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On multi-periodic solutions of quasilinear autonomous systems with an operator of differentiation on the Lyapunov's vector field

A quasilinear autonomous system with an operator of differentiation with respect to the characteristic directions of time and space variables associated with a Lyapunov's vector field is considered. The question of the existence of multi-periodic solutions on time variables is investigated, when the matrix of a linear system along characteristics has the property of exponential stability. And the non-linear part of the system is sufficiently smooth. In the note, on the basis of Lyapunov's method, the necessary properties of the characteristics of the system with the specified differentiation operator were substantiated; theorems on the existence and uniqueness of multi-periodic solutions of linear homogeneous and nonhomogeneous systems were proved; sufficient conditions for the existence of a unique multi-periodic solution of a quasilinear system were established. In the study of a nonlinear system, the method of contraction mapping was used.

Key words: multi-periodic solutions, autonomous system, operator of differentiation, Lyapunov's vector field.

Introduction

It is known that many phenomena connected by a continuous medium are described by systems of partial differential equations. In many cases, these systems are quasilinear, and these phenomena (sound, light, electromagnetic, gas and hydromechanical) are oscillatory-wave in nature. Consequently, the study of solutions of such systems with oscillatory properties over both time and space variables belong to an important part of the theory of equations in ordinary and partial derivatives. The foundations of this theory were laid in the classical works of A.M. Lyapunov, H.Poincaré and the fundamental research of Andronov-Witt-Khaykin, Krylov-Bogoliubov-Mitropolsky-Samoilenko, Kolmogorov-Arnold-Moser, etc.

A peculiar approach to the problems of the theory of oscillations was proposed in the works of V. Kharasakhal and D.U. Umbetzhanov [1–8], based on a deep connection between an almost periodic function of one variable and a periodic function of many variables, called a multi-periodic function, where the problems are quasi-periodic solutions of ordinary differential equations, are studied on the basis of multi-periodic solutions of systems of the partial differential equations of the first order. In this connection, we note that many quite serious results, known from oscillatory solutions of ordinary differential equations, they are extended to the case of multi-periodic solutions of partial differential equations [9–20], which were further developed in the articles [21–23].

We note, that some information on multi-periodic solutions of systems of the partial differential equations is contained in the literature review of the fundamental work [24], where the number of papers by one of the authors is presented.

We also note, that many theoretical questions of physics and technology are based on oscillatory processes. In particular, we pay attention to the works [25, 26], where an interesting research was conducted of problems from hydromechanics and control theory related to oscillatory processes described by the differential and integro-differential equations. These equations are attractive because it is possible for them to consider the problem of multi-periodic solutions and use the methods outlined in this article.

Of particular interest is the work [27], where the equations with a differentiation operator along the directions of a vector field on a torus are considered and conditions for the existence of their periodic solutions are established. Note that the differential operator under consideration is similar to the differentiation operator, which given in this note.

The methods of Poincaré-Lyapunov and Hamilton-Jacobi for integrating and researching of the periodic solutions are the basis of the methodology for studying the problem of this work. It is obvious, that the sources of multi-periodic solutions of the differential equations are their periodic solutions with different rationally incommensurable frequencies. In this regard, our attention is drawn to the problems studied in the articles [28, 29] and some commonality of their study methods with the methods of this work.

One of the common ways to investigate the periodic solutions is to use the methods of boundary value problems for the differential equations. In the works [30–35] for investigating the oscillatory solutions of some equations of various types of mathematical physics was used, a technique calling the method of parameterization. We note that the equations under consideration are representable as systems of equations of first-order derivatives.

In this article, we consider the quasilinear system of equations with a differentiation operator along the directions of the vector fields, where the characteristic directions of the differentiation operator along the time and space variables are independent, with the space variables being differentiated along the directions defined by the Lyapunov's system.

In the case of a non-autonomous system, the frequencies of the desired multi-periodic oscillations are mainly determined by the system itself. Consequently, the frequencies and their number are known in advance.

In this autonomous case, the main difficulty of the considering problem is related to the uncertainty of the frequency of periodic oscillations, which are components of the desired multi-periodic oscillations. This difficulty was surmountable that the characteristic vector field satisfies the conditions of the Lyapunov's system. Although, systems of the partial differential equations that do not contain time variables are often found in the scientific literature, but the problem of this note on the formulation is new and is being investigated for the first time.

We consider the autonomous system

$$Dx = P(\zeta)x + f(\zeta, x), \tag{1}$$

with differentiation operator

$$D = \frac{\partial}{\partial \tau} + \left\langle e, \frac{\partial}{\partial \overline{\tau}} \right\rangle + \left\langle J\zeta + \psi(\zeta), \frac{\partial}{\partial \zeta} \right\rangle, \tag{2}$$

where $x = (x_1, \ldots, x_n) \in R^n$ are unknown vector-functions with respect to the time $\tau \in R$, $\overline{\tau} = (\tau_1, \ldots, \tau_m) \in R^m$ and space $\zeta = (\zeta_0, \ldots, \zeta_k)$, $\zeta_j = (\xi_j, \eta_j) \in R^2$, $j = \overline{0, k}$, variables; $\left\langle e, \dfrac{\partial}{\partial \overline{\tau}} \right\rangle$ is the scalar product of m-vectors $e = (1, \ldots, 1)$ and $\dfrac{\partial}{\partial \overline{\tau}} = \left(\dfrac{\partial}{\partial \tau_1}, \ldots, \dfrac{\partial}{\partial \tau_m} \right)$; J is a $(2k+2)$-dimensional constant matrix; $\psi(\zeta)$ is a $(2k+2)$-vector-function given in a δ-neighborhood R_δ^{2k+2} of a point $\zeta = 0$ in Euclidean space R^{2k+2}; $\dfrac{\partial}{\partial \zeta} = \left(\dfrac{\partial}{\partial \zeta_0}, \ldots, \dfrac{\partial}{\partial \zeta_k} \right)$, $\dfrac{\partial}{\partial \zeta_j} = \left(\dfrac{\partial}{\partial \xi_j}, \dfrac{\partial}{\partial \eta_j} \right)$, $j = \overline{0, k}$, is a vector operator.

The matrix $P(\zeta) = [p_{ij}(\zeta)]_1^n$ is holomorphic in the R_ε^{2n+2} neighborhood of the point $\zeta = 0$:

$$P(\zeta) = \sum_{j=0}^{+\infty} \frac{1}{j!} \left\langle \zeta, \frac{\partial}{\partial \zeta} \right\rangle^j P(0), \, \zeta \in R_\varepsilon^{2k+2}, \tag{3}$$

where $\varepsilon > 0$ is some constant and $\delta = \delta(\varepsilon) > 0$ is sufficiently small.

The vector-function $f(\zeta, x)$ has the following properties of continuity and smoothness

$$f(\zeta, x) \in C_\zeta^{(e)} \left(R_\varepsilon^{2k+2} \times R_\Delta^n \right) \tag{4}$$

with bounded matrix of Jacobi

$$\left| \frac{\partial f(\zeta, x)}{\partial x} \right| \leq c, \; (\zeta, x) \in \overline{R}_\varepsilon^{2k+2} \times \overline{R}_\Delta^n, \tag{5}$$

where $c > 0$ is a constant, $\overline{R}_\varepsilon^{2k+2} \times \overline{R}_\Delta^n$ is the closure of the region $R_\varepsilon^{2k+2} \times R_\Delta^n$.

Thus, set the problem to clarify the conditions the $(\theta, \overline{\theta})$-periodicity of solutions of the system (1) when conditions (3), (4), and (5) are performed.

The differentiation operator along the directions of the diagonal of time and space variables on the Lyapunov's vector field

Differentiation by the operator D is conducted along directions of vector fields of time variables

$$\frac{d\overline{\tau}}{d\tau} = e \tag{6}$$

and space variables

$$\frac{d\zeta}{d\tau} = J\zeta + \psi(\zeta), \tag{7}$$

associated with the time variable $\tau \in R$.

The characteristic of the vector equation (6), outgoing from the point $\overline{\tau}_0 = (\tau_1^0, \dots, \tau_m^0)$ when $\tau = \tau_0$ is determined by the relation $\overline{\tau} = \overline{\tau}_0 + e(\tau - \tau_0)$. For our purpose, it's useful to take as the initial point $\overline{\tau}_0 = e\tau_0$. Therefore, we have

$$\overline{\tau} = e\tau. \tag{8}$$

It should also be noted here that the dimension m of the time vector $\overline{\tau}$ is related to the dimension of the common frequency basis of the family periodic solutions of the autonomous system (7), which cannot be specified in advance. In our case, we note that $m = k$.

The vector field (7) can be characterized by the following properties:

a) The matrix J can be represented in the form

$$J = diag[\nu_0 I_2, \dots, \nu_k I_2], \ \langle q, \nu \rangle \neq 0, \ q \in Z^{k+1}, \ q \neq 0, \tag{9}$$

where $I_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a two-dimensional symplectic unit, $\nu_j, j = \overline{0,k}$, are incommensurable frequencies, $q = (q_0, \dots, q_k) \in Z^{k+1}$ is an integer vector, $\nu = (\nu_0, \dots, \nu_k)$ is vector, Z is the set of integers.

b) The vector function $\psi(\zeta)$ is formed by a given scalar holomorphic function $\Psi(\zeta)$ in some δ-neighborhood R_δ^{2k+2} of the point $\zeta = 0$ in Euclidean space R^{2k+2} by applying an operator $I\frac{\partial}{\partial\zeta}$ with $(2k+2)$-matrix $I = diag[I_2, \dots, I_2]$, whose decomposition of the function of which $\Psi(\zeta)$ begins with a homogeneous form of at least the third degree:

$$\begin{gathered} \psi(\zeta) = I\tfrac{\partial}{\partial\zeta}\Psi(\zeta), \zeta \in R_\delta^{2k+2}, \\ \Psi(\zeta) = \sum_{j=3}^{+\infty} \tfrac{1}{j!} \left\langle \zeta_j, \tfrac{\partial}{\partial\zeta} \right\rangle^j \Psi(0). \end{gathered} \tag{10}$$

It is obvious, that the vector field (7) under the conditions (9) and (10) belongs to the class of Lyapunov's systems.

By conditions (9) and (10) can be represented system (7) to the scalar form

$$\begin{cases} \dfrac{d\xi_j}{d\tau} = -\nu_j \eta_j - \dfrac{\partial \Psi(\zeta)}{\partial \eta_j}, \\ \dfrac{d\eta_j}{d\tau} = \nu_j \xi_j + \dfrac{\partial \Psi(\zeta)}{\partial \xi_j}, \ j = \overline{0,k}. \end{cases} \tag{11}$$

with the first integral

$$H(\zeta) = \sum_{j=0}^{k} \frac{\nu_j}{2}(\xi_j^2 + \eta_j^2) + \Psi(\zeta). \tag{12}$$

By the first integral (12) the system (11) can be written in the form of a canonical system

$$\begin{cases} \dfrac{d\xi_j}{d\tau} = -\dfrac{\partial H(\zeta)}{\partial \eta_j}, \\ \dfrac{d\eta_j}{d\tau} = \dfrac{\partial H(\zeta)}{\partial \xi_j}, \ j = \overline{0,k}. \end{cases} \tag{13}$$

According to the Lyapunov's method [1, 2], the variables $(\xi_l, \eta_l), l \neq j, 0 \leq l \leq k$ as functions (ξ_j, η_j) with a fixed number j can be determined from the system of the partial differential equations

$$\begin{cases} \frac{\partial H}{\partial \xi_j}\frac{\partial x_l}{\partial \eta_j} - \frac{\partial H}{\partial \eta_j}\frac{\partial x_l}{\partial \xi_j} = -\frac{\partial H}{\partial \eta_j}, \\ \frac{\partial H}{\partial \xi_j}\frac{\partial y_l}{\partial \eta_j} - \frac{\partial H}{\partial \eta_j}\frac{\partial y_l}{\partial \xi_j} = \frac{\partial H}{\partial \xi_j}, \; l \neq j, \; 0 \leq l \leq k, \end{cases} \tag{14}$$

where $H = H(\zeta)$ is the Hamiltonian (12) of the systems (13).

System (14) with the initial condition $(\xi_l, \eta_l) = (0, 0)$ for $(\xi_j, \eta_j) = 0$, under conditions (9) and (10) allows an unique holomorphic solution $(\xi_l^*(\xi_j, \eta_j), \eta_l^*(\xi_j, \eta_j)) = \zeta_l^*(\xi_j, \eta_j)$ in the sufficiently small neighborhood R_δ^2 of the point $\zeta_l = 0$ in the plane R^2 for fixed values $l \neq j$.

We obtain the function

$$H_j(\xi_j, \eta_j) = H\left(\zeta_0^*(\xi_j, \eta_j), \ldots, \zeta_{j-1}^*(\xi_j, \eta_j), \zeta_j, \zeta_{j+1}^*(\xi_j, \eta_j), \ldots, \zeta_k^*(\xi_j, \eta_j)\right). \tag{15}$$

by substituting found solutions $\zeta_0^*(\xi_j, \eta_j), \ldots, \zeta_{j-1}^*(\xi_j, \eta_j), \zeta_{j+1}^*(\xi_j, \eta_j), \ldots, \zeta_k^*(\xi_j, \eta_j)$ of the systems (14) to the Hamiltonian $H(\zeta)$. Also, we set up a function

$$g_j(\xi_j, \eta_j) = 1 + \sum_{l \neq j} \left(\frac{\partial \xi_l^*(\xi_j, \eta_j)}{\partial \xi_j} \frac{\partial \eta_l^*(\xi_j, \eta_j)}{\partial \eta_j} - \frac{\partial \xi_l^*(\xi_j, \eta_j)}{\partial \eta_j} \frac{\partial \eta_l^*(\xi_j, \eta_j)}{\partial \xi_j} \right). \tag{16}$$

On the basis of functions (15) and (16), we consider the system of ordinary differential equations

$$\begin{cases} g_j(\xi_j, \eta_j)\frac{d\xi_j}{d\tau_j} = -\frac{\partial H_j(\xi_j, \eta_j)}{\partial \eta_j}, \\ g_j(\xi_j, \eta_j)\frac{d\eta_j}{d\tau_j} = \frac{\partial H_j(\xi_j, \eta_j)}{\partial \xi_j}, \end{cases} \tag{17}$$

which is a Lyapunov's system corresponding to the frequency ν_j. Therefore, system (17) defines a two-parameter family of periodic solutions

$$\begin{aligned} \xi_j &= h_j^{'}(\tau_j - \tau_0, \xi_j^0, \eta_j^0), \\ \eta_j &= h_j^{''}(\tau_j - \tau_0, \xi_j^0, \eta_j^0) \end{aligned} \tag{18}$$

with arbitrary initial values $(\xi_j, \eta_j)\big|_{\tau_j = \tau_0} = (\xi_j^0, \eta_j^0)$ from a sufficiently small neighborhood R_δ^2, and a period

$$\theta_j = \frac{2\pi}{\nu_j}\left(1 + c_j^{(1)} H_j\left(\xi_j^0, \eta_j^0\right) + c_j^{(2)}\left[H_j\left(\xi_j^0, \eta_j^0\right)\right]^2 + \ldots\right), \tag{19}$$

which the coefficients $c_j^{(1)}, c_j^{(2)}, \ldots$ didn't depend on the initial data, and they are $\theta_j = \frac{2\pi}{\nu_j}$ when $(\xi_j^0, \eta_j^0) = (0, 0)$.

Thus, by changing j from 0 to k and using the Lyapunov's method, we obtain all $1 + k$ periodic solutions (18) of the system (17). These solutions are components of a multi-periodic solution ζ of the system (7) in the form

$$\zeta = \left(h_0^{'}\left(\tau - \tau_0, \xi_0^0, \eta_0^0\right), h_0^{''}\left(\tau - \tau_0, \xi_0^0, \eta_0^0\right), \ldots, h_k^{'}\left(\tau_k - \tau_0, \xi_k^0, \eta_k^0\right), h_k^{''}\left(\tau_k - \tau_0, \xi_k^0, \eta_k^0\right)\right) \equiv$$

$$\equiv h\left(\tau - \tau_0, \tau_1 - \tau_0, \ldots, \tau_k - \tau_0, \xi_0^0, \eta_0^0, \ldots, \xi_k^0, \eta_k^0\right) \tag{20}$$

with initial condition

$$\zeta\big|_{\tau = \tau_1 = \ldots = \tau_k = \tau_0} = \left(\xi_0^0, \eta_0^0, \ldots, \xi_k^0, \eta_k^0\right) = \zeta_0 \tag{21}$$

and vector-period $\theta = (\theta_0, \theta_1, \ldots, \theta_k)$ with components (19) on a vector variable $\overline{\tau} = (\tau, \tau_1, \ldots, \tau_k)$.

We are passing to the vector notation from (20)–(21), and then the solution ζ of the system (7) is represented in the form

$$\zeta = h(\tau - \tau_0, \overline{\tau} - e\tau_0, \zeta_0), \tag{22}$$

where $h(0, \overline{0}, \zeta_0) = \zeta_0$. Equation (22) together with equation (8), represent the characteristics of the operator (2). Thus, the following assertion is substantiated.

1^0. Under conditions (9) and (10), operator (2) is the operator of differentiation with respect to τ of the functions $x(\tau, \overline{\tau}, \zeta)$ of the along direction of the main diagonal (8) of the time variables and along multi-periodically closed curves (22) with respect to space variables.

Therefore, the function Dx along the characteristic, given by relation (22), determines the rate of change of the function $x = x(\zeta)$ with respect to τ:

$$Dx\big|_{\zeta=h} = \frac{dx(h)}{d\tau}.$$

The statement 1^0 allows us to go from the differential equations with operator D to the integral equations, defined along the characteristics.

By the uniqueness property the solutions of the system (7), from the equation of characteristics (22) we have the expression

$$\zeta_0 = h(\tau_0 - \tau, e\tau_0 - \overline{\tau}, \zeta), \tag{23}$$

which is the first integral of the operator $D : Dh = 0$.

In addition, based on the same property, we obtain the group property of the characteristic in the form

$$h\left(\tau - s, \overline{\tau} - es, h(s - \tau_0, es - e\tau_0, \zeta_0)\right) = h\left(\tau - \tau_0, \overline{\tau} - e\tau_0, \zeta_0\right) \tag{24}$$

with $s \in R$.

Then we have the following statement.

2^0. Under the conditions of paragraph 1^0 the function $x\left(s - \tau_0, \overline{\tau} - e\tau_0, h(s - \tau_0, es - e\tau_0, \zeta_0)\right)$ taking into account property (24), is proceed to the function $x\left(s - \tau, es - \overline{\tau}, h(s - \tau, es - \overline{\tau}, \zeta)\right)$ with parameter $s \in R$, and variables $(\tau, \overline{\tau}, \zeta)$, where the given function defined along the characteristic

$$\tau = s - \tau_0,$$

$$\overline{\tau} = es - e\tau_0,$$

$$\zeta = h(s - \tau_0, es - e\tau_0, \zeta_0)$$

with a parameter s based on the first integrals of the systems (6) and (7) in the form

$$\tau_0 = \tau,$$

$$e\tau_0 = \overline{\tau},$$

$$\zeta_0 = h(\tau_0 - \tau, e\tau_0 - \overline{\tau}, \zeta),$$

obtained by relations (8) and (23).

Paragraph 2^0 allows leaving expressions defined along the characteristics of the operator D to the space of variables $(\tau, \overline{\tau}, \zeta)$.

Further, by the periodicity the characteristics (18) of the operator D in the period (19), the property of multi-periodicity of the vector-function (22) can be represented as

$$h(\tau + \theta, \overline{\tau} + q\overline{\theta}, \zeta) = h(\tau, \overline{\tau}, \zeta), \tag{25}$$

where the vector-period $(\theta, \overline{\theta}) = (\theta_0, \theta_1, \ldots, \theta_k)$ with the components $\theta = \theta_0, \overline{\theta} = (\theta_1, \ldots, \theta_k)$ is defined by the relation (19), and the periods $\theta_j, j = \overline{0, k}$ depend on the initial data $\zeta = \zeta_0$ of the characteristics of the operator (2), $q \in Z^m$.

We note, that when the question of multi-periodicity, we consider periods $(\theta, \overline{\theta})$ in the space of variables $(\tau, \overline{\tau}, \zeta) \in R \times R^k \times R_\delta^{k+1}$, by replacing the initial data ζ_0 to the corresponding value (23), and we get functions with respect to variables $(\tau, \overline{\tau}, \zeta)$, and parameter $\tau_0 \in R$:

$$\begin{aligned} \theta &= \theta(\tau_0, \tau, \overline{\tau}, \zeta), \\ \overline{\theta} &= \overline{\theta}(\tau_0, \tau, \overline{\tau}, \zeta). \end{aligned} \tag{26}$$

Since expressions (23) represent the first integral, that is, we have identity

$$Dh(\tau_0-\tau, e\tau_0-\overline{\tau},\zeta)=0, \tag{27}$$

then periods (26) are also first integrals, and therefore, we have identity relations

$$D\theta(\tau_0,\tau,\overline{\tau},\zeta)=0,$$
$$D\overline{\theta}(\tau_0,\tau,\overline{\tau},\zeta)=0. \tag{28}$$

Thus, as a consequence of the identities (27) and (28), we can formulate the following statement.

3^0. Under the conditions of the preceding paragraphs, any smooth function $f(h)$ has the properties

$$Df\big(h\left(s-\tau, es-\overline{\tau},\zeta\right)\big)=0, \tag{29}$$

$$f\big(h\left(s-\tau+\theta, es-\overline{\tau}+q\overline{\theta},\zeta\right)\big)=f\big(h\left(s-\tau, es-\overline{\tau},\zeta\right)\big), \tag{30}$$

where $s\in R$ is the parameter, and $q\in Z^m$, $h(s-\tau, es-\overline{\tau},\zeta)$ is the integral function (23).

In deducing relation (29) it was taken into account that if h is the first integral, and then the smooth function $f(h)$ is also an integral. Identity (30) follows from identity (25).

Homogeneous linear system

We consider a linear system

$$Dx=P(\zeta)x, \tag{31}$$

with respect to the unknown vector-function $x=(x_1,\ldots,x_n)$, where the operator D is defined by the formula (2) with properties (9) and (10); $P(\zeta)=[p_{ij}(\zeta)]_1^n$ is holomorphic matrix in the R_ε^{2n+2} neighborhood of the point $\zeta=0$, and satisfies condition (3).

The system (31) along the characteristics (22) represents a system of the ordinary differential equations with the multi-periodic matrix $P(h(\tau-s,\overline{\tau}-es,\zeta_0))$ with respect to $(\tau,\overline{\tau})$ with period $(\theta,\overline{\theta})$.

Then it is possible to determine the matrix X of the linear system (31)on the basis of the integral equation

$$X(\tau_0,\tau,\overline{\tau},\zeta)=E+\int\limits_{\tau_0}^{\tau}P(h(s-\tau,es-\overline{\tau},\zeta))X(\tau_0,s,es,\zeta)\,ds, \tag{32}$$

where E is the unit n-matrix, $\tau_0\in R$, $\tau\in R$, $\overline{\tau}\in R^k$, $\zeta\in R_\varepsilon^{2k+2}$ for sufficiently small $\varepsilon>0$, $X(\tau_0,\tau_0,e\tau_0,\zeta)=E$.

Obviously, the matriciant $X(\tau_0,\tau,\overline{\tau},\zeta)$ is holomorphic with respect to ζ by virtue of (3) and $(\theta,\theta,\overline{\theta})$-periodic by $(\tau_0,\tau,\overline{\tau})$

$$X(\tau_0+\theta,\tau+\theta,\overline{\tau}+e\theta,\zeta)=X(\tau_0,\tau,\overline{\tau},\zeta). \tag{33}$$

Further, suppose that the matrix $P(\zeta)$ provides the property of the exponential stability of system (31) in the form

$$|X(\tau_0,\tau,\overline{\tau},\zeta)|\le ae^{-\alpha(\tau-\tau_0)},\ \tau\ge\tau_0 \tag{34}$$

with constants $a\ge 1,\alpha>0$, where $\tau_0\in R$. If we take into account the solution $x=x(\tau_0,\tau,\overline{\tau},\zeta)$ of the system (31) with an initial condition that turns into the initial smooth function $u(\zeta)$, when $\tau=\tau_0$ in the form

$$x\big|_{\tau=\tau_0}=u(\zeta)\in C_\zeta^{(1)}\left(R_\varepsilon^{2k+2}\right)$$

expressed by

$$x(\tau_0,\tau,\overline{\tau},\zeta)=X(\tau_0,\tau,\overline{\tau},\zeta)u(h(\tau_0-\tau,e\tau_0-\overline{\tau},\zeta)), \tag{35}$$

then from the condition (34) implies the absence of the multi-periodic solution of the system (31), that is different from zero.

Lemma 1. Let conditions (3), (9), (10), and (34) be satisfied. Then the homogeneous linear system (31) has no multi-periodic solution, except for the trivial one.

Proof. Indeed, from the representation of solutions (35) and condition (34) follows that for fixed values ζ any solution with nonzero initial data $u\ne 0$ is unbounded.

Therefore, such a solution cannot be multi-periodicity. It only follows that $u=0$ is the only multi-periodic solution of the system (31).

Nonhomogeneous linear system

Now we consider the system in the form

$$Dx = P(\zeta)x + f(\zeta) \tag{36}$$

with free term $f(\zeta)$ of holomorphic in R_ε^{2k+2}:

$$f(\zeta) = \sum_{j=0}^{+\infty} \frac{1}{j!} \left\langle \zeta, \frac{\partial}{\partial \zeta} \right\rangle^j f(0), \zeta \in R_\varepsilon^{2k+2}, \tag{37}$$

where $\varepsilon > 0$ is some constant.

Theorem 1. Let conditions (3), (9), (10), (34), and (37) be satisfied. Then the system (36) has the unique $(\theta, \overline{\theta})$-periodic solution holomorphic with respect to $\zeta \in R_\delta^{2k+2}$ for sufficiently small $\delta = \delta(\varepsilon) > 0$

$$x^*(\tau, \overline{\tau}, \zeta) = \int_{-\infty}^{\tau} X(s, \tau, \overline{\tau}, \zeta) f(h(s - \tau, es - \overline{\tau}, \zeta))\, ds, \tag{38}$$

where $\delta = \delta(\varepsilon)$ is chosen such that, for $\zeta \in R_\delta^{2k+2}$ the inequality $|h(\tau, \overline{\tau}, \zeta)| < \varepsilon$ is satisfied.

Proof. Indeed, by a direct verification, we see, that function (38), which is determined by condition (34), in the form of an improper integral is the solution of the system (36). In this case, it is necessary to take into account that the matriciant X satisfies the matrix equation (32), and the integral h has the properties (22) and (23). The periodicity of the solution (37) with respect to τ with a period θ is checked on the basis of the property (33), and the $\overline{\theta}$-periodicity with respect to $\overline{\tau}$ follows from the property (25). The property of holomorphy follows from the holomorphy of the integral h of the matrix P, and the function f given in conditions (3), (10), and (37).

In conclusion, we note that if conditions (3) and (37) about the holomorphy of the matrix $P(\zeta)$ and vector-function $f(\zeta)$ are replaced by the conditions of their continuous differentiability, then generalizing Theorem 1, we can get the result about the existence of the multi-periodic solution (38), without the property of holomorphy.

Therefore, we have the following theorem.

Theorem 2. Let the matrix function $P(\zeta)$ be continuously differentiable in R_ε^{2k+2}:

$$P(\zeta) \in C_\zeta^{(e)} \left(R_\varepsilon^{2k+2}\right), \tag{39}$$

and the free term $f(\zeta)$ of the system also has the same property:

$$f(\zeta) \in C_\zeta^{(e)} \left(R_\varepsilon^{2k+2}\right), \tag{40}$$

where $C_\zeta^{(e)} \left(R_\varepsilon^{2k+2}\right)$ is the class of smooth functions of order $e = (1, \ldots, 1)$ in R_ε^{2k+2}. If satisfied conditions (9), (10) and (34), then relation (38) represents the unique $(\theta, \overline{\theta})$-periodic solution of the system (36) for sufficiently small $\delta = \delta(\varepsilon) > 0$, when $\zeta \in R_\delta^{2k+2}$.

The proof of the Theorem 2 is similar to the proof of the Theorem 1, with the difference, that the smoothness of the solutions is everywhere provided by the conditions (39), (40).

Now, additionally we consider the case when the free term f, except $\zeta \in R_\delta^{2k+2}$, depends on $(\tau, \overline{\tau}) \in R \times R^k$.

Then we have a non-autonomous system of equations

$$Dx = P(\zeta)x + f(\tau, \overline{\tau}, \zeta), \tag{41}$$

where the vector-function $f(\tau, \overline{\tau}, \zeta)$ has the property

$$f(\tau + \theta, \overline{\tau} + q\overline{\theta}, \zeta) = f(\tau, \overline{\tau}, \zeta) \in C_{\tau, \overline{\tau}, \zeta}^{(1, \overline{e}, e)} \left(R \times R^k \times R_\varepsilon^{2k+2}\right), \tag{42}$$

$q \in Z^k$, $\overline{e} = (1, \ldots, 1)$ is k-vector, $e = (1, \ldots, 1)$ is $(2k+2)$-vector, $\varepsilon = const > 0$.

Theorem 3. Let conditions (9), (10), (34), (39), and (42) be satisfied. Then system (41) allows the unique $(\theta, \overline{\theta})$-periodic solution in the form

$$x^*(\tau, \overline{\tau}, \zeta) = \int_{-\infty}^{\tau} X(s, \tau, \overline{\tau}, \zeta) f(s, es, h(s - \tau, es - \overline{\tau}, \zeta))\, ds, \tag{43}$$

where $(\tau, \overline{\tau}, \zeta) \in R \times R^k \times R_\delta^{2k+2}$, $\delta = \delta(\varepsilon)$ is a sufficiently small positive number.

The proof is conducted similarly to the proof of the Theorems 1 and 2, and therefore we will not prove that theorem.

Here, since the free term f is periodically with respect to $(\tau, \overline{\tau})$ with the same periods $(\theta, \overline{\theta})$ as the integral $h(\tau, \overline{\tau}, \zeta)$, then the oscillations described by the system (41), and doesn't undergo any other changes. We will need this case when studying a nonlinear autonomous system.

Nonlinear system

Let us consider the question of the existence of the multi-periodic solution of the system (1) satisfying conditions (3), (4), and (5). From conditions (4) and (5) follows that

$$|f(\zeta, x) - f(\zeta, y)| \leq c|x - y|, \tag{44}$$

$$|f(\zeta, x)| \leq b + c|x|, \tag{45}$$

where $(\zeta, x) \in \overline{R}_{\varepsilon}^{2k+2} \times \overline{R}_{\Delta}^{n}$. Let the constants α, a, b, c and Δ be related by

$$a(b + c\Delta) < \alpha\Delta. \tag{46}$$

The value $\delta = \delta(\varepsilon) > 0$ is chosen such that, $R_{\delta}^{2k+2} \subset R_{\varepsilon}^{2k+2}$.

We consider the space $S_{\delta,\Delta}^{\theta,\overline{\theta}}$ of vector-functions $x(\tau, \overline{\tau}, \zeta)$, which continuous for each $(\tau, \overline{\tau}, \zeta) \in R \times R^k \times R_{\delta}^{2k+2}$, $(\theta, \overline{\theta})$-periodic for $(\tau, \overline{\tau})$ and bounded by number $\Delta > 0$ on the norm

$$\|x\| = \sup_{R \times R^k \times \overline{R}_{\delta}^{2k+2}} \max_{j} |x_j(\tau, \overline{\tau}, \zeta)| \leq \Delta.$$

We define the operator in this space

$$(Fx)(\tau, \overline{\tau}, \zeta) = \int_{-\infty}^{\tau} X(s, \tau, \overline{\tau}, \zeta) f\left(h(s - \tau, es - \overline{\tau}, \zeta), x(s, es, h(s - \tau, es - \overline{\tau}, \zeta))\right) ds. \tag{47}$$

Lemma 2. Let conditions (4), (5), (9), (10), (34), (39) and (46) be satisfied. Then the operator F in the space $S_{\delta,\Delta}^{\theta,\overline{\theta}}$ has the unique fixed point for sufficiently small $\delta > 0$.

By virtue the conditions (34) and (45) the improper integral (47) converges uniformly. Consequently, by virtue property (4) the function $(Fx)(\tau, \overline{\tau}, \zeta)$ is continuous for all arguments.

After shifting τ to period θ, by virtue periodicity $x(\tau, \overline{\tau}, \zeta)$ with respect to τ, and the property (33) of the matriciant X, by replacing τ with $\tau + \theta$, we are convinced that function (47) is also θ-periodic with respect to τ. The periodicity of Fx with respect to $\overline{\tau}$ with the period $\overline{\theta}$ directly follows from the $\overline{\theta}$-periodicity of the matriciant X, given in (33), and the integral h by $\overline{\tau}$ according to property (25).

Using the estimates (34), (45), and (46) from expression (47), we obtain

$$|(Fx)(\tau, \overline{\tau}, \zeta)| \leq \frac{a}{\alpha}(b + c\Delta) < \Delta.$$

Consequently, the function Fx bounded by the number $\Delta > 0$. Thus, we were convinced that the operator F reflects into itself of the space $S_{\delta,\Delta}^{\theta,\overline{\theta}}$.

Further, on the basis of (44), from the representation (47) of the operator F we have the inequality

$$|(Fx)(\tau, \overline{\tau}, \zeta) - (Fy)(\tau, \overline{\tau}, \zeta)| \leq \frac{ac}{\alpha}\|x - y\|.$$

Consequently, by virtue of condition (46) $d = \dfrac{ac}{\alpha} < 1$, the estimate

$$|Fx - Fy| \leq d\|x - y\|$$

shows that the operator F is a contraction operator.

Obviously, the space $S^{\theta,\overline{\theta}}_{\delta,\Delta}$ is complete, and then the operator F has the unique fixed point in this space

$$x^*(\tau,\overline{\tau},\zeta) = (Fx^*)(\tau,\overline{\tau},\zeta) \in S^{\theta,\overline{\theta}}_{\delta,\Delta}. \tag{48}$$

Theorem 4. Under the conditions of the Lemma 2, the system (1) has the unique $(\theta,\overline{\theta})$-periodic solution for sufficiently small $\delta > 0$.

For the proof, we show, that the $(\theta,\overline{\theta})$-periodic by $(\tau,\overline{\tau})$ solution $x_*(\tau,\overline{\tau},\zeta)$ of the system (1) satisfies to the integral equation

$$x(\tau,\overline{\tau},\zeta) = (Fx)(\tau,\overline{\tau},\zeta). \tag{49}$$

Indeed, using this solution $x_*(\tau,\overline{\tau},\zeta)$, we consider a linear system

$$Dx = P(\zeta)x + f(\zeta, x_*(\tau,\overline{\tau},\zeta)) \tag{50}$$

by the form (41).

Therefore, in accordance with the Theorem 3, system (50), according to formula (43), has the unique $(\theta,\overline{\theta})$-periodic solution x of the operator expression

$$x = (Fx_*)(\tau,\overline{\tau},\zeta), \tag{51}$$

for sufficiently small $\delta > 0$.

Since the solution $x_*(\tau,\overline{\tau},\zeta)$ satisfies the system (1), it is also solution of the equation (50).

Consequently, by virtue of the uniqueness of $(\theta,\overline{\theta})$-periodic solutions from (51), we have

$$x_*(\tau,\overline{\tau},\zeta) = (Fx_*)(\tau,\overline{\tau},\zeta). \tag{52}$$

i.e. showed that $x_*(\tau,\overline{\tau},\zeta)$ is solution of the integral equation (49).

But as shown in the Lemma 2, it has the unique multi-periodic solution. Consequently, from the identities (48) and (52) we have

$$x_*(\tau,\overline{\tau},\zeta) = x^*(\tau,\overline{\tau},\zeta).$$

Thus, the solution $x^*(\tau,\overline{\tau},\zeta)$ has the smoothness property for all arguments and is determined by the integral equation (49). The theorem is completely proved.

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Ж.А. Сартабанов, Б.Ж. Омарова

Ляпунов векторлық өрісі бойынша дифференциалдау операторлы квазисызықты автономдық жүйенің көппериодты шешімі туралы

Ляпунов векторлық өрісімен байланысты уақыт және кеңістік айнымалылы сипаттама бағыты бойынша дифференциалдау операторлы квазисызықты автономдық жүйе қарастырылды. Сызықты жүйенің матрицанты сипаттама бойында экспоненциалды орнықтылық қасиетке ие болғанда уақыт айнымалысы бойынша көппериодты шешімнің бар болуы туралы сұрақ зерттелді. Ал жүйенің сызықты емес бөлігі жеткілікті жатық болады. Мақалада Ляпунов әдісі негізінде көрсетілген дифференциалдау операторлы жүйенің сипаттамасының қажетті қасиеттері негізделді; біртекті және біртексіз сызықты жүйенің көппериодты шешімінің бар болуы және жалғыздығы туралы теорема дәлелденді; квазисызықты жүйенің жалғыз көппериодты шешімінің бар болуының жеткілікті шарты анықталды. Сызықты емес жүйені зерттеу барысында сығушы бейнелеу әдісі қолданылды.

Кілт сөздер: көппериодты шешім, автономдық жүйе, дифференциалдау операторы, Ляпунов векторлық өрісі.

Ж.А. Сартабанов, Б.Ж. Омарова

О многопериодических решениях квазилинейных автономных систем с оператором дифференцирования по векторному полю Ляпунова

Рассмотрена квазилинейная автономная система с оператором дифференцирования по характеристическим направлениям временных и пространственных переменных, связанных с векторным полем Ляпунова. Исследован вопрос о существовании многопериодических по временным переменным решений, когда матрицант линейной системы вдоль характеристик обладает свойством экспоненциальной устойчивости. А нелинейная часть системы является достаточно гладкой. В статье на основе метода Ляпунова обоснованы необходимые свойства характеристик системы с указанным оператором дифференцирования; доказаны теоремы о существовании и единственности многопериодических решений линейных однородных и неоднородных систем; установлены достаточные условия существования единственного многопериодического решения квазилинейной системы. При исследовании нелинейной системы использован метод сжатых отображений.

Ключевые слова: многопериодическое решение, автономная система, оператор дифференцирования, векторное поле Ляпунова.

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