}\frac{d}{dt}\int\limits_{B_R}\big|\mathbf{U}_t\big|^{2p}d\mathbf{x}\leq p\Big(\int\limits_{B_R}\big|\mathbf{U}_t\big|^{2p}d\mathbf{x}\Big)^{\frac{2p-1}{2p}}\Big(\int\limits_{B_R}\big|\mathbf{f}_t\big|^{2p}d\mathbf{x}\Big)^{\frac{1}{2p}}.$$

From where, arguing as well as in the previous case, we come to the statement:
Theorem 2. If the input data to the original problem (1) satisfies the requirements **i**), **ii**), then for the solutions of the problem (10) the following estimates are valid:

$$\|\,\mathbf{U}\,\|_{\mathbf{C}^1(0,\infty;\mathbf{L}_\infty(R_3))}\leq q_{2,4}\,\|\,\mathbf{\Phi_1}\,\|_{\mathbf{L}_\infty(R_3)}+d_{2,4}\,\|\,\mathbf{f}\,\|_{\mathbf{C}^1(0,\infty;\mathbf{L}_\infty(R_3))}\equiv A_2,$$

$$\big\|E\big\|_{\mathbf{C}^1(0,\infty;\mathbf{L}_\infty(R_3))}\leq A_2,\ d_{2,4}=\mathrm{g}_{2,2}\mathrm{q}_{04}.$$

Next, we introduce the differential operator

$$D^\alpha\cdot=\frac{\partial^\alpha\cdot}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}},\ \alpha=\alpha_1+\alpha_2+\alpha_3,\ \alpha=\overline{1,3};\ \alpha_i\in\{0,1,2,3\},$$

For example, when $\alpha=1,\ D\cdot=\frac{\partial\cdot}{\partial x_1}\vee\frac{\partial\cdot}{\partial x_2}\vee\frac{\partial\cdot}{\partial x_3}$. Acting by the operator D on the problem (10) we obtain the extended Cauchy problem with respect to vector functions $\mathbf{U}_{tx_i}$:

$$\frac{\partial D\mathbf{U}_t}{\partial t}-\mu\Delta D\mathbf{U}_t-\nabla DE_t=D\mathbf{f}_t(t,\mathbf{x}), \tag{13a}$$

$$D\mathbf{U}_t(0,\mathbf{x})=D\mathbf{\Phi}_1(\mathbf{x}), \tag{13b}$$

Theorem 3. If the input of the problem (1) satisfies the requirements **i**), **ii**), then for solutions to the problem (13) the following estimates are valid:

$$\|\mathbf{U}\|_{C^1(0,\infty;\mathbf{W}^1_\infty(B_R))}\leq\|\mathbf{\Phi}_1\|_{\mathbf{W}^1_\infty(B_R)}+\|\mathbf{f}\|_{C^1(0,\infty;\mathbf{W}^1_\infty(B_R))}\equiv A_3.$$

$$\left\|E\right\|_{C^1(0,\infty;\mathbf{W}^1_\infty(B_R))} \leq A_3. \tag{14}$$

Proof. We denote $\mathbf{v} = D\mathbf{U}_t$. Multiply the equation (13a) by a vector function $p|D\mathbf{U}_t|^{2(p-1)}D\mathbf{U}_t$ and integrate domains B_R and simplify each term of the result, as in the proof of theorem 2, the second term from the left side taking into account (12), and the third term $\int\limits_{B_R} D\mathbf{U}_t\nabla|D\mathbf{U}_t|^{2p}\mathbf{dx} = 0$, due to the orthogonality of spaces $\mathring{\mathbf{J}}(B_R)$ and $\mathbf{G}(B_R)$. We estimate the right-hand side by Holder's inequality and in the end we get the estimate:

$$\frac{1}{2}\frac{d}{dt}\int\limits_{B_R} \left|D\mathbf{U}_t\right|^{2p} d\mathbf{x} \leq p\Big(\int\limits_{B_R} \left|D\mathbf{U}_t\right|^{2p} d\mathbf{x}\Big)^{\frac{2p-1}{2p}}\Big(\int\limits_{B_R} \left|D\mathbf{f}_t\right|^{2p} d\mathbf{x}\Big)^{\frac{1}{2p}}. \tag{15}$$

The inequality (15) is no different from (9), only in the places of the function under the integrals, in this case there are their derivatives, that is, $D\mathbf{U}_t$. Therefore, arguing literally, as after the inequality (9) of the theorem 1, we find the estimates (14). The theorem 3 is proved.
Corollary 1.

$$\|\mathbf{U}(t,\mathbf{x})\|_{\mathbf{C}^1(0,\infty;C(R_3))} \leq q_{2\kappa},\ q_{2\kappa} - const, \tag{16a}$$

$$\left\|E\right\|_{C^1(0,\infty;C(R_3))} \leq q_{2\kappa}. \tag{16b}$$

Proof. From the estimate (14), using embedding theorems Sobolev [7; 64], we find

$$\|\mathbf{U}\|_{\mathbf{C}^1(0,\infty;\mathbf{C}(B_R))} \leq d_1\|\mathbf{\Phi}\|_{\mathbf{C}(B_R)} + d_2\|\mathbf{f}\|_{\mathbf{C}^1(0,\infty;\mathbf{C}(B_R))}, \tag{17}$$

where $d_1 = \frac{M\|\mathbf{\Phi}\|_{\mathbf{W}^1_\infty(B_R)}}{\|\mathbf{\Phi}\|_{\mathbf{C}(B_R)}},\ d_2 = \frac{M\|\mathbf{f}\|_{C^1(0,\infty;\mathbf{W}^1_\infty(B_R)}}{\|\mathbf{f}\|_{C^1(0,\infty;\mathbf{C}(B_R))}}$, M- constant of the embedding theorem. From the inequality (17), taking into account the requirement **i**), **ii**) for input data, we find

$$\|\mathbf{U}(t,\mathbf{x})\|_{\mathbf{C}^1(0,\infty;C(B_R))} \leq q_{2\kappa}\frac{8\pi}{3}(1 - \frac{R^2}{(1+R)^3}) = d_{1,4}, \tag{18}$$

$$\left\|E\right\|_{C^1(0,\infty;\mathbf{C}(B_R))} \leq d_{1,4},\ \kappa = 4.$$

From here, passing to the limit at $R \to \infty$ we come to inequalities (16). The corollary 1 is proved.

Further, acting by the operator D^α on the problem (10) sequentially for $\alpha = 2, 3$ we get extended Cauchy problems with respect to a vector function $\mathbf{U}_{tx_ix_j},\ \mathbf{U}_{tx_ix_jx_k}$:

$$\frac{\partial D^\alpha \mathbf{U}_t}{\partial t} - \mu\Delta D^\alpha \mathbf{U}_t - \nabla D^\alpha E_t = D^\alpha \mathbf{f}_t(t,\mathbf{x}), \tag{19a}$$

$$D^\alpha \mathbf{U}_t(0,\mathbf{x}) = D^\alpha \mathbf{\Phi}_1(\mathbf{x}), \quad \alpha = 2, 3. \tag{19b}$$

Theorem 4. If the input to the problem (1) satisfies the requirements **i**), **ii**), then for the solutions of the extended problems (19), the estimates:

$$\|\mathbf{U}\|_{C^1(0,\infty;\mathbf{W}^\alpha_\infty(B_R))} \leq \|\mathbf{\Phi}_1\|_{\mathbf{W}^\alpha_\infty(B_R)} + \|\mathbf{f}\|_{C^1(0,\infty;\mathbf{W}^\alpha_\infty(B_R))} \equiv A_\alpha, \left\|E\right\|_{C^1(0,\infty;\mathbf{W}^\alpha_\infty(B_R))} \leq A_\alpha.\ \alpha = 2, 3, \tag{20}$$

From estimates (20), using embedding theorems and the requirement **i**), **ii**) for the input, we obtain the inequalities are similar (17), (18), then moving from there to the limit as $R \to \infty$ we find the estimates (21), (22) in Corollary 2:
Corollary 2.

$$\Big(\|\mathbf{U}(t,\mathbf{x})\|_{\mathbf{C}^1(Q)} \leq q_{3\kappa}\Big) \wedge \Big(\left\|E\right\|_{C^1(Q)} \leq q_{3\kappa}\Big),\ \alpha = 2,\ q_{m\kappa} - const. \tag{21}$$

$$\Big(\|\mathbf{U}(t,\mathbf{x})\|_{\mathbf{C}^1(0,\infty;C^2(R_3))} \leq q_{4\kappa}\Big) \wedge \Big(\left\|E\right\|_{C^1(0,\infty;C^2(R_3))} \leq q_{4\kappa}\Big),\ \alpha = 3. \tag{22}$$

Note that the number of all possible derivatives of the third the order of the vector function $D^3\mathbf{U}_t$ in spatial $\mathbf{x}$ is equal to ten.

As a result, the following main

Theorem 5. From the theorems 1–4 and Corollaries 1, 2 followed by boundedness, continuity and continuity of the first time derivative t vector functions $\mathbf{U}$ and kinetic energy density E, as well as the continuity of various derivatives of the first and second orders in the spatial variables $\mathbf{x}$ and satisfies the Laplace equation (6) for all $t \in (0, \infty)$, thus the function E is regular harmonic function in the finite domain B_R. From the general theory harmonic functions (h.f.) [8], [9] it follows that h.f. It has derivatives of any order and according to the statements proved to The function E also belongs to this class. Then in the domain B_R for each $t \in (0, \infty)$ based on the corollary of the Poisson formula and Harnack inequalities positive harmonic function E is constant over the spatial variables $\mathbf{x}$ for every $t \in (0, \infty)$.

0.3 On the existence and uniqueness of smooth solutions of the Navier-Stokes equations

From the theorem 5 it follows that the harmonic function $E(t, \mathbf{x})$ is constant inside the ball B_R right up to spherical ball surface ∂B_R, i.e.

$$E(t, \mathbf{x})\big|_{\partial B_R} = \frac{1}{2}(1+t)^{-\kappa}|\mathbf{\Phi}(R)|^2, \tag{23}$$

where $\kappa-$positive integer. Then the harmonic function E can be determined from the Dirichlet problem for the Laplace equation in the exterior of the sphere ∂B_R of radius R with constant boundary condition (23):

$$\Delta E(t, \mathbf{x}) = 0, \qquad E(t, \mathbf{x})\big|_{\partial B_R} = \frac{1}{2}(1+t)^{-\kappa}|\mathbf{\Phi}(R)|^2, \quad \forall t \in (0, \infty).$$

It is known [8; 231] that the solution to this problem can be written with using the Poisson formula:

$$E(t, \mathbf{x}) = \frac{|\mathbf{\Phi}(R)|^2}{8\pi(1+t)^\kappa} \int\limits_{\partial B_R} \frac{\rho^2 - R^2}{Rr^3} d(\partial B_R), \;\; R < \rho < \infty. \tag{24}$$

Hence, since $\rho > R$ we find the function E

$$E(t, \mathbf{x}) = \frac{|\mathbf{\Phi}(R)|^2}{2(1+t)^\kappa}\frac{R}{\rho}, \quad \rho = |\mathbf{x}|,$$

which is a continuous harmonic function of the form:

$$E(t, \mathbf{x}) = \begin{cases} \frac{|\mathbf{\Phi}(R)|^2}{2(1+t)^\kappa}, & \mathbf{x} \in \bar{B}_R, \\ \\ \frac{|\mathbf{\Phi}(R)|^2}{2(1+t)^\kappa}\frac{R}{\rho}, & \mathbf{x} \in R_3 \backslash B_R, \ \forall t \in (0, \infty), \end{cases}$$

which has continuous derivatives of all orders outside the sphere B_R.

Where from

$$\nabla E(t, \mathbf{x}) = \begin{cases} 0, & \mathbf{x} \in \bar{B}_R, \\ \\ c(t)\nabla(\frac{1}{\rho}), & \mathbf{x} \in R_3 \backslash B_R, \ \forall t \in (0, \infty), \end{cases} \tag{25}$$

where $c(t) = \frac{|\mathbf{\Phi}(R)|^2 R}{2(1+t)^\kappa}$.

Now from a non-linear system of Navier-Stokes equations (5), taking into account (25), we arrive at a linear system equations of parabolic type, i. e., to a system of disengaged heat equations with known right-hand sides

$$\mathbf{f}^{\mathrm{g}}(t,\mathbf{x}) = \begin{cases} \mathbf{f}, \quad \mathbf{x} \in \bar{B}_R, \\ \\ \mathbf{f} + c\nabla(\frac{1}{\rho}),\ \mathbf{x}\in R_3\backslash B_R,\ \forall t \in (0,\infty), \end{cases}$$

then the Cauchy problem for the obtained systems of equations taking into account initial conditions (1b) can be written as:

$$\frac{\partial \mathbf{U}}{\partial t} - \mu\Delta\mathbf{U} = \mathbf{f}^{\mathrm{g}}(t,\mathbf{x}), \tag{26a}$$

$$\mathbf{U}(0,\mathbf{x}) = \mathbf{\Phi}(\mathbf{x}). \tag{26b}$$

Where do we get the uniqueness solution to the problem (26), using Poisson formula obtained and justified by the Fourier transform for the heat equation, for example, in [8]:

$$U_\alpha(t,\mathbf{x}) = (2\sqrt{\pi})^3 \int\limits_0^t \int\limits_{R_3} \frac{1}{(t-\tau)^{\frac{3}{2}}} \exp\big(-\frac{r^2}{4\mu(t-\tau)}\big) f^{\mathrm{g}}_\alpha(\tau,\mathbf{y})d\mathbf{y}d\tau +$$

$$+\frac{1}{(2\sqrt{(\pi t)})^3} \int\limits_{R_3} \exp\big(-\frac{r^2}{4\mu t}\big)\Phi_\alpha(\mathbf{y})d\mathbf{y},\ r = |\mathbf{x}-\mathbf{y}|,\ \alpha = 1,2,3. \tag{27}$$

For $t > 0$ the function $U_\alpha(t,\mathbf{x})$ is infinite differentiable with respect to t and spatial variables $\mathbf{x}$ and that all derivatives can be obtained using differentiation, the Poisson formula (27) under the sign integral. □

0.4 On the estimation of the curl vector of a problem (1)

Multiply the Navier-Stokes equations (1a) by $2\mathbf{U}$ and integrate over the domain B_R

$$\frac{d}{dt}\int\limits_{B_R} |\mathbf{U}|^2 d\mathbf{x} - 2\mu\int\limits_{B_R}(\Delta\mathbf{U},\mathbf{U})d\mathbf{x} - \int\limits_{B_R}\mathbf{U}\nabla E d\mathbf{x} = 2\int\limits_{B_R}\mathbf{U}\mathbf{f}d\mathbf{x}. \tag{28}$$

From where we transform the second term on the left with integration by parts

$$-2\mu\int\limits_{B_R}(\Delta\mathbf{U},\mathbf{U})d\mathbf{x} = 2\mu\int\limits_{B_R}\sum_{\alpha=1}^3(\nabla U_\alpha)^2 d\mathbf{x} - \mu\int\limits_{\partial B_R}\frac{\partial}{\partial\mathbf{n}}\sum_{\alpha=1}^3 U_\alpha^2 d\mathbf{x} =$$

$$2\mu\int\limits_{B_R}\sum_{\alpha=1}^3(\nabla U_\alpha)^2 d\mathbf{x} - 2\mu\int\limits_{\partial B_R}\frac{\partial E}{\partial\mathbf{n}}d\mathbf{x} = 2\mu\int\limits_{B_R}\sum_{\alpha=1}^3(\nabla U_\alpha)^2 d\mathbf{x},$$

because $\int\limits_{\partial B_R}\frac{\partial E}{\partial\mathbf{n}}d\mathbf{x} = 0$ by harmonic property functions E, where ∂B_R is the spherical surface of the ball B_R (imaginary, of course). Third term $\int\limits_{B_R}\mathbf{U}\nabla E d\mathbf{x} = 0$ by virtue of orthogonality of spaces $\mathring{\mathbf{J}}(B_R)$ and $\mathbf{G}(B_R)$. The right side (28) is estimated by Cauchy-Bunyakovsky inequality and as a result we get:

$$\frac{d}{dt}\int\limits_{B_R}|\mathbf{U}|^2 d\mathbf{x} + 2\mu\int\limits_{B_R}\sum_{\alpha=1}^3(\nabla U_\alpha)^2 d\mathbf{x} \leq 2\Big(\int\limits_{B_R}|\mathbf{U}|^2 d\mathbf{x}\Big)^{\frac{1}{2}}\Big(\int\limits_{B_R}|\mathbf{f}|^2 d\mathbf{x}\Big)^{\frac{1}{2}}. \tag{29}$$

From here

$$\frac{d}{dt}\int\limits_{B_R}|\mathbf{U}|^2 d\mathbf{x} \leq 2\Big(\int\limits_{B_R}|\mathbf{U}|^2 d\mathbf{x}\Big)^{\frac{1}{2}}\Big(\int\limits_{B_R}|\mathbf{f}|^2 d\mathbf{x}\Big)^{\frac{1}{2}}.$$

Whence it follows that

$$\frac{d}{dt}\parallel \mathbf{U}(t)\parallel_{L_2(B_R)} \leq \parallel \mathbf{f}(t)\parallel_{L_2(B_R)}.$$

We integrate the last inequality ranging from 0 to t and will find

$$\sup_{t>0}\parallel \mathbf{U}(t)\parallel_{L_2(B_R)} \leq \parallel \mathbf{\Phi}\parallel_{L_2(B_R)} + \sup_{t>0}\parallel \mathbf{f}(t)\parallel_{L_2(B_R)} \equiv A_4. \tag{30}$$

Now, integrating (29) over $t \in (0,\infty)$ and taking into account (30), we find

$$\int\limits_0^t \sum_{\alpha=1}^{3}\|\nabla U_\alpha(\tau)\|^2_{L_2(B_R)} d\tau \leq \frac{1}{\mu}\Big(\parallel \mathbf{\Phi}\parallel^2_{L_2(B_R)} + A_4 \sup_{t>0}\|\mathbf{f}(t)\|_{L_2(B_R)}\Big). \tag{31}$$

Lemma 3. Occurs

$$\|rot\mathbf{U}(t)\|^2_{L_2(B_R)} = \sum_{\alpha=1}^{3}\|\nabla U_\alpha(t)\|^2_{L_2(B_R)}, \forall t \in (0,\infty). \tag{32}$$

Proof. Follows from identity

$$\sum_{\alpha=1}^{3}\frac{\partial \mathbf{U}}{\partial x_\alpha}\nabla U_\alpha = \sum_{\alpha=1}^{3}(\nabla U_\alpha)^2 - (rot\mathbf{U})^2.$$

It suffices to integrate this identity over the domain B_R with orthogonality of spaces $\mathring{\mathbf{J}}(B_R)$ and $\mathbf{G}(B_R)$. From (31), using (32),we obtain an estimate for the curl vector

$$\int\limits_0^t \|rot\mathbf{U}(\tau)\|^2_{L_2(B_R)} d\tau \leq Re\Big(\parallel \mathbf{\Phi}\parallel^2_{L_2(B_R)} + A_4 \sup_{t>0}\|\mathbf{f}(t)\|_{L_2(B_R)}\Big). \tag{33}$$

where $Re-$ is the Reynolds number. Hence it is not difficult to notice that with the $Re \to \infty$ curl vector is destroyed. □

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Ә.Ш. Ақыш (Ақышев)

Навье-Стокс теңдеулеріне Коши есебі

Ch. Feffermann жұмыстарында Навье-Стокс теңдеулеріне (НСТ) екі есеп қойылған, оның біреуі Коши есебі және ол «физикалық тұрғыдан ойластырылған тек шексіз тегіс функциялар болып табылатын шешімдер» деп тұжырымдайды. Автордың осы мақаласында Ch. Feffermannның жоғарыдағы аталған есебіне оң жауап алынған. Навье-Стокс теңдеулері үшін Коши есебінің жалқы шексіз тегіс шешуінің барлығы дәлелденген. Нәтижесінде автордың ертеректе көрсеткен, қысым мен кинетикалық энергияның арасындағы байланысқа негізделген. Навье-Стокс теңдеулері үшін Коши есебін тереңірек зерттеу нәтижесінде E функциясының тұйық үзіліссіздігі Лаплас теңдеуін қанағаттандырады және t бойынша бірінші, ал кеңістік айнымалылары $\mathbf{x}$ бойынша екінші туындыларының барлығының үзіліссіздігі көрсетіліп және R_3 кеңістігінде регуляр гармониялық функция екендігі көрсетілген. $E-$нің айқын түрі табылып, оның көмегімен Навье-Стокс теңдеулері жылдамдық векторының құраушылары бойынша сызықты параболалық теңдеулерге келтіріліп, Фурье түрлендіруінің әдісімен есептің t және кеңістік айнымалылары $\mathbf{x}$ бойынша шексіз тегіс дәл шешуі табылған. Құйын векторының Рейнольдс санымен байланыстыратын бағалау алынған.

Кілт сөздер: Навье-Стокс теңдеулері үшін Коши есебі, кинетикалық энергия тығыздығының R_3 кеңістігінде регуляр гармониялығы, құйын векторын Рейнольдс санымен байланыстыратын бағалау.

А.Ш. Акыш (Акишев)

Задача Коши для уравнений Навье-Стокса

В работах Ch. Feffermana ставятся две задачи для уравнений Навье-Стокса: одной из них является задача Коши, и он считает «физически осмысленными только те решения, которые являются бесконечно гладкими функциями». В данной статье автор получил положительные ответы на упомянутую выше задачу Ch. Feffermana. Им доказаны единственность и существование гладких решений задачи Коши для уравнений Навье-Стокса. За основу взято соотношение между давлением P и плотностью кинетической энергии E, ранее установленное автором. В результате углубленных исследований задачи Коши для уравнений Навье-Стокса показано, что E — ограниченная, непрерывная функция, удовлетворяющая уравнению Лапласа, имеющая непрерывные производные первого порядка по t и всевозможные вторые производные по пространственным переменным $\mathbf{x}$ и являющаяся регулярной гармонической функцией в пространстве R_3. Найден явный вид E, с помощью которого уравнения Навье-Стокса сведены к системе линейных параболических уравнений и выписаны решения преобразованием Фурье, бесконечно дифференцируемые по t и $\mathbf{x}$. Получена оценка, связывающая векторвихря с числом Рейнольдса.

Ключевые слова: задача Коши для уравнений Навье-Стокса, гармоничность плотности кинетической энергии, единственность и существование гладких решений уравнений Навье-Стокса, оценка, связывающая векторвихря с числом Рейнольдса.