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Asymptotic solutions of scalar integro-differential equations with partial derivatives and with rapidly oscillating coefficients

The work is devoted to the development of an asymptotic integration algorithm for the Cauchy problem for a singularly perturbed partial differential integro-differential equation with rapidly oscillating coefficients, which describe various physical processes in micro-inhomogeneous media.This direction in the theory of partial differential equations is developing intensively and finds numerous applications in radiophysics, electrical engineering, filtering theory, phase transition theory, elasticity theory, and other branches of physics, mechanics, and technology. For studies of such processes, asymptotic methods are usually used. It is known that currently rapidly developing numerical methods do not exclude asymptotic. This happens for a number of reasons. Firstly, a reasonably constructed asymptotics, especially its main term, carries information that is important for applications about the qualitative behavior of the solution and, in this sense, to some extent replaces the exact solution, which most often cannot be found. Secondly, as follows from the above, knowledge of the solution structure helps in the development of numerical methods for solving complex problems; therefore, the development of asymptotic methods contributes to the development of numerical methods. Regularization of the problem is carried out, the normal and unique solvability of general iterative problems is proved.

Keywords: singularly perturbed, partial integro differential equation, regularization of an integral, solvability of iterative problems.

Introduction

A mathematical description of physical processes in micro-inhomogeneousmedia suggests that the local characteristics of the latter depend on a small parameter which is a characteristic scale of the microstructure of the medium. To construct mathematical models of such processes, an asymptotic analysis of the problem is performed for It turns out that the limits of the solutions to the problem are described by some new differential equations that have relatively smoothly varying coefficients and are considered in simple domains.These equations are mathematical models of physical processes in micro-inhomogeneousmedia, and their coefficients are effective characteristics of such media. For mathematical studies of such processes, asymptotic methods are usually used. It is known that currently rapidly developing numerical methods do not exclude asymptotic ones. This happens for a number of reasons. Firstly, a reasonably constructed asymptotics, especially its main term, carries information that is important for applications about the qualitative behavior of the solution and, in this case to some extent replaces the exact solution, which most often cannot be found. Secondly, as follows from the above, knowledge of the solution structure helps in the development of numerical methods for solving complex problems; therefore, the development of asymptotic methods contributes to the development of numerical methods. Thirdly, for some problems, especially those related to fast oscillations, there are simply no effective numerical methods that give a sufficient degree of accuracy. The first of the problems with an irregular dependence in perturbation theory that arose in connection with the problems of celestial mechanics and electrical engineering were nonlinear equations, which are often called oscillating equations at present. Tasks of this kind arise everywhere where certain transient processes take place. Studies of oscillating and singularly perturbed oscillating systems described by ordinary differential equations to the splitting methods were carried out in [1–4] and regularization methods in [5–8]. An analysis of the main results of the study for systems of homogeneous and inhomogeneous differential equations prompted the idea to study singularly perturbed integro-differential equations with rapidly oscillating coefficients.A system of integro-differential equations in the absence of resonance is considered, i.e. when the integer linear combination of frequencies of the rapidly oscillating cosine does not coincide with the frequency of the spectrum of the limit operator [9, 10]. It should be noted that when developing an algorithm for constructing an asymptotic solution to the problem, the ideas of the regularization method used to study ordinary integro-differential equations [11–21] and integro-differential equations with partial derivatives [23–24] were used.

We consider the Cauchy problem for the integro-differential equation with partial derivatives:

$$\begin{array}{l} \varepsilon\frac{\partial y(x,t,\varepsilon)}{\partial x}=a(x)y(x,t,\varepsilon)+\int\limits_{x_0}^{x}K(x,t,s)y(s,t,\varepsilon)ds+h(x,t)+ \\ +\varepsilon g(x)cos\frac{\beta(x)}{\varepsilon}y(x,t,\varepsilon),\quad y(x_0,t,\varepsilon)=y^0(t)\quad ((x,t)\in[x_0,X]\times[0,T]), \end{array} \tag{1}$$

where $\beta'(x)>0$, $g(x)$, $a(x)$ is a scalar functions, $y^0(t)$ constant, $\varepsilon>0$ is a small parameter. Denote by $\lambda_1(x)=-a(x)$, $\beta'(x)$ is a frequency of rapidly oscillating cosine. In the following, functions $\lambda_2(x)=-i\beta'(x)$, $\lambda_3(x)=+i\beta'(x)$ will be called *the spectrum of a rapidly oscillating coefficient.*

We assume that the conditions are fulfilled:

(i) $a(x),g(x),\beta(x)\in C^\infty[x_0,X]$; $h(x,t)\in C^\infty[x_0,X]\times[0,T]$, the kernel $K(x,t,s)$ belongs to the space $K(x,t,s)\in C^\infty\{x_0<x<s<X,\ 0<t<T\}$;

(ii) $\lambda_1(x)\equiv a(x)\neq\lambda_j(x),\quad j=2,3,\quad \lambda_i(x)\neq 0,\ (\forall x\in[x_0,X]),\ i=1,2,3;$

(iii) $\lambda_1(x)\leq 0,\quad (\forall x\in[x_0,X])$;

(iv) for $\forall x\in[x_0,X]$ and $n_2\neq n_3$ inequalities

$$\begin{array}{l} n_2\lambda_2(x)+n_3\lambda_3(x)\neq\lambda_1(x), \\ \lambda_1(x)+n_2\lambda_2(x)+n_3\lambda_3(x)\neq\lambda_1(x),\ \ (\forall x\in[x_0,X]) \end{array}$$

for all multi-indices $n=(n_2,n_3)$ with $|n|\equiv n_2+n_3\geq 1$ (n_2 and n_3 are non-negative integers) are holds.

We will develop an algorithm for constructing a regularized [5] asymptotic solution of problem (1).

1 Regularization of problem (1)

Denote by $\sigma_j=\sigma_j(\varepsilon)$, independent of t magnitudes $\sigma_1=e^{-\frac{i}{\varepsilon}\beta(t_0)}$, $\sigma_1=e^{+\frac{i}{\varepsilon}\beta(t_0)}$, and rewrite system (1) as

$$\begin{array}{l} \varepsilon\frac{\partial y(x,t,\varepsilon)}{\partial x}-\lambda_1(x)y(x,t,\varepsilon)-\varepsilon\frac{g(t)}{2}\left(e^{-\frac{i}{\varepsilon}\int_{t_0}^{t}\beta'(\theta)d\theta}\sigma_1+e^{+\frac{i}{\varepsilon}\int_{t_0}^{t}\beta'(\theta)d\theta}\sigma_2\right)y(x,t,\varepsilon)- \\ -\int_{x_0}^{x}K(x,t,s)\,y(s,t,\varepsilon)ds=h(x,t),\ y(x_0,t,\varepsilon)=y^0,\ \ ((x,t)\in[x_0,X]\times[0,T]). \end{array} \tag{2}$$

We introduce regularizing variables (see [5, 6]):

$$\tau_j=\frac{1}{\varepsilon}\int_{x_0}^{x}\lambda_j(\theta)\,d\theta\equiv\frac{\psi_j(x)}{\varepsilon},\ j=\overline{1,3},$$

and instead of problem (2), consider the problem

$$\begin{array}{l} \varepsilon\frac{\partial\tilde{y}}{\partial x}+\sum\limits_{j=1}^{3}\lambda_j(t)\frac{\partial\tilde{y}}{\partial\tau_j}-\lambda_1(x)\tilde{y}-\varepsilon\frac{g(t)}{2}\left(e^{\tau_2}\sigma_1+e^{\tau_3}\sigma_2\right)\tilde{y}- \\ -\int_{x_0}^{x}K(x,t,s)\,\tilde{y}(s,t,\frac{\psi(s)}{\varepsilon},\varepsilon)ds=h(x,t),\ ,\tilde{y}(x,t,\tau,\varepsilon)|_{x=x_0,\tau=0}=y^0,\ \ ((x,t)\in[x_0,X]\times[0,T]), \end{array} \tag{3}$$

for the function $\tilde{y}=\tilde{y}(x,t,\tau,\varepsilon)$, where is indicated: $\tau=(\tau_1,\tau_2,\tau_3)$, $\psi=(\psi_1,\psi_2,\psi_3)$. It is clear that if $\tilde{y}=\tilde{y}(x,t,\tau,\varepsilon)$ is a solution to problem (3), then the vector function $y=\tilde{y}\left(x,\frac{\psi(x)}{\varepsilon},\varepsilon\right)$ is an exact solution to problem (2), therefore, problem (3) is extended with respect to problem (2). However, it cannot be considered fully regularized, since it does not regularize the integral term $J\tilde{y}=\int_{x_0}^{x}K(x,t,s)\,\tilde{y}(s,\frac{\psi(s)}{\varepsilon},\varepsilon)ds$. To regularize the integral operator, we introduce a class M_ε that is asymptotically invariant with respect to the operator $J\tilde{y}$ (see [5], p. 62). Recall the corresponding concept.

Definition 1. A class M_ε is said to be asymptotically invariant (with $\varepsilon\to+0$) with respect to an operator P_0 if the following conditions are fulfilled:

1) $M_\varepsilon\subset D(P_0)$ with each fixed $\varepsilon>0$;

2) the image $P_0g(x,\varepsilon)$ of any element $g(x,\varepsilon)\in M_\varepsilon$ decomposes in a power series

$$P_0g(x,\varepsilon)=\sum_{n=0}^{\infty}\varepsilon^n g_n(x,\varepsilon)(\varepsilon\to+0,g_n(x,\varepsilon)\in M_\varepsilon,n=0,1,...),$$

convergent asymptotically for $\varepsilon\to+0$ (uniformly with $x\in[x_0,X]$).

From this definition it can be seen that the class M_ε depends on the space U, in which the operator P_0 is defined. In our case $P_0 = J$. For the space U we take the space of vector functions $y(x,t,\tau)$, represented by sums

$$y(x,t,\tau,\sigma) = y_0(x,t,\sigma) + \sum_{i=1}^{3} y_i(x,t,\sigma)\, e^{\tau_i} + \sum_{2\le |m| \le N_y}^{*} y^m(x,t,\sigma)\, e^{(m,\tau)} +$$

$$+ \sum_{1\le |m| \le N_y}^{*} y^{e_1+m}(x,t,\sigma)\, e^{(e_1+m,\tau)}, y_i(x,t,\sigma), y^m(x,t,\sigma), y^{e_1+m}(x,t,\sigma) \in C^\infty[x_0,X]\times[0,T], \qquad (4)$$

$$m = (0,m_2,m_3), 1 \le |m| \equiv m_2+m_3 \le N_y, i = \overline{1,3},$$

where is denoted: $\lambda(x) \equiv (\lambda_1,\lambda_2,\lambda_3)$, $(m,\lambda(x)) \equiv m_2\lambda_2(x) + m_3\lambda_3(x)$, $(e_1+m,\lambda(x)) \equiv \lambda_1(x) + m_2\lambda_3(x) + +m_3\lambda_3(x)$; an asterisk $*$ above the sum sign indicates that the summation for $|m| \ge 1$ it occurs only over multi-indices $m = (0,m_2,m_3)$ with $m_2 \ne m_3$, $e_1 = (1,0,0)$, $\sigma = (\sigma_1,\sigma_2)$.

Note that here the degree N_y of the polynomial $y(x,t,\tau,\sigma)$ relative to the exponentials e^{τ_1} depends on the element y. In addition, the elements of space U depend on bounded in $\varepsilon > 0$ terms of constants $\sigma_1 = \sigma_1(\varepsilon)$ and $\sigma_2 = \sigma_2(\varepsilon)$, and which do not affect the development of the algorithm described below, therefore, in the record of element (4) of this space U, we omit the dependence on $\sigma = (\sigma_1,\sigma_2)$ for brevity. We show that the class $M_\varepsilon = U|_{\tau=\psi(t)/\varepsilon}$ is asymptotically invariant with respect to the operator J. The image of the operator on the element (4) of the space U has the form

$$Jy(x,t,\tau) = \int_{x_0}^{x} K(x,t,s)y_0(s,t)ds + \sum_{i=1}^{3}\int_{x_0}^{x} K(x,t,s)y_i(s,t)e^{\frac{1}{\varepsilon}\int_{x_0}^{s}\lambda_i(\theta)d\theta}ds +$$

$$+ \sum_{2\le|m|\le N_z}^{*}\int_{x_0}^{x} K(x,t,s)y^m(s,t)e^{\frac{1}{\varepsilon}\int_{x_0}^{s}(m,\lambda(\theta))d\theta}ds + \sum_{1\le|m|\le N_z}^{*}\int_{x_0}^{x} K(x,t,s)\, y^{e_1+m}(s,t)\, e^{\frac{1}{\varepsilon}\int_{x_0}^{s}(e_1+m,\lambda(\theta))d\theta}ds.$$

Integrating in parts, we will have

$$J_i(x,t,\varepsilon) = \int_{x_0}^{x} K(x,t,s)y_i(s,t)e^{\frac{1}{\varepsilon}\int_{x_0}^{s}\lambda_i(\theta)d\theta}ds = \varepsilon\int_{x_0}^{x}\frac{K(x,t,s)y_i(s,t)}{\lambda_i(s)}de^{\frac{1}{\varepsilon}\int_{x_0}^{s}\lambda_i(\theta)d\theta} =$$

$$= \varepsilon\left[\left.\frac{K(x,t,s)y_i(s,t)}{\lambda_i(s)}e^{\frac{1}{\varepsilon}\int_{x_0}^{s}\lambda_i(\theta)d\theta}\right|_{s=x_0}^{s=x} - \int_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{K(x,t,s)y_i(s,t)}{\lambda_i(\theta)}\right)e^{\frac{1}{\varepsilon}\int_{x_0}^{s}\lambda_i(\theta)d\theta}ds\right] =$$

$$= \varepsilon\left[\frac{K(x,t,x)y_i(x,t)}{\lambda_i(x)}e^{\frac{1}{\varepsilon}\int_{x_0}^{x}\lambda_i(\theta)d\theta} - \frac{K(x,t,x_0)y_i(x_0,t)}{\lambda_i(x_0)}\right] - \varepsilon\int_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{K(x,t,s)y_i(s,t)}{\lambda_i(s)}\right)e^{\frac{1}{\varepsilon}\int_{x_0}^{s}\lambda_i(\theta)d\theta}ds.$$

Continuing this process further, we obtain the decomposition

$$J_i(x,t,\varepsilon) = \sum_{\nu=0}^{\infty}(-1)^\nu\varepsilon^{\nu+1}\left[\left(I_i^\nu\left(K(x,t,s)y_i(s,t)\right)\right)_{s=x}e^{\frac{1}{\varepsilon}\int_{x_0}^{x}\lambda_i(\theta))\,d\theta} - \left(I_i^\nu\left(K(x,t,s)y_i(s,t)\right)\right)_{s=x_0}\right],$$

$$I_i^0 = \frac{1}{\lambda_i(s)}\cdot, I_i^\nu = \frac{1}{\lambda_i(s)}I_i^{\nu-1} (\nu \ge 1,\, i = \overline{1,3}).$$

Applying the integration operation in parts to integrals

$$J_m(x,t,\varepsilon) = \int_{x_0}^{x} K(x,t,s)y^m(s,t)e^{\frac{1}{\varepsilon}\int_{x_0}^{s}(m,\lambda(\theta))d\theta}ds,$$

$$J_{e_1+m}(x,t,\varepsilon) = \int_{x_0}^{x} K(x,t,s)y^{e_1+m}(s,t)\, e^{\frac{1}{\varepsilon}\int_{x_0}^{s}(e_1+m,\lambda(\theta))d\theta}ds,$$

we note that for all multi-indices $m = (0,m_2,m_3)$, $m_2 \ne m_3$, inequalities

$$(m,\lambda(x)) \equiv m_2\lambda_2(x) + m_3\lambda_3(x) \ne 0 \quad \forall x \in [x_0,X],\ \ m_2+m_3 \ge 2.$$

are satisfied. In addition, for the same multi-indices $m=(0,m_2,m_3)$ we have

$$(e_1+m,\lambda(x))\neq 0\ \forall x\in[x_0,X],\ m_2\neq m_3,\ |m|=m_2+m_3\geq 1.$$

Indeed, if $(e_1+m,\lambda(x))=0$ for some $x\in[x_0,X]$ and $m_2\neq m_3$, $m_2+m_3\geq 1$, then $m_2\lambda_2(x)+m_3\lambda_3(x)= =-\lambda_1(x),\ \ m_2+m_3\geq 1$, which contradicts condition (iv). Therefore, integration by parts in integrals is possible. Performing it, we will have:

$$J_m(x,t,\varepsilon)=\int\limits_{t_0}^{x}K(x,t,s)y^m(s,t)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(m,\lambda(\theta))d\theta}ds=\varepsilon\int\limits_{x_0}^{x}\frac{K(x,t,s)y^m(s,t)}{(m,\lambda(s))}de^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(m,\lambda(\theta))d\theta}=$$

$$=\varepsilon\left[\frac{K(x,t,s)y^m(x,t)}{(m,\lambda(s))}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(m,\lambda(\theta))d\theta}\Bigg|_{s=x_0}^{s=x}-\varepsilon\int\limits_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{K(x,t,s)y^m(s,t)}{(m,\lambda(s))}\right)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(m,\lambda(\theta))d\theta}\right]-$$

$$=\varepsilon\left[\frac{K(x,t,x)y^m(x,t)}{(m,\lambda(x))}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}(m,\lambda(\theta))d\theta}-\frac{K(x,t,x_0)y^m(x_0,t)}{(m,\lambda(x_0))}\right]-$$

$$-\varepsilon\int\limits_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{K(x,t,s)y^m(s,t)}{(m,\lambda(s))}\right)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(m,\lambda(\theta))d\theta}ds.$$

Therefore, the image of the operator J on the element (4) of the space U is represented as a series

$$Jz\left(t,\tau\right)=\int_{t_0}^{t}K\left(t,s\right)z_0\left(s\right)ds+\sum_{i=1}^{4}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}[(I_i^{\nu}\left(K\left(t,s\right)z_i\left(s\right)\right))_{s=t}e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_i(\theta))d\theta}-$$
$$-\left(I_i^{\nu}\left(K\left(t,s\right)z_i\left(s\right)\right)\right)_{s=t_0}]+\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}[(I_m^{\nu}\left(K\left(t,s\right)z^m\left(s\right)\right))_{s=t}e^{\frac{1}{\varepsilon}\int_{t_0}^{t}(m,\lambda(\theta))d\theta}-$$
$$-\left(I_m^{\nu}\left(K\left(t,s\right)z^m\left(s\right)\right)\right)_{s=t_0}]+\sum_{j=1}^{2}\sum_{1\leq|m|\leq N_z}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}[\left(I_{j,m}^{\nu}\left(K\left(t,s\right)z^{e_j+m}\left(s\right)\right)\right)_{s=t}\times$$
$$\times e^{\frac{1}{\varepsilon}\int_{t_0}^{t}(e_j+m,\lambda(\theta))}-\left(I_{j,m}^{\nu}\left(K\left(t,s\right)z^{e_j+m}\left(s\right)\right)\right)_{s=t_0}]_{\tau=\psi(t)/\varepsilon}.$$

Continuing this process, we obtain the series

$$J_m(x,t,\varepsilon)=\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[(I_m^{\nu}\left(K(x,t,s)y^m(s,t)\right))_{s=t}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}(m,\lambda(\theta))d\theta}-\right.$$
$$\left.-\left(I_m^{\nu}\left(K(x,t,s)y^m(s,t)\right)\right)_{s=t_0}\right],$$

$$I_m^0=\frac{1}{(m,\lambda(s))}.,I_m^{\nu}=\frac{1}{(m,\lambda(s))}\frac{\partial}{\partial s}I_m^{\nu-1}\,(\nu\geq 1,|m|\geq 2),$$

$$J_{e_1+m}(x,t,\varepsilon)=\int\limits_{x_0}^{x}K(x,t,s)y^{e_1+m}(s,t)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(e_1+m,\lambda(\theta))d\theta}ds=$$
$$=\varepsilon\int\limits_{x_0}^{s}\tfrac{K(x,t,s)y^{e_1+m}(s,t)}{(e_1+m,\lambda(s))}de^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(e_1+m,\lambda(\theta))d\theta}=$$
$$=\varepsilon\left[\tfrac{K(x,t,s)y^{e_1+m}(s,t)}{(e_1+m,\lambda(s))}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(e_1+m,\lambda(\theta))d\theta}\Bigg|_{s=x_0}^{s=x}-\right.$$
$$\left.-\varepsilon\int\limits_{x_0}^{x}\left(\tfrac{\partial}{\partial s}\tfrac{K(x,t,s)y^{e_1+m}(s,t)}{(e_1+m,\lambda(s))}\right)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(e_1+m,\lambda(\theta))d\theta}ds\right]-$$

$$=\varepsilon\left[\frac{K(x,t,x)y^{e_1+m}\left(x,t\right)}{(e_1+m,\lambda(x))}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}(e_1+m,\lambda(\theta))d\theta}-\frac{K(x,t,x_0)y^{e_1+m}\left(x_0,t\right)}{(e_1+m,\lambda(x_0))}\right]-$$

$$-\varepsilon\int\limits_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{K(x,t,s)y^{e_1+m}(s,t)}{(e_1+m,\lambda(s))}\right)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}(e_1+m,\lambda(\theta))d\theta}ds.$$

Continuing this process, we obtain the series

$$J_{e_1+m}(x,t,\varepsilon)=\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I^{\nu}_{e_1+m}\left(K(x,t,s)y^{e_1+m}(s,t)\right)\right)_{s=x}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}(e_1+m,\lambda(\theta))d\theta},\right.$$
$$\left.-\left(I^{\nu}_{e_1+m}\left(K(x,t,s)y^{e_1+m}(s,t)\right)\right)_{s=x_0}\right],$$

$$I^{0}_{e_1+m}=\frac{1}{(e_1+m,\lambda(s))}.,I^{\nu}_{j,m}=\frac{1}{(e_1+m,\lambda(s))}\frac{\partial}{\partial s}I^{\nu-1}_{m}(\nu\geq 1,\,|m|\geq 1),$$

Therefore, the image of the operator j on the element (4) of the space U is represented as a series

$$Jy(x,t,\tau)=\int\limits_{x_0}^{x}K(x,t,s)y_0(s,t)ds+\sum_{i=1}^{3}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I^{\nu}_{i}\left(K(x,t,s)y_i(s,t)\right)\right)_{s=t}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_i(\theta))\,d\theta}-\right.$$
$$\left.-\left(I^{\nu}_{i}\left(K(x,t,s)y_i(s,t)\right)\right)_{s=t_0}\right]+$$
$$+\sum_{2\leq|m|\leq N_Y}^{*}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I^{\nu}_{m}\left(K(x,t,s)y^{m}(s,t)\right)\right)_{s=t}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}(m,\lambda(\theta))\,d\theta}-\left(I^{\nu}_{m}\left(K(x,t,s)y^{m}(s,t)\right)\right)_{s=t_0}\right]+$$
$$+\sum_{1\leq|m|\leq N_Y}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I^{\nu}_{e_1+m}\left(K(x,t,s)y^{e_1+m}(s,t)\right)\right)_{s=x}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}(e_1+m,\lambda(\theta))d\theta}-\right.$$
$$\left.-\left(I^{\nu}_{e_1+m}\left(K(x,t,s)y^{e_1+m}(s,t)\right)\right)_{s=x_0}\right].$$

It is easy to show (see, for example, [25], pp. 291-294) that this series converges asymptotically for $\varepsilon\to+0$ (uniformly in $(x,t)\in[x_0,X]\times[0,T]$). This means that the class M_ε is asymptotically invariant (for $\varepsilon\to+0$) with respect to the operator J.

We introduce operators $R_\nu:\ U\to U$, acting on each element $y(x,t,\tau)\in U$ of the form (4) according to the law:

$$R_0y\,(x,t,\tau)=\int\limits_{x_0}^{x}K(x,t,s)y_0(s,t)ds, \qquad (5_0)$$

$$R_1y(x,t,\tau)=\sum_{i=1}^{3}\left[\left(I^{0}_{i}\left(K(x,t,s)y_i(s,t)\right)\right)_{s=x}e^{\tau_i}-\left(I^{0}_{i}\left(K(x,t,s)y_i(s,t)\right)\right)_{s=x_0}\right]+$$
$$+\sum_{1\leq|m|\leq N_z}^{*}\left[\left(I^{0}_{m}\left(K(x,t,s)y^{m}\,(s,t)\right)\right)_{s=x}e^{(m,\tau)}-\left(I^{0}_{m}\left(K(x,t,s)y^{m}\,(s,t)\right)\right)_{s=x_0}\right]+ \qquad (5_1)$$
$$+\sum_{1\leq|m|\leq N_z}^{*}\left[\left(I^{0}_{e_1+m}\left(K(x,t,s)y^{e_1+m}\,(s,t)\right)\right)_{s=x}e^{(e_1+m,\tau)}-\left(I^{0}_{e_1+m}\left(K(x,t,s)y^{e_1+m}\,(s,t)\right)\right)_{s=x_0}\right],$$

$$R_{\nu+1}y(x,t,\tau)=\sum_{i=1}^{3}\left[\left(I^{\nu}_{i}\left(K(x,t,s)y_i\,(s,t)\right)\right)_{s=x}e^{\tau_i}-\left(I^{\nu}_{i}\left(K(x,t,s)y_i\,(s,t)\right)\right)_{s=x_0}\right]+$$
$$+\sum_{2\leq|m|\leq N_y}^{*}\left[\left(I^{\nu}_{m}\left(K(x,t,s)y^{m}\,(s,t)\right)\right)_{s=x}e^{(m,\tau)}-\left(I^{\nu}_{m}\left(K(x,t,s)y^{m}\,(s,t)\right)\right)_{s=x_0}\right]+ \qquad (5_{\nu+1})$$
$$+\sum_{1\leq|m|\leq N_z}^{*}\left[\left(I^{\nu}_{e_1+m}\left(K(x,t,s)z^{e_1+m}\,(s,t)\right)\right)_{s=x}e^{(e_1+m,\tau)}-\left(I^{\nu}_{e_1+m}\left(K(x,t,s)z^{e_1+m}\,(s,t)\right)\right)_{s=x_0}\right],\nu\geq 1.$$

Now let $\tilde{y}\left(x,t,\tau,\varepsilon\right)$ be an arbitrary continuous function on $(x,t,\tau)\in[x_0,X]\times[0,T]\times\{Re\lambda_1(x)\}$ with asymptotic expansion

$$\tilde{y}(x,t,\tau,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k y_k(x,t,\tau),\quad y_k(x,t,\tau)\in U, \tag{6}$$

converging as $\varepsilon\to+0$ (uniformly in $(x,t,\tau)\in[x_0,X]\times[0,T]\times\{Re\lambda_1(x)\}$). Then the image $J\tilde{y}\left(x,t,\tau,\varepsilon\right)$ of this function is decomposed into an asymptotic series

$$J\tilde{z}\left(x,t,\tau,\varepsilon\right)=\sum_{k=0}^{\infty}\varepsilon^k Jy_k\left(x,t,\tau\right)=\sum_{r=0}^{\infty}\varepsilon^r\sum_{s=0}^{r}R_{r-s}y_s\left(x,t,\tau\right)|_{\tau=\psi(x)/\varepsilon}.$$

This equality is the basis for introducing an extension of an operator J on series of the form (6):

$$\tilde{J}\tilde{y}\left(x,t,\tau,\varepsilon\right)\equiv\tilde{J}\left(\sum_{k=0}^{\infty}\varepsilon^k y_k\left(x,t,\tau\right)\right)\triangleq\sum_{r=0}^{\infty}\varepsilon^r\sum_{s=0}^{r}R_{r-s}y_s\left(x,t,\tau\right).$$

Although the operator $\tilde{J}$ is formally defined, its utility is obvious, since in practice it is usual to construct the N–th approximation of the asymptotic solution of the problem (2), in which impose only N-th partial sums of the series (6), which have not a formal, but a true meaning. Now you can write a problem that is completely regularized with respect to the original problem (2):

$$\begin{gathered}L_\varepsilon\tilde{y}(x,t,\tau,\varepsilon)\equiv\varepsilon\frac{\partial\tilde{y}}{\partial x}+\sum_{j=1}^{3}\lambda_j(x)\frac{\partial\tilde{y}}{\partial\tau_j}-\lambda_1(x)\tilde{y}-\tilde{J}\tilde{y}-\varepsilon\frac{g(x)}{2}(e^{\tau_2}\sigma_1+e^{\tau_3}\sigma_2)\tilde{y}=\\=h(x,t),\quad\tilde{y}(x_0,t,0,\varepsilon)=y^0,\quad\left((x,t)\in[x_0,X]\times[0,T]\right).\end{gathered}\tag{7}$$

2 Solvability of iterative problems

Substituting the series (6) into (7) and equating the coefficients with the same degrees ε, we obtain the following iterative problems:

$$Ly_0\left(x,t,\tau\right)\equiv\sum_{j=1}^{3}\lambda_j\left(x\right)\frac{\partial y_0}{\partial\tau_j}-\lambda_1(x)y_0-R_0y_0=h\left(x,t\right),\,y_0\left(x_0,t,0\right)=y^0(t);\tag{8_0}$$

$$Ly_1\left(x,t,\tau\right)=-\frac{\partial y_0}{\partial x}+\frac{g(x)}{2}\left(e^{\tau_2}\sigma_1+e^{\tau_3}\sigma_2\right)y_0+R_1y_0,\;y_1\left(x_0,t,0\right)=0;\tag{8_1}$$

$$Ly_2\left(x,t,\tau\right)=-\frac{\partial y_1}{\partial x}+\frac{g(x)}{2}\left(e^{\tau_2}\sigma_1+e^{\tau_3}\sigma_2\right)y_1+R_1y_1+R_2y_0,\;y_0\left(x_0,t,0\right)=0;\tag{8_2}$$

$$\cdots$$

$$Ly_k\left(x,t,\tau\right)=-\frac{\partial y_{k-1}}{\partial x}+\frac{g(x)}{2}\left(e^{\tau_2}\sigma_1+e^{\tau_3}\sigma_2\right)y_{k-1}+R_ky_0+...+R_1y_{k-1},\;\;y_k\left(x_0,t,0\right)=0,\,k\geq1.\tag{8_k}$$

Each of the iterative problems (8_k) can be written as

$$Lz\left(x,t,\tau\right)\equiv\sum_{j=1}^{3}\lambda_j\left(x\right)\frac{\partial y}{\partial\tau_j}-\lambda_1(x)y-R_0y=h\left(x,t,\tau\right),\,y\left(x_0,t,0\right)=y^*,\tag{9}$$

where

$$h(x,t,\tau)=h_0(x,t)+\sum_{i=1}^{3}h_i(x,t)e^{\tau_i}+\sum_{2\leq|m|\leq N_y}^{*}h^m(x,t)e^{(m,\tau)}+\sum_{1\leq|m|\leq N_y}^{*}h^{e_1+m}(x,t)e^{(e_1+m,\tau)}\in U,$$

is the known vector function of space U, y^* is the known constant vector of the complex space C, and the operator R_0 has the form (see (5_0))

$$\begin{gathered}R_0y(x,t,\tau)\equiv R_0\left[y_0(x,t)+\sum_{i=1}^{3}y_i(x,t)e^{\tau_i}+\sum_{2\leq|m|\leq N_y}^{*}y^m(x,t)e^{(m,\tau)}+\sum_{1\leq|m|\leq N_y}^{*}y^{e_1+m}(x,t)e^{(e+m,\tau)}\right]\triangleq\\\triangleq\int_{x_0}^{x}K(x,t,s)y_0(s,t)ds.\end{gathered}$$

We will determine the solution of equation (9) as an element (4) of the space U:

$$y(x,t,\tau)=y_0(x,t)+\sum_{i=1}^{3}y_i(x,t)e^{\tau_i}+\sum_{2\le|m|\le N_z}^{*}y^m(x,t)e^{(m,\tau)}+\sum_{1\le|m|\le N_H}^{*}y^{e_1+m}(x,t)e^{(e_1+m,\tau)}\equiv$$

$$\equiv y_0(x,t)+\sum_{i=1}^{3}y_i(x,t)e^{\tau_i}+\sum_{2\le|m|\le N_y}^{*}y^m(x,t)e^{(m,\tau)}+\sum_{2\le|m^1|\le N_H}^{*}y^{m^1}(x,t)e^{(m^1,\tau)}, \tag{10}$$

where for convenience introduced multi-indices

$$m^1=e_1+m\equiv(1,m_2,m_3)\,,\ \left|m^1\right|=1+m_2+m_3\ge 2,$$

m_2 and m_3 are non-negative integer numbers. Substituting (10) into equation (9), and equating here the free terms and coefficients separately for identical exponents, we obtain the following equations:

$$-\lambda_1(x)y_0(x,t)-\int\limits_{x_0}^{x}K(x,t,s)y_0(s,t)ds=h_0(x,t), \tag{11}$$

$$\left[\lambda_i(x)-\lambda_1(x)\right]y_i(x,t)=h_i(x,t),\quad i=\overline{1,3}, \tag{11_i}$$

$$\left[(m,\lambda(x))-\lambda_1(x)\right]y^m(x,t)=h^m(x,t),\quad m_2\ne m_3,\quad 2\le|m|\le N_h, \tag{11_m}$$

$$\left[\left(m^1,\lambda(x)\right)-\lambda_1(x)\right]y^{m^1}(x,t)=h^{m^1}(x,t),\quad m_2\ne m_{3,}\ \ 1\le\left|m^1\right|\le N_h. \tag{12}$$

The equation (11) can be written as

$$y_0(x,t)=\int\limits_{t_0}^{t}\left(-\lambda_1^{-1}(x)K(x,t,s)\right)y_0(s,t)ds-\lambda_1^{-1}(x)h_0(x,t). \tag{11_0}$$

Due to the smoothness of the kernel $-\lambda_1^{-1}(x)K(x,t,s)$ and heterogenei ty $-\lambda_1^{-1}(x)h_0(x,t)$, this Volterra integral system has a unique solution $y_0(x,t)\in C^\infty[x_0,X]\times[0,T]$. The equations ($11_2$) - ($11_3$) also have unique solutions

$$y_i(x,t)=\left[\lambda_i(x)-\lambda_1(x)\right]^{-1}h_i(x,t)\in C^\infty[x_0,X]\times[0,T],\ \ i=2,3,$$

since $\lambda_i(x)\ne\lambda_1(x)$, $i=2,3$. Equation (11_1) are solvable in space $C^\infty[x_0,X]\times[0,T]$ if and only

$$h_1(x,t)\equiv 0\ \ (\forall(x,t)\in[x_0,X]\times[0,T])\,. \tag{13}$$

Further, since $(m,\lambda(x))\equiv m_2\lambda_2(x)+m_3\lambda_3(x)\ne\lambda_1(x),\ \ |m|=m_2+m_3\ge 2$ (see condition (iv) the absence of resonance), the equation (11_m) has a unique solution

$$y^m(x,t)=\left[(m,\lambda(x))-\lambda_1(x)\right]^{-1}h^m(x,t)\in C^\infty[x_0,X]\times[0,T],\ \ 2\le|m|\le N_h.$$

We now equation (12). Let us show that when $\left|m^1\right|\ge 1$ the functions $\left(m^1,\lambda(x)\right)\ne\lambda_1(x)$. Indeed, let $\left(m^1,\lambda(t)\right)=\lambda_1(t),\quad\left|m^1\right|\ge 1$. Then

$$\lambda_1(x)+m_2\lambda_2(x)+m_3\lambda_3(x)=\lambda_1(x)\ \Leftrightarrow\ m_2\lambda_2(x)+m_3\lambda_3(x)=0\ \Leftrightarrow\ m_2\ne m_3,\ \ m_2+m_3\ge 1,$$

which cannot be (see definition of class U). Thus, equation (12) for $\left|m^1\right|\ge 1$ has a unique solution

$$z^{m^1}(x,t)=\left[\left(m^1,\lambda(x)\right)-\lambda_1(x)\right]^{-1}h^{m^1}(x,t),\ \ 1\le\left|m^1\right|\le N_h,$$

inn class $C^\infty[x_0,X]\times[0,T]$.

We have proved the following statement.

Theorem 1. Let conditions (i)-(ii), (iv) be fulfilled and the right-hand side

$$h(x,t,\tau)=h_0(x,t)+\sum_{i=1}^{3}h_i(x,t)e^{\tau_i}+\sum_{2\le|m|\le N_z}^{*}h^m(x,t)e^{(m,\tau)}+\sum_{1\le|m|\le N_H}^{*}h^{e_1+m}(x,t)e^{(e_1+m,\tau)}\in U$$

of equation (9) belongs to the space U. Then, for the solvability of equation (9) in space U, it is necessary and sufficient that condition (13) is satisfied.

Under constraint (13), equation (9) has the following solution in space U:

$$y(x,t,\tau)=y_0(x,t)+\xi(x,t)e^{\tau_1}+\sum_{i=2}^{3}\left[\lambda_i(x)-\lambda_1(x)\right]^{-1}H_i(x,t)e^{\tau_i}+$$

$$+\sum_{2\le|m|\le N_H}^{*}\left[(m,\lambda(x))-\lambda_1(x)\right]^{-1}H^m(x,t)e^{(m,\tau)}+ \tag{14}$$

$$\sum_{1\le|m|\le N_H}^{*}\left[\left(m^1,\lambda(x)\right)-\lambda_1(x)\right]^{-1}H^{e_1+m}(x,t)e^{(e_1+m,\tau)},$$

where $\xi(x,t)\in C^{\infty}[x_0,X]\times[0,T]$ are arbitrary function, $y_0(x,t)$ is the solution of an integral equation (11_0), $m\equiv(0,m_2,m_3)$, $m_2\neq m_3$, $|m|=m_2+m_3\geq 1$.

Subject the solution (14) to the initial condition $y(x_0,t,0)=y_*(t)$. Then we have

$$\xi(x_0,t)-\lambda_1^{-1}(x_0)h_0(x_0,t)+\sum_{i=2}^{3}[\lambda_i(x_0)-\lambda_1(x_0)]^{-1}h_i(x_0,t)+$$

$$+\sum_{2\le|m|\le N_h}^{*}\left[(m,\lambda(x_0))-\lambda_1(x_0)\right]^{-1}h^m(x_0,t)+\sum_{1\le|m|\le N_h}^{*}\left[\left(m^1,\lambda(x_0)\right)-\lambda_1(x_0)\right]^{-1}h^{e_1+m}(x_0,t)=y_*\Leftrightarrow$$

$$\Leftrightarrow\quad\xi(x_0,t)=y_*+\lambda_1^{-1}(x_0)h_0(x_0,t)-\sum_{i=2}^{3}[\lambda_i(x_0)-\lambda_1(x_0)]^{-1}h_i(x_0,t)- \tag{15}$$

$$-\sum_{2\le|m|\le N_h}^{*}\left[(m,\lambda(x_0))-\lambda_1(x_0)\right]^{-1}h^m(x_0,t)-\sum_{1\le|m|\le N_h}^{*}\left[\left(m^1,\lambda(x_0)\right)-\lambda_1(x_0)\right]^{-1}h^{e_1+m}(x_0,t).$$

However, the functions $\xi(x,t)$ were not found completely. An additional requirement is required to solve problem (13). Such a requirement is dictated by iterative problems (8_k), from which it can be seen that the natural additional constraint is the condition

$$-\frac{\partial y}{\partial x}+\frac{g(x)}{2}(e^{\tau_2}+e^{\tau_3})y+R_1y+p(x,t,\tau)\equiv 0,\qquad(\forall(x,t)\in[x_0,X]\times[0,T]), \tag{16}$$

where $p(x,t,\tau)=p_0(x,t)+\sum\limits_{i=1}^{3}p_i(x,t)e^{\tau_i}+\sum\limits_{2\le|m|\le N_z}^{*}p^m(x,t)e^{(m,\tau)}+\sum\limits_{1\le|m|\le N_H}^{*}p^{e_1+m}(x,t)e^{(e_1+m,\tau)}\in U$ is the known vector-function. The right part of this equation:

$$G(x,t,\tau)\equiv-\frac{\partial y}{\partial t}+\frac{g(x)}{2}\left(e^{\tau_2}\sigma_1+e^{\tau_3}\sigma_2\right)y+Q(x,t,\tau)=$$

$$=-\frac{\partial}{\partial x}\left[y_0(x,t)+\sum_{i=1}^{3}y_i(x,t)e^{\tau_i}+\sum_{2\le|m|\le N_y}^{*}y^m(x,t)e^{(m,\tau)}+\sum_{1\le|m|\le N_y}^{*}y^{e_1+m}(x,t)e^{(e_1+m,\tau)}\right]+$$

$$+\frac{g(x)}{2}\left(e^{\tau_2}\sigma_1+e^{\tau_3}\sigma_2\right)\left[y_0(x,t)+\sum_{i=1}^{3}y_i(x,t)e^{\tau_i}+\sum_{2\le|m|\le N_y}^{*}y^m(x,t)e^{(m,\tau)}+\right.$$

$$\left.+\sum_{1\le|m|\le N_y}^{*}y^{e_1+m}(x,t)e^{(e_1+m,\tau)}\right]+p(x,t,\tau),$$

may not belong to space U, if $y = y(x,t,\tau) \in U$. Indeed, taking into account the form (14) of the function $y = y(x,t,\tau) \in U$, we will have

$$Z(x,t,\tau) \equiv G(x,t,\tau) + \frac{\partial y}{\partial x} = \frac{g(x)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left[y_0(x,t) + \sum_{i=1}^{3} y_i(x,t)e^{\tau_i} + \right.$$

$$\left. + \sum_{2\le|m|\le N_z}^{*} y^m(x,t)e^{(m,\tau)} + \sum_{1\le|m|\le N_z}^{*} y^{e_1+m}(x,t)e^{(e_j+m,\tau)}\right] =$$

$$= \frac{g(x)}{2} y_0(x,t)\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right) + \sum_{i=2}^{3}\frac{g(x)}{2} y_i(x,t)\left(e^{\tau_1+\tau_2}\sigma_1 + e^{\tau_1+\tau_3}\sigma_2\right) +$$

$$+\frac{g(x)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left[\sum_{2\le|m|\le N_z}^{*} y^m(x,t)e^{(m,\tau)} + \sum_{1\le|m|\le N_z}^{*} y^{e_1+m}(x,t)e^{(e_1+m,\tau)}\right] + p(x,t,\tau).$$

Here are terms with exponents

$$\begin{gathered} e^{\tau_2+\tau_3} = e^{(m,\tau)}|_{m=(0,1,1)},\;\; e^{\tau_2+(m,\tau)}(\;m_2+1=m_3),\; e^{\tau_3+(m,\tau)}(m_3+1=m_2), \\ e^{\tau_2+(e_1+m,\tau)}(\;m_2+1=m_3),\; e^{\tau_3+(e_1+m,\tau)}(m_3+1=m_2) \end{gathered} \tag{$*$}$$

do not belong to space U, since in multi-index $m = (0, m_2, m_3)$ of the space U must be $m_2 \ne m_3$, $m_2 + m_3 \ge 1$. Then, according to the well-known theory (see [5], p. 234), we embed these terms in the space U according to the following rule (see $(*)$):

$$\begin{gathered} \widehat{e^{\tau_3+\tau_2}} = 1,\; \widehat{e^{\tau_2+(m,\tau)}} = 1\,(m_2+1=m_3,\; m_2\ne m_3),\; \widehat{e^{\tau_3+(m,\tau)}} = 1\;(m_3+1=m_2, m_2\ne m_3), \\ \widehat{e^{\tau_2+(e_1+m,\tau)}} = e^{\tau_1}(m_2+1=m_3, m_2\ne m_3),\; \widehat{e^{\tau_3+(e_1+m,\tau)}} = e^{\tau_2}(m_3+1=m_2, m_2\ne m_3). \end{gathered} \tag{$**$}$$

In $Z(x,t,\tau)$ need of embedding only the terms

$$M(x,t,\tau) \equiv \sum_{i=2}^{3}\frac{g(x)}{2} y_i(x,t)\left(e^{\tau_i+\tau_2}\sigma_1 + e^{\tau_i+\tau_3}\sigma_2\right) + \frac{g(x)}{2} y_1(x,t)\left(e^{\tau_1+\tau_2}\sigma_1 + e^{\tau_1+\tau_3}\sigma_2\right),$$

$$S(x,t,\tau) \equiv \frac{g(x)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left[\sum_{2\le|m|\le N_z}^{*} y^m(x,t)e^{(m,\tau)} + \sum_{1\le|m|\le N_z}^{*} y^{e_1+m}(x,t)e^{(e_1+m,\tau)}\right].$$

We describe this embedding in more detail, taking into account formulas $(**)$:

$$M(x,t,\tau) \equiv \sum_{k=1}^{2}\frac{g(x)}{2} y_k(x,t)\left(e^{\tau_k+\tau_2}\sigma_1 + e^{\tau_k+\tau_3}\sigma_2\right) + \frac{g(x)}{2} y_1(x,t)\left(e^{\tau_1+\tau_2}\sigma_1 + e^{\tau_1+\tau_3}\sigma_2\right) =$$

$$= \frac{g(x)}{2}\left[y_1(x,t)e^{\tau_1+\tau_2}\sigma_1 + y_1(x,t)e^{\tau_1+\tau_3}\sigma_2 + y_2(x,t)e^{2\tau_2}\sigma_1 + \right.$$

$$\left. + y_2(x,t)e^{\tau_2+\tau_3}\sigma_2 + y_3(x,t)e^{\tau_3+\tau_2}\sigma_1 + y_3(x,t)e^{2\tau_3}\sigma_2\right] \quad \Rightarrow$$

$$\Rightarrow \widehat{M}(x,t,\tau) = \frac{g(x)}{2}\left[y_1(x,t)e^{\tau_1+\tau_2}\sigma_1 + y_1(x,t)e^{\tau_1+\tau_3}\sigma_2 + \right.$$

$$\left. + y_2(x,t)e^{2\tau_2}\sigma_1 + y_2(x,t)\sigma_2 + y_3(x,t)\sigma_1 + y_3(x,t)e^{2\tau_3}\sigma_2\right].$$

(note that in $\widehat{M}(x,t,\tau)$ there are no members containing e^{τ_1} measurement exponents $|m| = 1$);

$$S(x,t,\tau) \equiv \frac{g(x)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left[\sum_{2\le|m|\le N_z}^{*} y^m(x,t)e^{(m,\tau)} + \sum_{1\le|m|\le N_z}^{*} y^{e_1+m}(x,t)e^{(e_1+m,\tau)}\right] =$$

$$= \frac{g(x)}{2}\left[\sum_{2\le|m|\le N_z}^{*} y^m(x,t)\left(e^{\tau_2+(m,\tau)}\sigma_1+e^{\tau_3+(m,\tau)}\sigma_2\right)+\right.$$

$$\left.+\sum_{1\le|m|\le N_z}^{*} y^{e_1+m}(x,t)\left(e^{(e_1+m,\tau)+\tau_2}\sigma_1+e^{(e_1+m,\tau)+\tau_3}\sigma_2\right)\right] \Rightarrow$$

$$\Rightarrow \widehat{S}(x,t,\tau)=\frac{g(x)}{2}\left[\sum_{\substack{2\le|m|\le N_z,\\ m_2+1=m_3}} y^m(x,t)\sigma_1+\sum_{\substack{2\le|m|\le N_z,\\ m_3+1=m_2}} z^m(x,t)\sigma_2+\sum_{\substack{2\le|m|\le N_z,\\ m_2+1\ne m_3, m_3+1\ne m_2}}^{*} y^m(x,t)e^{(m,\tau)}+\right.$$

$$+\left(\sum_{\substack{1\le|m|\le N_z,\\ m_2+1=m_3}} y^{e_1+m}(x,t)\sigma_1+\sum_{\substack{1\le|m|\le N_z,\\ m_3+1=m_2}} y^{e_1+m}(x,t)\sigma_2\right)e^{\tau_1}+\sum_{\substack{1\le|m|\le N_z,\\ m_2+1\ne m_3, m_3+1\ne m_2}}^{*} y^{e_1+m}(x,t)e^{(e_1+m,\tau)}.$$

After embedding, the right-hand side of equation (16) will look like

$$\widehat{G}(x,t,\tau)=-\frac{\partial}{\partial x}\left[y_0(x,t)+\sum_{i=1}^{3} y_i(x,t)e^{\tau_i}+\sum_{2\le|m|\le N_z}^{*} y^m(x,t)e^{(m,\tau)}+\sum_{1\le|m|\le N_z}^{*} y^{e_1+m}(x,t)e^{(e_1+m,\tau)}\right]+$$

$$+\widehat{M}(x,t,\tau)+\widehat{S}(x,t,\tau)+R_1 y(x,t,\tau)+p(x,t,\tau),$$

moreover, in $\widehat{S}(x,t,\tau)$ the coefficient at e^{τ_1} do not depend on $y_1(x,t)$. As indicated in [5], the embedding $G(x,t,\tau)\to\widehat{G}(x,t,\tau)$ will not affect the accuracy of the construction of asymptotic solutions of problem (2), since $\widehat{Z}(x,t,\tau)|_{\tau=\psi(x)/\varepsilon}\equiv Z(x,t,\tau)|_{\tau=\psi(x)/\varepsilon}$.

We show that the problem (9) has the unique solution in the space U if (16) is satisfied.

Theorem 2. Let the conditions (i)-(iv) take place and the right-hand side

$$h(x,t,\tau)=h_0(x,t)+\sum_{i=1}^{3} h_i(x,t)e^{\tau_i}+\sum_{2\le|m|\le N_z}^{*} h^m(x,t)e^{(m,\tau)}+\sum_{1\le|m|\le N_H}^{*} h^{e_1+m}(x,t)e^{(e_1+m,\tau)}\in U$$

satisfy the condition (13). Then the problem (9) is uniquely solvable in the space U under the additional condition (16).

Proof. To use the condition (16), we calculate the expression $-\frac{\partial y}{\partial x}+\frac{g(x)}{2}(e^{\tau_2}+e^{\tau_3})y+R_1y+p(x,t,\tau)$. Since

$$-\frac{\partial}{\partial x}\left[y_0(x,t)+\sum_{i=1}^{3} y_i(x,t)e^{\tau_i}+\sum_{2\le|m|\le N_z}^{*} y^m(x,t)e^{(m,\tau)}+\sum_{1\le|m|\le N_z}^{*} y^{e_1+m}(x,t)e^{(e_1+m,\tau)}\right]+$$

$$+\widehat{M}(x,t,\tau)+\widehat{S}(x,t,\tau)+\sum_{i=1}^{3}\left[\left(I_i^0\left(K(x,t,s)y_i(s,t)\right)\right)_{s=x}e^{\tau_i}-\left(I_i^0\left(K(x,t,s)y_i(s,t)\right)\right)_{s=x_0}\right]+$$

$$+\sum_{1\le|m|\le N_z}^{*}\left[\left(I_m^0\left(K(x,t,s)y^m(s,t)\right)\right)_{s=x}e^{(m,\tau)}-\left(I_m^0\left(K(x,t,s)y^m(s,t)\right)\right)_{s=x_0}\right]+$$

$$+\sum_{1\le|m|\le N_z}^{*}\left[\left(I_{e_1+m}^0\left(K(x,t,s)y^{e_1+m}(s,t)\right)\right)_{s=x}e^{(e_1+m,\tau)}-\left(I_{e_1+m}^0\left(K(x,t,s)y^{e_1+m}(s,t)\right)\right)_{s=x_0}\right]+$$

$$+p(x,t,\tau).$$

therefore(16) takes the form

$$\frac{\partial\left(\xi(x,t)\right)}{\partial x}+\frac{K(x,t,x)}{\lambda_1(x)}\xi(x,t)+p_1(x,t)e^{\tau_1}+p_0(x,t)=0.$$

Taking into account the initial condition (15), this equation has the unique solution

$$\xi(x,t) = e^{q(x,t)}\left[\xi(x_0,t) + \int\limits_{x_0}^{x} p_i(s,t)e^{-q(s,t)}ds\right],$$

where $q(x,t) = \int\limits_{x_0}^{x} \lambda_1^{-1}(s)K(s,t,s)ds$. Hence, under the conditions of Theorem 2, the solution (14) in the space U is uniquely determined.

Applying theorems 1 and 2 to iterative problems (we construct the series (8_k), with coefficients in the class U. Let $y_{\varepsilon N}(x,t) = \sum\limits_{k=0}^{N} \varepsilon^k y_k\left(x,t,\psi(x,\varepsilon)\right)$ be the restriction of the N-th partial sum of this series at $\tau = \psi(x,\varepsilon)$. As well as in [2], it is not difficult to prove the following

Theorem 3. Let the conditions (i)-(iv) be satisfied. Then for $\varepsilon \in (0,\varepsilon_0]$, where $\varepsilon_0 > 0$ is sufficiently small, the problem (1) has the unique solution $y(x,t,\varepsilon) \in [x_0,X]\times[0,T]$, and the estimate

$$\|y(x,t,\varepsilon - y_{\varepsilon N}(x,t)\|_{C([x_0,X]\times[0,T])} \le C_N\varepsilon^{N+1} \quad (N = 0,1,2,\ldots),$$

takes place, where the constant $C_N > 0$ does not depend on $\varepsilon \in (0,\varepsilon_0]$.

3 Construction of the solution of the first iteration problem in space U

Theorem 1, we will try to find a solution to the first iteration problem (8_0). Since the right side $h(x,t)$ of the equation (8_0) satisfies condition (13), this system has (according to (14)) a solution in space U in the form

$$y_0(x,t,\tau) = y_0^{(0)}(x,t) + \alpha_1^{(0)}(x,t)e^{\tau_1}, \tag{17}$$

where $y_0^{(0)}(x,t)$ is the solution of the integrated equation

$$y_0^{(0)}(x,t) = \int\limits_{x_0}^{x}\left(-a^{-1}(x)K(x,t,s)\right)y_0^{(0)}(s,t)ds - a^{-1}(x)h(x,t),$$

where $\alpha_1^{(0)}(x,\ t) \in C^\infty\left[x_0,\ X\right]\times[0,\ T]$ are arbitrary functions. Subjecting (17) to the initial condition $y_0(x_0,t,0) = y^0$, we will have

$$y_0^{(0)}(x_0,t) + \alpha_1^{(0)}(x_0,t) = y^0 \quad\Leftrightarrow\quad \alpha_1^{(0)}(x_0,t) = y^0 + a^{-1}(x_0)h(x_0,t).$$

To fully compute the functions $\alpha_1^{(0)}(x,t)$, we proceed to the next iteration problem (8_1). Substituting into it the solution (14) of the equation (8_0) we arrive at the following equation:

$$\begin{gathered} Ly_1(x,t,\tau) = -\frac{d}{dx}y_0^{(0)}(x,t) - \frac{d}{dx}\left(\alpha_1^{(0)}(x,t)\right)e^{\tau_1} + \\ +\frac{g(x)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left(y_0^{(0)}(x,t) + \alpha_1^{(0)}(x,t)e^{\tau_1}\right) + \\ +\frac{K(x,t,x)\alpha_1^{(0)}(x,t)}{\lambda_1(x)}e^{\tau_1} - \frac{K\left(x,t,x_0\right)\alpha_1^{(0)}(x_0,t)}{\lambda_1(x_0)}, \end{gathered} \tag{18}$$

(here we used the expression (5_1) for $R_1y(x,t,\tau)$ and took into account that for $y(x,t,\tau) = y_0(x,t,\tau)$ only the terms with e^{τ_1} remain in the sum (5_1)). It is not difficult to see that the right side

$$\begin{gathered} H(x,t,\tau) = -\frac{d}{dx}y_0^{(0)}(x,t) - \frac{d}{dx}\left(\alpha_1^{(0)}(x,t)\right)e^{\tau_1} + \\ +\frac{g(x)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left(y_0^{(0)}(x,t) + \alpha_1^{(0)}(x,t)e^{\tau_1}\right) + \end{gathered}$$

$$+\frac{K(x,t,x)\alpha_1^{(0)}(x,t)}{\lambda_1(x)}e^{\tau_1}-\frac{K\left(x,t,x_0\right)\alpha_1^{(0)}(x_0,t)}{\lambda_1(x_0)},$$

of equation (18) belongs to space U. Equation (18) is solvable in this space U if and only if condition (13) are satisfied, which in our case take the form

$$-\frac{d}{dx}\left(\alpha_1^{(0)}(x,t)\right)+\frac{K(x,t,x)}{\lambda_1(x)}\alpha_1^{(0)}(x,t)=0.$$

Attaching to this system the initial conditions $\alpha_1^{(0)}(x_0,t)=y^0+a^{-1}(x_0)h(x_0,t)$, we find uniquely functions

$$\alpha_1^{(0)}(x,t)=\alpha_1^{(0)}(x_0,t)exp\left\{\int\limits_{x_0}^{x}\left(\frac{K(s,t,s)}{\lambda_1(s)}\right)ds\right\},$$

therefore, we uniquely calculate the solution (17) of the problem (9_0) in the space U. Moreover, the main term of the asymptotic of the solution to problem (2) has the form

$$y_{\varepsilon 0}(x,t)=y_0^{(0)}(x,t)+\alpha_1^{(0)}(x_0,t)exp\left\{\int\limits_{x_0}^{x}\left(\frac{K(s,t,s)}{\lambda_1(s)}\right)ds\right\}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_1(\theta)d\theta},$$

where $\alpha_1^{(0)}(x_0,\ t)=y^0+a^{-1}(x_0)h(x_0,\ t)$, $y_0^{(0)}(x,\ t)$ is the solution of the integrated system $y_0^{(0)}(x,t)=$ $=\int\limits_{x_0}^{x}\left(-a^{-1}(x)K(x,t,s)\right)y_0^{(0)}(s,t)ds-a^{-1}(x)h(x,t)$.

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Б.Т. Калимбетов, Х.Ф. Етмишев

Жылдам осцилляцияланатын коэффициентті дербес туындылы скаляр интегро-дифференциалдық теңдеулердің асимптотикалық шешімдері

Мақалада микро-біртекті емес ортадағы әртүрлі физикалық процестерді сипаттайтын жылдам осцилляцияланатын коэффициентті дербес туындылы сингуляр ауытқыған интегро-дифференциалдық теңдеу үшін Коши есебін асимптотикалық интегралдау алгоритмін әзірлеуге арналған. Бұл дербес туындылы дифференциалдық теңдеулер теориясындағы бағыт қарқынды дамып келеді және радиофизикада, электротехникада, сүзу теориясында, фазалық өту теориясында, серпімділік теориясында және физиканың, механика мен техниканың басқа да бөлімдерінде көптеген қолданыс табады. Мұндай процестерді зерттеу үшін әдетте асимптотикалық әдістер қолданылады. Қазіргі уақытта қарқынды дамып келе жатқан сандық әдістер асимптотикалық әдістерді жоққа шығармайтыны белгілі. Бұл бірнеше себептерге байланысты болады. Біріншіден, негізделген асимптотикалық шешім, әсіресе оның негізгі мүшесі, шешімнің сапасы туралы қосымша ақпаратты алуға және осы мағынада көбінесе табуы мүмкін болмаған нақты шешімді ауыстыруға мүмкіндік береді. Екіншіден, жоғарыда айтылғандай, шешім құрылымын білу күрделі есептерді шешудің сандық әдістерін жасауға көмектеседі, сондықтан асимптотикалық әдістердің дамуы сандық әдістердің дамуына ықпал етеді. Есептің регуляризациясы жүргізілген, жалпы итерациялык есептердің қалыпты және бір мәнді шешілімділігі дәлелденген.

Кілт сөздер: сингуляр ауытқу, дербес туындылы дифференциалдық теңдеулер, интегралды регуляризациялау, итерациялық есептердің шешімділігі.

Б.Т. Калимбетов, Х.Ф. Етмишев

Асимптотические решения скалярных интегро-дифференциальных уравнений с частными производными и с быстро осциллирующими коэффициентами

Статья посвящена разработке алгоритма асимптотического интегрирования задачи Коши для сингулярно возмущенного интегро-дифференциального уравнения в частных производных с быстро осциллирующими коэффициентами, описывающими различные физические процессы в микронеоднородных средах. Это направление в теории уравнений с частными производными интенсивно развивается и находит многочисленные применения в радиофизике, электротехнике, теории фильтрации, теории фазовых переходов, теории упругости и других разделах физики, механики и техники. Для исследования таких процессов обычно используются асимптотические методы. Известно, что бурно развивающиеся в настоящее время численные методы не исключают асимптотических. Это происходит по ряду причин. Во-первых, разумно построенная асимптотика, особенно ее главный член, несет существенную для приложений информацию о качественном поведении решения и в этом смысле в определенной мере заменяет точное решение, которое чаще всего не может быть найдено. Во-вторых, как это следует из сказанного выше, знание структуры решения помогает при разработке численных методов решения сложных задач, поэтому развитие асимптотических методов способствует развитию численных методов. Произведена регуляризация задачи, доказана нормальная и однозначная разрешимость общих итерационных задач.

Ключевые слова: сингулярное возмущение, дифференциальное уравнение с частными производными, регуляризация интеграла, разрешимость итерационных задач.

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