DOI 10.31489/2019M4/75-80
UDC 510.67

A.R. Yeshkeyev, M.T. Omarova

Ye.A. Buketov Karaganda State University, Kazakhstan
(E-mail: omarovamt_963@mail.ru)

Companions of (n_1, n_2)-Jonsson theory

In given work are considered model-theoretical properties of companions of (n_1, n_2)-Jonsson theory. Also were considered a communications between center and (n_1, n_2)-Jonsson theory. Herewith considered theories is perfect in the sense of the existence of appropriate model companion. In given article introduced new concepts: (n_1, n_2)-Jonsson theory, $D - \alpha$-model companion. New results are shown with respect to model companions of α-Jonsson theory and 1-perfect 1-Jonsson theory.

Keywords: Jonsson theory, model complete, n-model complete, nearly n-model complete, $D - \alpha$-model companion, model completion, (n_1, n_2)-Jonsson theory.

In the work [1] were determined generalized Jonsson theories and in the language of the introduced definitions were given a descriptions of all generalized Jonsson theories of Boolean algebras. In [2] were defined the concepts of 1-model completeness and near model completeness, and in this work, it was pointed out about the possibility of transferring these two concepts to arbitrary n. In fact, there is a direct connection between these two works are related to model completeness, thereby determining a utensils of essence of given works to one of classical directions in the model theory, determined by the studies of Abraham Robinson. In general, the development of this subjects after the beginning of the 70 was naturally suspended due to the fact that the main trends of development of model theory were based on technology and concepts, related to the study of complete theories. On the other hand, the main ideas of Robinson's directions relate to the study of inductive theories, which generally are not complete. The special subclass of inductive theories is the class of Jonsson theories. Toward this class can be attributed basic algebraic examples of theories, which are play important role in the various modern sections of mathematics. For example, theories of groups, theories of Abelian groups, theories of fields of fixed characteristic, theories of Boolean algebras, theories of polygons, etc. The given examples of theories show the relevance of studying model-theoretical properties of Jonsson theories.

In the classic textbook, in the form of reference book [3] can find the definition of Jonsson theory, later on, the study of Jonsson theories was developed in the following list of works: [4–6].

In [1], the concept of a semantic model is used in a substantial way, and this concept by its existence is connected with an additional axiom about the existence of a strongly unattainable cardinal to the existing system of axioms of Zermelo-Frenkel set theory. In the work [7] a new definition of the semantic model was given, in the framework of which developed and develops the further research of Jonsson theories [8–10]. In this definition of the semantic model in work [7] is no requirement about the existence of a strongly unattainable cardinal. The greatest success at the study of Jonsson theories can be achieved in case of saturation of semantic model. As it was determined, in [11] such theories, by analogy with [1], will be called perfect Jonsson theories. These theories have many good semantic properties related to the theory. For example, the theory of algebraically closed fields of a fixed characteristic is model companion of theory of fields of the same characteristic. A concept of model companion was defined by A. Robinson and this concept is closely related with the concept of model completeness.

In connection with the above, we want to consider the properties of Jonsson theories, which use the concept of n-model completeness from work [2] for some n and the corresponding concept of a model companion, using the results from [1], but within the framework of a new approach to Jonsson theories [11–14] using the new definition of semantic model and companions in study of model completeness from [2].

We give the necessary definitions of concepts which we will use.

We will start with definition of Jonsson theory.

Definition 1 [11]. A theory T is called Jonsson if the following conditions are satisfied:

1) T has infinite models;
2) T is inductive;
3) T has the joint embedding property (JEP);

4) T has the amalgam property (AP).

Following definitions (1-5) and facts (theorems 1-2) allows the reader to get acquainted with the inner structure of semantic model as part of the definition from [7].

Definition 2 [7]. Let $\kappa \geq \omega$. A model $\mathfrak{M}$ of theory T is called κ-universal for T if every model T of strictly less power κ isomorphically embedded in $\mathfrak{M}$.

Definition 3 [7]. Let $\kappa \geq \omega$. A model $\mathfrak{M}$ of theory T is called κ-homogeneous for T if for any two models $\mathfrak{A}$ and $\mathfrak{A}_1$ of theory T, which are submodels of $\mathfrak{M}$, power strictly less, than κ, and isomorphic $f : \mathfrak{A} \to \mathfrak{A}_1$, for every extension $\mathfrak{B}$ of model $\mathfrak{A}$, which is submodel of $\mathfrak{M}$ and model of T of strictly less power κ exists extension $\mathfrak{B}_1$ of model $\mathfrak{A}_1$, which is submodel of $\mathfrak{M}$, and isomorphic $g : \mathfrak{B} \to \mathfrak{B}_1$, continuing f.

Definition 4 [7]. κ-homogeneous-universal model for theory T of power κ, where $\kappa \geq \omega$, is called homogeneous-universal model for T.

Theorem 1 [7]. Every Jonsson theory T has κ^+-homogeneous-universal model of power 2^κ. Inversely, if T is inductive, has infinite model and has ω^+-homogeneous-universal model, then a theory T is Jonsson theory.

Theorem 2 [7]. Let T be a Jonsson theory. Two models $\mathfrak{M}$ and $\mathfrak{M}_1$ κ-homogeneous-universal for T are elementary equivalent.

Definition 5 [7]. ω^+-homogeneous-universal model of theory T is called semantic model C_T of Jonsson theory T.

For any Jonsson theory a semantic model always exists, therefore it plays an important role as a semantic invariant.

From definition of semantic model follows that:

Proposition 1 [15]. Any two semantic models of Jonsson theory T is elementary equivalent between themselves.

Lemma 1 [15]. Semantic model C_T of Jonsson theory T is T-existential closed.

Definition 6 [15]. Elementary theory of semantic model C of Jonsson theory T is called semantic completion (center) T^* of this T, i.e. $T^* = Th(C_T)$.

As we have already noticed, greatest progress in learning of Jonsson theories, as a rule, can be achieved provided that of perfection of Jonsson theory.

Definition 7 [15]. A Jonsson theory T is called perfect if every semantic model of T is saturated model of T^*.

It is well know, that only at work with perfect Jonsson theories a class of existential closed models of considered theory is elementary.

Theorem 3 [15]. Let E_T be a class of all existential closed models of theory T. If a Jonsson theory T is perfect, then $E_T = \operatorname{Mod} T^*$, where $T^* = Th(C_T)$.

A concept of model completeness introduced by A. Robinson is played large role in the study of model companions of various types of classical algebras.

Definition 8 [2]. A theory T is model complete if for any $B, D \in ModT$ and B is a submodel of D, then $B \prec D$.

Definition 9 [2]. $B \subseteq_1 D$ satisfied if B is a submodel of D, for every $\forall$-formulas (equivalently, $\exists$-formulas) $\psi(\overline{x})$ and for every $\overline{b} \in B$ will performed $B \models \psi(\overline{b})$ provided that $D \models \psi(\overline{b})$.

Generalization of definition 8 of model completeness, namely, definition 10, was consider in [2] by the authors using the concept (definition 9).

Definition 10 [2]. A theory T is 1-model complete if for any $B, D \in ModT$ and $B \subseteq_1 D$, then $B \prec D$.

One of the interesting properties of classical model theory is a property of quantifier eliminable which is also associated with a special case of model companion. In [2] was determined generalization of concept of quantifier eliminable, namely, definition 11.

Definition 11 [2]. A theory T is nearly model complete if for any formulas $\psi(\overline{x})$ exists a formula $\varphi(\overline{x})$ which is Boolean combination of $\forall$-formulas such that $T \models \forall\overline{x}[\psi \leftrightarrow \varphi]$.

Moreover in work [2] criterion was obtained (proposition 2.).

Proposition 2 [2]. A theory T is 1-model complete iff for any formulas $\psi(\overline{x})$ exists a formula $\varphi(\overline{x})$ which is a $\forall$-formulas such that $T \models \forall\overline{x}[\psi \leftrightarrow \varphi]$.

On the other hand, in work [1] was considered a generalization of Jonsson theory and the main tool of this generalization was a concept of Γ-embedding which is generalizated a concept of isomorphic embedding with respect to considered formulas. Instead of boolean combination of atomic formulas is considered a formulas with quantifier prenix of length α. In place of Boolean combination of atomic formulas we consider a formulas with quantifier prenix of length α. Under Γ we understand a kind of formulas, for example, $\Gamma = \Pi_\alpha$.

A set of all formulas (is a view of formula $\forall\exists\ldots\psi$) denote by Π_n, $\Sigma_n = \{\psi | \neg\psi \in \Pi_n\}$.

Definition 12 [1]. A map $f : A \to B$ is called a Γ-embedding if for any $\overline{a} \in A$ and $\psi(\overline{x}) \in \Gamma$ from $A \models \psi(\overline{a})$ implies $B \models \psi(f(\overline{a}))$.

A concept of model companion was determined by A. Robinson, and it is played important role in the study of various types of algebras, theories of which has model companion [1] (chapter 4).

Definition 13 [3]. A theory T is called model companion of T if:

1) T and T^* are mutually model consistent;

2) T^* is model complete.

Using next theorem we understand a value of concept of model companion for any Jonsson theory, semantic model of which is saturated.

Theorem 4 [15]. Let T be arbitrary Jonsson theory, then the following conditions are equivalent:

1) T is perfect;

2) T^* is model companion of T.

Using concept of finite diagram from work [16], T.G. Mustafin is determined a concept of model completion for generalized Jonsson theory. In the future on throughout of all paper in the results concerning work [1], as a semantic model we use a model as part of the definition 5.

Definition 14 [1]. 1. A set $D(\mathfrak{B}) = \cup_{n<\infty}\{Th(\mathfrak{B}, \overline{b}) | \ \overline{b} \in |\mathfrak{B}|^n\}$ is called finite diagram of system $\mathfrak{B}$.

2. Algebraic system $\mathfrak{A}$ is called a $D(\mathfrak{B})$-system if satisfied $Th(\mathfrak{A}) = Th(\mathfrak{B})$ and $D(\mathfrak{A}) \subseteq D(\mathfrak{B})$.

3. If T is arbitrary theory, then any this model is called a $D(T)$-model.

In the future we will consider that $D = D(T)$ or $D = D(\mathfrak{B})$ for some model $\mathfrak{B}$ of theory T.

Using Γ-embeddings at work [1] was determined special case of α-model companion, namely, of concept of α-model completion which can be obtained from definitions 15 and 16 from work [1].

Definition 15 [1]. We say that a theory T is $D - \alpha$-model complete if a theory $T \cup Th_{\Pi_\alpha}(B, |B|)$ is complete with respect to D for any model $B \models T$.

Definition 16 [1]. Let T_1, T_2 be arbitrary theories of one language. A theory T_2 is called $D - \alpha$-model completion of T_1 if:

1) any model of T_1 is Π_α-embeddable in some D-model of T_2, and conversely, every D-model of T_2 is Π_α-embeddable in suitable (or some) model of T_1;

2) T_2 is $D - \alpha$-model complete;

3) a theory $T_2 \cup Th_{\Pi_\alpha}(B, |B|)$ is complete with respect to D for any model B of T_1.

In the future we will say that if satisfied condition (1) from definition 16, then considered theories are $D - \Pi_\alpha$-mutually model consistent, where $D = D(T)$ or $D = D(\mathfrak{B})$ for some model $\mathfrak{B}$ of theory T.

This theorem is α-Jonsson generalization of criterion of perfectness of Jonsson theory (theorem 4).

Proposition 3 [1]. Let T be arbitrary α-Jonsson theory, then the following conditions are equivalent:

1) T is perfect;

2) T^* is α-model completion of T.

Proceed to the main result of given paper. For this we must define a concept of (n_1, n_2)-Jonsson theory. Let n_1, n_2 be arbitrary natural numbers.

Definition 17. Theory T is called (n_1, n_2)-Jonsson theory if it is n_1-model complete and nearly n_2-model complete theory.

It is clear from definition 17 that $n_1 \geq n_2$. If $n_2 = 0$, then ал center of Jonsson theory T^* admit elimination of quantifiers. If n_1=0, then Jonsson theory T^* is model complete theory. Note that $(0, n)$-Jonsson theory is perfect for any natural n.

In other words, n_1-model completeness denote by the index n_1, but near model completeness denote by the index n_2. It is clear that it 2 various indices, and they may be dependentes, i.e. considered theories can be with one index or simultaneously with two indices.

We determine a concept of α-model companion of α-Jonsson theory.

Definition 18. Let T_1, T_2 be α-Jonsson theories of one language. A theory T_2 is called $D-\alpha$-model companion of T_1 if:

1) any model of T_1 is Π_α-embeddable in some D-model of T_2, and conversely, every D-model of T_2 is Π_α-embeddable in suitable (or some) model of T_1;

2) T_2 is $D - \alpha$-model complete.

Theorem 5. Every α-Jonsson theory T has no more than one α-model companion.

Proof. Let's say on the contrary, i.e. α-Jonsson theory has as minimum two various α-model companion, or example, T_1, T_2. Hence, theory T is perfect. Then by definition a theories T_1 and T_2 are mutually model consistent. By criterion of perfectness of α-Jonsson theory we can conclude that theories T_1 and T_2 are mutually model consistent with T^*, where T^* is center of theory T. And this means that T_1 and T_2 are cosemantic among themselves, and means they are equal.

By criterion of perfectness (proposition 3.) we can be conclude that α-Jonsson theory (when α=1) is 1-perfect if the theory has 1-model companion.

Next theorem allows you to get a description of 1-perfect 1-Jonsson theory in the sense of work [1].

Theorem 6. 1-Jonsson theory is 1-perfect iff the following conditions are equivalence:

1) a theory T has 1-model companion T^m in sense of work [1];

2) a theory $T^m = T^c$, where T^c is a center of theory T;

3) a theories $T^m = T^c$ and T are $D - \Pi_1$-mutually model consistent, where $D = D(C)$, is semantic model of theory T. A theory T^m is 1-model complete in sense of work [2].

Proof. From (1) to (2) and from (2) to (1) follows from proposition 3 at α=1.

We prove from (1) to (3). Since a theory T has 1-model companion T^m, then by definition 18, when α=1, we have that T^m is $D-1$-model complete, where $D = D(C)$, and C is semantic model of theory T. A theory T^m is $D-1$-model complete if a theory $T \cup Th_{\Pi_1}(C, |C|)$ is complete. Since any model of theory T isomorphically embedded in model C, then easy to notice that by universality of formulas Π_1 and $D-1$-model completeness all models of theory T with respect to Π_1-sentences are elementary equivalent. We need to show that from what $A \subseteq_1 B$ follows that $A \preceq B$ for any $A, B \in ModT$. Suppose the contrary. This means that there are such that $A, B \in ModT$, but it is not true that $A \npreceq B$. This is equivalent to that $B \notin ModD(A)$, but it is not true, since $D(A) \subseteq D(C)$ and B is a model $D(C)$ by $D-1$-model completeness of theory T.

We prove from (3) to (1). Suppose the contrary. This means that a theory T is not 1-perfect, and this means that it is non-perfect in sense of work [1] (proposition 3.), i.e. she does not have $D-1$-model completion. But by (3) a theories T^m and T are $D - \Pi_1$-mutually model consistent, where $D = D(C)$, C is semantic model of theory T, moreover T^m is 1-model complete in sense of work [2]. But then we can use a criterion (proposition 2.), which say that: a Jonsson theory T is 1-model complete iff for any formulas $\psi(\overline{x})$ exists a formula $\varphi(\overline{x})$ which is a $\forall$-formulas such that $T \models \forall\overline{x}[\psi \leftrightarrow \varphi]$. 1-nonperfectness is means that there is such type p, consisting from universal formulas with constants from some subset X of model C, which is not implemented in C. Without loss of generality, we can assume that X is contained in some model A of theory T^m. Since a type p is consistent set, then there is elementary extension A' of model A, which is implemented a type p. But since p is consisted from $\forall$-formulas, then p is implemented and in A. By $D - \Pi_1$-model compatibility A is invested in C, and this means, that a type p is implemented and in C. And from that one we can conclude about avalibility of the contradiction.

References

1 Мустафин Т.Г. Обобщенные условия Йонсона и описание обобщенно-йонсоновских теорий булевых алгебр / Т.Г. Мустафин // Матем. тр. — 1998. — № 2. — С. 135–197.

2 Kueker D.W. Nearly model complete theories. / D.W. Kueker, P. Turnquist // Mathematical Logic Quarterly. — 1999. — No. 3. — P. 291–298.

3 Теория моделей: справочная книга по математической логике: [В 4-х ч.]. Ч.1. / Под ред. Дж. Барвайса. — М.: Наука, 1982. — 392 с.

4 Yeshkeyev A.R. Properties of lattices of the existential formulas of Jonsson fragments / A.R. Yeshkeyev, M.T. Kasymetova // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 79. — No. 3. — P. 25–32.

5 Yeshkeyev A.R. On lattice of existential formulas for fragment of Jonsson set / A.R. Yeshkeyev, O.I. Ulbrikht // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 79. — No. 3. — P. 33–39.

6 Yeshkeyev A.R. Strongly minimal Jonsson sets and their properties / A.R. Yeshkeyev // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 80. — No. 4. — P. 47–51.

7 Mustafin Y.T. Quelques properties des theories de Jonsson. / Y.T. Mustafin // The Journal of Symbolic logic. — 2002. — No. 2. — P. 528-536.

8 Ешкеев А.Р. JSp-косемантичность и JSB-свойство абелевых групп / А.Р. Ешкеев, О.И. Ульбрихт // Сибирские электронные математические известия. — 2016. – Т. 13. – С. 861–874.

9 Yeshkeyev A.R. The Properties of Similarity for Jonsson's Theories and Their Models / A.R. Yeshkeyev // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 80. — No. 4. — P. 52–59.

10 Yeshkeyev A.R. Companions of the fragments in the Jonsson set / A.R. Yeshkeyev // Bulletin of the Karaganda University. Mathematics series. — 2017. — Vol. 85. — No. 1. — P. 41–45.

11 Ешкеев А.Р. Йонсоновские теории и их классы моделей: моногр. / А.Р. Ешкеев, М.Т. Касыметова. — Караганда: Изд-во КарГУ, 2016. – 346 с.

12 Yeshkeyev A.R. dcl-cl-atomic and prime sets / A.R. Yeshkeyev, A.K. Issayeva // Bulletin of the Karaganda University. Mathematics series. — 2019. — Vol. 93. — No. 1. — P. 88–94.

13 Yeshkeyev A.R. Properties of hybrids of Jonsson theories / A.R. Yeshkeyev, N.M. Mussina // Bulletin of the Karaganda University. Mathematics series. — 2018. — Vol. 92. — No. 4. — P. 99–104.

14 Yeshkeyev A.R. Model-theoretic properties of the #-companion of a Jonsson set / A.R. Yeshkeyev, M.T. Kassymetova, N.K. Shamataeva // Eurasian mathematical journal. — 2018. — Vol. 9. — No. 2. — P. 68–81.

15 Ешкеев А.Р. Йонсоновские теории: учеб. пос. / А.Р. Ешкеев. — Караганда: Изд-во КарГУ, 2009. — 249 с.

16 Shelah S. Finite diagrams stable in power / S. Shelah // Ann. Math. Logic. — 1971. — No. 3. — P. 271–362.

А.Р. Ешкеев, М.Т. Омарова

(n_1, n_2)-Йонсон теориясының компаньондары

Мақалада (n_1, n_2)-йонсон теориясы компаньондерінің модельдік-теоретикалық қасиеттері қарастырылған. Сонымен бірге (n_1, n_2)-йонсон теориясымен орталық арасындағы байланыс зерттелген. Бұл ретте қарастырылып отырған теориялар модельдi компаньоннің мағынасына сәйкес келеді. Авторлар жаңа ұғымдар енгізді, атап айтқанда: (n_1, n_2)-йонсон теориясы; $D - \alpha$-модельдi компаньон. α-йонсон және 1-кемел 1-йонсон теориясының модельдік-компаньондарына қатысты жаңа нәтижелер көрсетілген.

Кілт сөздер: йонсондық теория, модельдi толық, n-модельдi толық, дерлік n-модельдi толық, $D - \alpha$-модельдi компаньон, модельдi толықтыру, (n_1, n_2)-йонсондық теория.

А.Р. Ешкеев, М.Т. Омарова

Компаньоны (n_1, n_2)-йонсоновских теорий

В статье рассмотрены теоретико-модельные свойства компаньонов (n_1, n_2)-йонсоновской теории. Также изучены связи между центром и (n_1, n_2)-йонсоновской теорией. При этом рассматриваемые теории являются совершенными в смысле существования соответственного модельного компаньона. Авторами введены новые понятия, а именно: (n_1, n_2)-йонсоновская теория; $D - \alpha$-модельный компаньон. Показаны новые результаты относительно модельных компаньонов α-йонсоновской и 1-совершенной 1-йонсоновской теорий.

Ключевые слова: йонсоновская теория, модельно полная, n-модельно полная, почти n-модельно полная, $D - \alpha$-модельный компаньон, модельное пополнение, (n_1, n_2)-йонсоновская теория.

References

1 Mustafin, T.G. (1998). Obobshchennye usloviia Ionsona i opisanie obobshchenno-ionsonovskikh teorii bulevykh alhebr [Generalized conditions of Jonsson and description of generalized Jonsson theories of Boolean algebras]. *Matematicheskie trudy — Mathematical works, 2,* 135–197 [in Russian].

2 Kueker, D.W. & Turnquist, P. (1999). Nearly model complete theories. *Mathematical Logic Quarterly, 3,* 291–298.

3 Barwise, Ed.J. (1982). *Teoriia modelei [Model theory].* Ed.J. Barwise, Yu.L. Yershov (Ed.). (Parts 1-4). Moscow: Nauka [in Russian].

4 Yeshkeyev, A.R. & Kasymetova, M.T. (2015). Properties of lattices of the existential formulas of Jonsson fragments. *Bulletin of the Karaganda University. Mathematics series, Vol. 79, 3,* 25–32.

5 Yeshkeyev, A.R., & Ulbrikht, O.I. (2015). On lattice of existential formulas for fragment of Jonsson set. *Bulletin of the Karaganda University. Mathematics series, Vol. 79, 3,* 33–39.

6 Yeshkeyev, A.R. (2015). Strongly minimal Jonsson sets and their properties. *Bulletin of the Karaganda University. Mathematics series, Vol. 80, 4,* 47–51.

7 Mustafin, Ye.T. (2002). Quelques properties des theories de Jonsson. *The Journal of Symbolic logic, 2,* 528–536.

8 Yeshkeyev, A.R., & Ulbrikht, O.I. (2016). JSp-kosemantichnost i JSB svoistvo abelevykh hrupp [JSp-cosemantic and JSB-property of abelian groups]. *Sibirskie elektronnye matematicheskie izvestiia – Siberian Electronic Mathematical Reports, Vol. 13,* 861–874 [in Russian].

9 Yeshkeyev, A.R. (2015). The Properties of Similarity for Jonsson's Theories and Their Models. *Bulletin of the Karaganda University. Mathematics series, Vol. 80, 4,* 52–59.

10 Yeshkeyev, A.R. (2017). Companions of the fragments in the Jonsson set. *Bulletin of the Karaganda University. Mathematics series, Vol. 85, 1,* 41–45.

11 Yeshkeyev, A.R., & Kassymetova, M.T. (2016). *Ionsonovskie teorii i ikh klassy modelei [Jonsson theories and their classes of models].* Karaganda: Izdatelstvo Karahandinskoho hosudarstvennoho universiteta [in Russian].

12 Yeshkeyev, A.R., & Issayeva, A.K. (2019). dcl-cl-atomic and prime sets. *Bulletin of the Karaganda University. Mathematics series, Vol. 93, 1,* 88–94.

13 Yeshkeyev, A.R., & Mussina, N.M. (2018). Properties of hybrids of Jonsson theories. *Bulletin of the Karaganda University. Mathematics series, Vol. 92, 4,* 99–104.

14 Yeshkeyev, A.R., Kassymetova, M.T., & Shamataeva, N.K. (2018). Model-theoretic properties of the #-companion of a Jonsson set. *Eurasian mathematical journal, Vol. 9, 2,* 68–81.

15 Yeshkeyev, A.R. (2009). *Ionsonovskie teorii [Jonsson theories].* Karaganda: Izdatelstvo Karahandinskoho hosudarstvennoho universiteta [in Russian].

16 Shelah, S. (1971). Finite diagrams stable in power. *Ann. Math. Logic, 3,* 271–362.